Münster J. of Math. **5** (2012), 73–98
urn:nbn:de:hbz:6-88399593108

Münster Journal of Mathematics

Free group actions from the viewpoint of dynamical systems

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(Communicated by Siegfried Echterhoff)

Abstract. A dynamical system is a triple (A, G, α), consisting of a unital locally convex algebra A, a topological group G and a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$, which induces a continuous action of G on A. In this paper we present a new characterization of free group actions (in classical differential geometry), involving dynamical systems and representations of the corresponding transformation groups. Indeed, given a dynamical system (A, G, α), we provide conditions including the existence of "sufficiently many" representations of G which ensure that the corresponding action

$$\sigma : \Gamma_A \times G \to \Gamma_A, \quad (\chi, g) \mapsto \chi \circ \alpha(g)$$

of G on the spectrum Γ_A of A is free. In particular, the case of compact abelian groups is discussed very carefully and involves an application to the noncommutative geometry of principal torus bundles.

1. Introduction

Since the *Erlanger Programm* of Felix Klein, the defining concept in the study of a geometry has been its symmetry group. In classical differential geometry the symmetries of a manifold are measured by Lie groups, i.e., one studies smooth group actions of a Lie group G acting by diffeomorphisms on a manifold M. Of particular interest is the class of smooth group actions which are free and proper: In fact, by a classical result of differential geometry having a free and proper action of a Lie group G on a manifold P is equivalent to saying that P carries the structure of a (smooth) principal bundle with structure group G.

The origin of this paper is the question of whether there is a way to translate the geometric concept of principal bundles to noncommutative differential geometry. From a geometrical point of view it is, so far, not sufficiently well understood what a "noncommutative principal bundle" should be. Still, there are several approaches towards the noncommutative geometry of principal bundles: For example, there is a well-developed abstract algebraic approach known as Hopf-Galois extensions which uses the theory of Hopf algebras (cp. [19] or [9, Chap. VII]). Another topologically oriented approach can be found in [4]; here the authors use C^*-algebraic methods to develop a theory of principal noncommutative torus bundles based on Green's Theorem (cp. [7, Cor. 15]). Furthermore, the authors of [2] introduce C^*-algebraic analogs of freeness and properness. In [23] we have developed a geometrically oriented approach to the noncommutative geometry of principal bundles based on dynamical systems and the representation theory of the corresponding transformation groups.

The starting point of the last approach is the observation that (smooth) group actions may also be studied from the viewpoint of dynamical systems. Since we are interested in principal bundles, i.e., in free and proper smooth group actions, it is reasonable to ask if there exist natural (algebraic) conditions on a dynamical system (A, G, α) which ensure that the corresponding action

$$\sigma : \Gamma_A \times G \to \Gamma_A, \quad (\chi, g) \mapsto \chi \circ \alpha(g)$$

of G on the spectrum Γ_A of A is free. An important remark in this context is that the freeness condition of a group action (let's say of a group G) is pretty similar to the condition appearing in the definition of a family of point separating representations of G.

We now give a rough outline of the results that can be found in this paper, without going too much into detail.

A dynamical system (A, G, α) is called smooth if G is a Lie group and the group homomorphism $\alpha : G \to \mathrm{Aut}(A)$ induces a smooth action of G on A. The goal of Section 2 is to show that smooth group actions may also be studied from the viewpoint of smooth dynamical systems, i.e., that each smooth group action induces in a natural way a smooth dynamical system and vice versa.

In Section 3 we introduce the concept of a *free dynamical system*. In fact, given a dynamical system (A, G, α) and a representation (π, V) of G, we first associate the "generalized space of sections"

$$\Gamma_A V := \left\{ s \in A \otimes V \mid (\forall g \in G)(\alpha(g) \otimes \mathrm{id}_V)(s) = (\mathrm{id}_A \otimes \pi(g)^{-1})(s) \right\}.$$

Then we call a dynamical system (A, G, α) *free*, if A is commutative and G admits a family $(\pi_j, V_j)_{j \in J}$ of point separating representations of G such that the evaluation maps defined on $\Gamma_A V_j$ are surjective onto V_j (evaluation with respect to elements of Γ_A). We obtain the following theorem:

Theorem. (Freeness of the induced action) *If (A, G, α) is a free dynamical system, then the induced action*

$$\sigma : \Gamma_A \times G \to \Gamma_A, \ (\chi, g) \mapsto \chi \circ \alpha(g)$$

of G on the spectrum Γ_A of A is free.

Interpreting each $\Gamma_A V_j$ as a (possibly singular) vector bundle over Γ_A / G, this result means that the induced action of G on Γ_A is free if and only if every fiber of each $\Gamma_A V_j$ is "full", i.e., isomorphic to V_j.

In Section 4 we apply the results of Section 3 to dynamical systems arising from group actions in classical geometry. In particular, we will see how this leads to a new characterization of free group actions. For this purpose we have to restrict our attention to Lie groups that admit a family of finite-dimensional continuous point separating representations.

Theorem. (Characterization of free group actions) *Let P be a manifold, G a compact Lie group and $(C^\infty(P), G, \alpha)$ a smooth dynamical system. Then the following statements are equivalent:*

(a) The smooth dynamical system $(C^\infty(P), G, \alpha)$ is free.
(b) The induced smooth group action $\sigma : P \times G \to P$ is free.

In particular, in this situation the two concepts of freeness coincide.

From a geometrical point of view the previous theorem means that it is possible to test the freeness of a (smooth) group action $\sigma : P \times G \to P$ in terms of surjective maps defined on spaces of sections of associated (singular) vector bundles.

Section 5 is devoted to a more careful discussion of free dynamical systems with compact abelian transformation groups. Indeed, we show that the generalized spaces of sections associated to the dual group (which separates the points) are exactly the corresponding isotypic components and that the surjectivity condition is, for example, fulfilled if each isotypic component contains an invertible element; a requirement which is in the spirit of actions having "large" isotypic components (cp. [17]). Further, we explain how this observation leads to a natural concept of "trivial noncommutative principal bundles" (cp. [24]).

In Section 6 we introduce a stronger version of freeness for dynamical systems than the one given in Section 3. In fact, instead of considering arbitrary families $(\pi_j, V_j)_{j \in J}$ of (continuous) point separating representations of a topological group G, we restrict our attention to families $(\pi_j, \mathcal{H}_j)_{j \in J}$ of *unitary irreducible* point separating representations. At this point, we recall that each locally compact group G admits a family of continuous unitary irreducible point separating representations (cp. Theorem 3.3). In particular, we show that "strongly graded" dynamical systems are free. At the end we discuss a connection to the theory of Hopf-Galois extensions

The goal of Section 7 is to study some topological aspects of (free) dynamical systems. In particular, we provide conditions which ensure that a dynamical system induces a topological principal bundle. Section 8 is dedicated to an open problem and an application of this open problem to the structure theory of C^*-algebras.

In the appendix we discuss some properties of the spectrum of the algebra of smooth functions on a manifold.

Preliminaries and notations

All manifolds appearing in this paper are assumed to be finite-dimensional, paracompact, second countable and smooth if not mentioned otherwise. For the necessary background on principal bundles we refer to [13] or [12]. Furthermore, all algebras are assumed to be complex. If A is an algebra, we write $\Gamma_A := \mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{C}) \setminus \{0\}$ (with the topology of pointwise convergence on A) for the spectrum of A. Moreover, a dynamical system is a triple (A, G, α) consisting of a unital locally convex algebra A, a topological group G and a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$, which induces a continuous action of G on A.

2. Dynamical systems in classical differential geometry

Since the *Erlanger Programm* of Felix Klein, the defining concept in the study of a geometry has been its symmetry group. In classical differential geometry the symmetries of a manifold are measured by Lie groups, i.e., one studies smooth group actions of a Lie group G acting by diffeomorphisms on a manifold M. The goal of this section is to show that smooth group actions may also be studied from the viewpoint of (smooth) dynamical systems, i.e., that each smooth group action induces in a natural way a (smooth) dynamical system and vice versa. We start with the following proposition:

Proposition 2.1. *If $\sigma : M \times G \to M$ is a smooth (right-) action of a Lie group G on a finite-dimensional manifold M (possibly with boundary) and E is a locally convex space, then the induced (left-) action*

$$\alpha : G \times C^\infty(M, E) \to C^\infty(M, E),\ \alpha(g, f)(m) := (g.f)(m) := f(\sigma(m, g))$$

of G on the locally convex space $C^\infty(M, E)$ is smooth.

Proof. We first recall from [16, Prop. I.2] that the evaluation map

$$\mathrm{ev}_M : C^\infty(M, E) \times M \to E,\ (f, m) \mapsto f(m)$$

is smooth. Next, [16, Appendix A, Lemma A3] implies that the action map α is smooth if and only if the map

$$\alpha^\wedge : C^\infty(M, E) \times M \times G \to E,\ (f, m, g) \mapsto f(\sigma(m, g))$$

is smooth. Since

$$\alpha^\wedge = \mathrm{ev}_M \circ (\mathrm{id}_{C^\infty(M,E)} \times \sigma),$$

we conclude that $\alpha^\wedge$ is smooth as a composition of smooth maps. □

The previous proposition immediately leads us to the following definition:

Definition 2.2. (Smooth dynamical systems) We call a dynamical system (A, G, α) *smooth* if G is a Lie group and $\alpha : G \to \mathrm{Aut}(A)$ induces a smooth action of G on A.

Example 2.3. (Classical group actions) As the previous discussion shows, a classical example of such a smooth dynamical system is induced by a smooth action $\sigma : M \times G \to M$ of a Lie group G on a manifold M. In particular, each principal bundle (P, M, G, q, σ) induces a smooth dynamical system $(C^\infty(P), G, \alpha)$, consisting of the Fréchet algebra of smooth functions on the total space P, the structure group G and a group homomorphism $\alpha : G \to \mathrm{Aut}(C^\infty(P))$, induced by the smooth action $\sigma : P \times G \to P$ of G on P. For further examples of smooth dynamical systems we refer the interested reader to [23].

The following proposition characterizes the fixed point algebra of a smooth dynamical system, which is induced from a principal bundle, as the algebra of smooth functions on the corresponding base space:

Proposition 2.4. *Let (P, M, G, q, σ) be a principal bundle and $(C^\infty(P), G, \alpha)$ be the induced smooth dynamical system. Then the map*

$$\Psi : C^\infty(P)^G \to C^\infty(M) \text{ defined by } \Psi(f)(q(p)) := f(p)$$

is an isomorphism of Fréchet algebras.

Proof. First we observe that the map Ψ is well-defined and a homomorphism of algebras. Further, the universal property of submersions implies that $\Psi(f)$ defines a smooth function on M.

Next, if $\Psi(f) = 0$, then the G-invariance of f implies that $f = 0$. Hence, Ψ is injective. To see that Ψ is surjective, we choose $h \in C^\infty(M)$ and put $f := h \circ q$. Then $f \in C^\infty(P)^G$ and $\Psi(f) = h$. The claim now follows the continuity of Ψ and $\Psi^{-1} = q^*$. □

In the following we will show that if M is a manifold, then each smooth dynamical system of the form $(C^\infty(M), G, \alpha)$ induces a smooth action of the Lie group G on M. As a first step we endow $\Gamma_{C^\infty(M)}$ with the structure of a smooth manifold:

Lemma 2.5. *If M is a manifold, then there is a unique smooth structure on $\Gamma_{C^\infty(M)}$ for which the map*

$$\Phi : M \to \Gamma_{C^\infty(M)}, \ m \mapsto \delta_m$$

becomes a diffeomorphism.

Proof. Proposition A.2 implies that the map Φ is a homeomorphism. Therefore, Φ induces a unique smooth structure on $\Gamma_{C^\infty(M)}$ such that Φ becomes a diffeomorphism. □

The following observation is well-known, but by a lack of a reference, we give the proof:

Lemma 2.6. *A continuous map $f : M \to N$ between manifolds M and N is smooth if and only if the composition $g \circ f : M \to \mathbb{R}$ is smooth for each $g \in C^\infty(N, \mathbb{R})$.*

Proof. The "if"-direction is clear. The proof of the other direction is divided into three parts:

(i) We first note that the map f is smooth if and only if for each $m \in M$ there is an open m-neighborhood U such that $f_{|U} : U \to M$ is smooth. Therefore, let $m \in M$, $n := f(m)$ and (ψ, V) be a chart around n. We now choose an open n-neighborhood W such that $\overline{W} \subseteq V$ and a smooth function $h : N \to \mathbb{R}$ satisfying

$h_{|\overline{W}} = 1$ and $\mathrm{supp}(h) \subseteq V$. We further choose an open m-neighborhood U such that $f(U) \subseteq W$ (here, we use the continuity of the map f). Since the inclusion map $i : W \to N$ is smooth, it remains to prove that $f_{|U} : U \to W$ is smooth.

(ii) A short observation shows that the map $f_{|U} : U \to W$ is smooth if and only if the map $\psi \circ f_{|U} : U \to \mathbb{R}^n$ is smooth. If $\psi = (\psi_1, \ldots, \psi_n)$, then the last function is smooth if and only if each of its coordinate functions $\psi_i \circ f_{|U} : U \to \mathbb{R}$ is smooth.

(iii) For fixed $i \in \{1, \ldots, n\}$, we now show that the coordinate function $\psi_i \circ f_{|U} : U \to \mathbb{R}$ is smooth. For this we note that $h_i := h \cdot \psi_i$ defines a smooth $\mathbb{R}$-valued function on N satisfying $h_{i|W} = \psi_i$. Hence, the assumption implies that the map $h_i \circ f : M \to \mathbb{R}$ is smooth. Since the restriction of a smooth map to an open subsets is smooth again, we conclude from $f(U) \subseteq W$ that

$$(h_i \circ f)_{|U} = \psi_i \circ f_{|U}$$

is smooth as desired. This proves the lemma. □

Proposition 2.7. *If M is a manifold, G a Lie group and $(C^\infty(M), G, \alpha)$ a smooth dynamical system, then the homomorphism $\alpha : G \to \mathrm{Aut}(C^\infty(M))$ induces a smooth (right-) action*

$$\sigma : M \times G \to M, \ (\delta_m, g) \mapsto \delta_m \circ \alpha(g)$$

of the Lie group G on the manifold P. Here, we have identified M with the set of characters via the map Φ from Lemma 2.5.

Proof. The proof of this proposition is divided into two parts:

(i) As a first step we again use [16, Prop. I.2], which states that the evaluation map

$$\mathrm{ev}_M : C^\infty(M) \times M \to \mathbb{K}, \ (f, m) \mapsto f(m).$$

is smooth. From this we conclude that the map σ is continuous (cp. Proposition 7.3).

(ii) In view of part (i), we may use Lemma 2.6 to verify the smoothness of σ. Indeed, the map σ is smooth if and only if the map

$$\sigma_f : M \times G \to \mathbb{R}, \ (\delta_m, g) \mapsto \sigma(\delta_m, g)(f) = (\alpha(g, f))(m)$$

is smooth for each $f \in C^\infty(M, \mathbb{R})$. Therefore, we fix $f \in C^\infty(M, \mathbb{R})$ and note that

$$\sigma_f = \mathrm{ev}_M \circ (\mathrm{id}_M \times \alpha_f),$$

where

$$\alpha_f : G \to C^\infty(M), \ g \mapsto \alpha(g, f)$$

denotes the smooth orbit map of f. Hence, the map σ_f is smooth as a composition of smooth maps. Since f was arbitrary, the map σ is smooth. □

Remark 2.8. (Inverse constructions) Note that the constructions of Proposition 2.1 and Proposition 2.7 are inverse to each other.

Remark 2.9 (Principal bundles)**.** Since we are in particularly interested in principal bundles, it is reasonable to ask if there exist natural (algebraic) conditions on a smooth dynamical system $(C^\infty(P), G, \alpha)$ which ensure the freeness of the induced action σ of G on P of Proposition 2.7. In fact, if this is the case and if the action is additionally proper, then we obtain a principal bundle $(P, P/G, G, \mathrm{pr}, \sigma)$, where $\mathrm{pr} : P \to P/G,\ p \mapsto p.G$ denotes the corresponding orbit map. We will treat this question in the next section.

3. Free dynamical systems

In this section we introduce the concept of a *free dynamical system.* Loosely speaking, we call a dynamical system (A, G, α) free, if the unital locally convex algebra A is commutative and the topological group G admits a family $(\pi_j, V_j)_{j\in J}$ of point separating representations of G such that the evaluation maps defined on the "generalized spaces of sections" $\Gamma_A V_j := (A \otimes V_j)^G$ are surjective onto V_j (evaluation with respect to elements of Γ_A) . We will in particular see how this condition implies that the induced action

$$\sigma : \Gamma_A \times G \to \Gamma_A,\ (\chi, g) \mapsto \chi \circ \alpha(g)$$

of G on the spectrum Γ_A of A is free. We start with some basics from the representation theory of (topological) groups, which will later be important for deducing the freeness property:

Definition 3.1 (Separating representations)**.** Let G be topological group. We say that a family $(\pi_j, V_j)_{j\in J}$ of (continuous) representations of G *separates the points of* G if for each $g \in G$ with $g \neq 1_G$, there is a $j \in J$ such that $\pi_j(g) \neq \mathrm{id}_{V_j}$. A short observation shows that this condition is equivalent to the statement: If $g \in G$ is such that $\pi_j(g) = \mathrm{id}_{V_j}$ for all $j \in J$, then $g = 1_G$.

Remark 3.2 (Faithful representations)**.** We recall that each faithful representation (π, V) of a topological group G separates the points of G.

An important class of groups that admit a family of separating representations is given by the locally compact groups:

Theorem 3.3 (Gelfand-Raikov)**.** *Each locally compact group G admits a family of continuous unitary irreducible representations that separates the points of G.*

Proof. A proof of this statement can be found in [26]. □

Definition 3.4 ("Generalized space of sections")**.** Let A be a unital locally convex algebra and G a topological group. If (A, G, α) is a dynamical system and (π, V) a (continuous) representation of G, then there is a natural (continuous) action of G on the tensor product $A \otimes V$ defined on simple tensors by the assignment $g.(a \otimes v) := (\alpha(g).a) \otimes (\pi(g).v)$. We write

$$\Gamma_A V := \big\{ s \in A \otimes V \mid (\forall g \in G)(\alpha(g) \otimes \mathrm{id}_V)(s) = (\mathrm{id}_A \otimes \pi(g)^{-1})(s) \big\}$$

for the set of fixed elements under this action.

Lemma 3.5. *Let (A, G, α) be as in Definition 3.4. If A^G is the corresponding fixed point algebra and (π, V) a continuous representation of G, then the map*

$$\rho : \Gamma_A V \times A^G \to \Gamma_A V,\ (a \otimes v, b) \mapsto ab \otimes v$$

defines on $\Gamma_A V$ the structure of a locally convex A^G-module.

Proof. According to [23, Prop. D.2.5], $A \otimes V$ carries the structure of a locally convex A^G-module. Thus, a short calculation shows that the same holds for the restriction to the (closed) subspace $\Gamma_A V$. □

Remark 3.6. Let (P, M, G, q, σ) be a principal bundle. Further, let (π, V) be a smooth representation of G defining the associated vector bundle $\mathbb{V} := P \times_\pi V$ over M (for infinite-dimensional V, we just have to note that bundle charts of (P, M, G, q, σ) induce bundle charts for the associated algebra bundle). If we write

$$C^\infty(P, V)^G := \{f : P \to V \mid (\forall g \in G)\ f(p.g) = \pi(g^{-1}).f(p)\}$$

for the space of equivariant smooth functions, then the map

$$\Psi_\pi : C^\infty(P, V)^G \to \Gamma\mathbb{V} \text{ defined by } \Psi_\pi(f)(q(p)) := [p, f(p)],$$

is a topological isomorphism of $C^\infty(M)$-modules. Indeed, a proof of this statement can be found in [23, Cor. 3.3.7].

Example 3.7. (The classical case) Let (P, M, G, q, σ) be a principal bundle and $(C^\infty(P), G, \alpha)$ be the corresponding smooth dynamical system from Example 2.3. If (π, V) a finite-dimensional representation of G, then an easy observation shows that $C^\infty(P) \otimes V \cong C^\infty(P, V)$ (as Fréchet spaces) and further that

$$\Gamma_{C^\infty(P)} V = (C^\infty(P) \otimes V)^G \cong C^\infty(P, V)^G.$$

In particular, we conclude from Remark 3.6 that $\Gamma_{C^\infty(P)} V$ is topologically isomorphic to $\Gamma\mathbb{V}$ as $C^\infty(M)$-module.

Remark 3.8. The previous Example justifies to consider the A^G-module $\Gamma_A V$ as the *generalized space of sections* associated to the dynamical system (A, G, α) and the representation (π, V) of G.

We now come to the central definition of this section. Note that A is assumed to be a commutative algebra, since our considerations depend on the existence of enough characters:

Definition 3.9. (Free dynamical systems) Let A be a commutative unital locally convex algebra and G a topological group. A dynamical system (A, G, α) is called *free* if there exists a family $(\pi_j, V_j)_{j \in J}$ of point separating representations of G such that the map

$$\mathrm{ev}^j_\chi := \mathrm{ev}^{V_j}_\chi : \Gamma_A V_j \to V_j,\ a \otimes v \mapsto \chi(a) \cdot v$$

is surjective for all $j \in J$ and all $\chi \in \Gamma_A$.

Theorem 3.10. (Freeness of the induced action) *If (A, G, α) is a free dynamical system, then the induced action*

$$\sigma : \Gamma_A \times G \to \Gamma_A,\ (\chi, g) \mapsto \chi \circ \alpha(g)$$

of G on the spectrum Γ_A of A is free.

Proof. We divide the proof of this theorem into four parts:

(i) In order to verify the freeness of the map σ, we have to show that the stabilizer of each element of Γ_A is trivial: Consequently, we fix $\chi_0 \in \Gamma_A$ and let $g_0 \in G$ with $\chi_0 \circ \alpha(g_0) = \chi_0$.

(ii) Since (A, G, α) is assumed to be a free dynamical system, there exists a family $(\pi_j, V_j)_{j\in J}$ of point separating representations of G for which the map

$$\mathrm{ev}^j_\chi : \Gamma_A V_j \to V_j,\ a \otimes v \mapsto \chi(a) \cdot v$$

is surjective for all $j \in J$ and all $\chi \in \Gamma_A$. In particular, we can choose $j \in J$, $v \in V_j$ and $s \in \Gamma_A V_j$ with $\mathrm{ev}^j_{\chi_0}(s) = v$. We recall that the element s satisfies the equation

$$(1) \qquad (\alpha(g_0) \otimes \mathrm{id}_{V_j})(s) = (\mathrm{id}_A \otimes \pi_j(g_0^{-1}))(s).$$

(iii) Applying $\chi_0 \otimes \mathrm{id}_{V_j}$ to the left of equation (1) leads to

$$((\chi_0 \circ \alpha(g_0)) \otimes \mathrm{id}_{V_j})(s) = (\chi_0 \otimes \pi_j(g_0^{-1}))(s).$$

Thus, we conclude from $\chi_0 \circ \alpha(g_0) = \chi_0$ that

$$(\chi_0 \otimes \mathrm{id}_{V_j})(s) = (\chi_0 \otimes \pi_j(g_0^{-1}))(s) = \pi_j(g_0^{-1})((\chi_0 \otimes \mathrm{id}_{V_j})(s)).$$

(iv) We finally note that $s \in \Gamma_A V_j$ implies that

$$(\chi_0 \otimes \mathrm{id}_{V_j})(s) = \mathrm{ev}^j_{\chi_0}(s) = v.$$

In view of part (iii) this shows that $v = \pi_j(g_0)(v)$. As $j \in J$ and $v \in V_j$ were arbitrary, we conclude that $\pi_j(g_0) = \mathrm{id}_{V_j}$ for all $j \in J$ and therefore that $g_0 = 1_G$ (cp. Definition 3.1). This completes the proof. □

Remark 3.11. (a) In some situations, for example when considering infinite-dimensional representations, it might be more interesting and convenient to consider a completed version of $\Gamma_A V$, i.e.,

$$\widehat{\Gamma}_A V := \{s \in A \widehat{\otimes} V \mid (\forall g \in G)(\alpha(g) \widehat{\otimes}\, \mathrm{id}_V)(s) = (\mathrm{id}_A \widehat{\otimes} \pi(g)^{-1})(s)\},$$

where $\widehat{\otimes}$ denotes the completed projective tensor product. It is worth noticing that in this setting all the previous constructions remain true in an appropriated sense.

(b) Let (P, M, G, q, σ) be a principal bundle and $(C^\infty(P), G, \alpha)$ be the corresponding smooth dynamical system. If V is a complete infinite-dimensional space and (π, V) a smooth representation of G, then [8, Chap. 2, §3.3, Thm. 13] implies that $C^\infty(P)\widehat{\otimes} V \cong C^\infty(P, V)$ (as locally convex spaces) and further that

$$\widehat{\Gamma}_{C^\infty(P)} V = (C^\infty(P) \widehat{\otimes} V)^G \cong C^\infty(P, V)^G.$$

Remark 3.12. (Relation to Hopf-Galois extensions)

(a) A similar construction as in Definition 3.4, but in the dual language of Hopf-Galois extensions, can be found in [2, Sec. 3.1].

(b) Next, let A be a complete commutative unital locally convex algebra, G a compact Lie group and (A, G, α) a dynamical system. If the associated coaction (cp. [20, Thm. 9.2.4]) is a Hopf-Galois extension, i.e., if the map

$$\Delta : A\widehat{\otimes}_{A^G} A \to A\widehat{\otimes} C^\infty(G) = C^\infty(G, A), \;\; \Delta(a \otimes a')(g) = a \cdot \alpha(g, a') \tag{2}$$

is an isomorphism of locally convex spaces, then the dynamical system (A, G, α) is free. Here, $\widehat{\otimes}$ denotes the completed projective tensor product and the identification $A\widehat{\otimes} C^\infty(G) = C^\infty(G, A)$ follows, for example, from [8, Chap. 2, §3.3, Thm. 13]. To prove the freeness of the dynamical system (A, G, α) we first choose an element $\chi \in \Gamma_A$ (turning $\mathbb{C}$ into a module over A and any of its subalgebras) and a finite-dimensional representation (π, V) of G. Then a short calculation shows that $\mathbb{C}\widehat{\otimes}_{A^G}\Gamma_A V$ is naturally isomorphic to V. Indeed, we first observe that

$$\mathbb{C}\widehat{\otimes}_{A^G}\Gamma_A V \cong (\mathbb{C}\widehat{\otimes}_A A)\widehat{\otimes}_{A^G}\Gamma_A V \cong \mathbb{C}\widehat{\otimes}_A (A\widehat{\otimes}_{A^G}\Gamma_A V).$$

Moreover, a short calculation shows that

$$A\widehat{\otimes}_{A^G}\Gamma_A V \cong A\widehat{\otimes}_{A^G}(A \otimes V)^G \cong (A\widehat{\otimes}_{A^G} A \otimes V)^G,$$

where the group G acts trivially on the first factor. Since A is complete and V is finite-dimensional, we conclude that

$$A\widehat{\otimes}_{A^G}(A \otimes V) \cong A\widehat{\otimes}_{A^G}(A\widehat{\otimes} V) \cong (A\widehat{\otimes}_{A^G} A)\widehat{\otimes} V \cong (A\widehat{\otimes}_{A^G} A) \otimes V.$$

Thus, using the map Δ from (2) leads to

$$(A\widehat{\otimes}_{A^G} A) \otimes V \cong C^\infty(G, A \otimes V).$$

The space of fixed elements for the corresponding action of G on $C^\infty(G, A \otimes V)$ is isomorphic to $A \otimes V$, the isomorphism given by evaluating a smooth G-invariant function in the unit element of G. By summarizing, we finally obtain

$$\mathbb{C}\widehat{\otimes}_{A^G}\Gamma_A V \cong \mathbb{C}\widehat{\otimes}_A (A \otimes V) \cong \mathbb{C} \otimes V \cong V.$$

We proceed by noticing that this isomorphism maps an element $z\widehat{\otimes}_{A^G} s$ of $\mathbb{C}\widehat{\otimes}_{A^G}\Gamma_A V$ to $z \cdot \mathrm{ev}_\chi(s) \in V$ proving the desired freeness of the dynamical system (A, G, α) (note that each compact Lie group admits a faithful finite-dimensional representation (cp. [10, Thm. 11.3.9]).

(c) When working with $\widehat{\Gamma}_A V$ (cp. Remark 3.11 (a)), then a similar argument as in part (b) applied to a complete (possibly infinite-dimensional) representation (π, V) of a general Lie group G shows that $\mathbb{C}\widehat{\otimes}_{A^G}\widehat{\Gamma}_A V$ is naturally isomorphic to V.

(d) Another relation in the case of compact abelian Lie groups can be found at the end of Section 6.

4. A new characterization of free group actions in classical geometry

In this section we apply the results of the previous section to dynamical systems arising from group actions in classical geometry. In particular, we will see how this leads to a new characterization of free group actions. We start with the following lemma:

Lemma 4.1. *If G is a (possibly infinite-dimensional) Lie group, then the left-regular representation $(\lambda, C^\infty(G))$ of G on $C^\infty(G)$ is smooth and separates the points of G.*

Proof. An easy calculation shows that $(\lambda, C^\infty(G))$ separates the points of G. Its smoothness is a consequence of [5, Rem. 12.4 and Lemma 12.5]. □

Remark 4.2. We recall that it is exactly the *linear Lie groups* that admit a family of *finite-dimensional* continuous point separating representations. In fact, a reference for this statement is [10, Thm. 15.2.7].

Theorem 4.3. *Let P be a manifold and G be a Lie group. Then the following assertions hold:*

(a) If the smooth dynamical system $(C^\infty(P), G, \alpha)$ is free and, in addition, the induced action σ of G on P is proper, then we obtain a principal bundle $(P, P/G, G, \mathrm{pr}, \sigma)$ (cp. Remark 2.9).

(b) Conversely, if (P, M, G, q, σ) is a principal bundle, then the corresponding smooth dynamical system $(C^\infty(P), G, \alpha)$ is free.

Proof. (a) We first recall that the induced action $\sigma : P \times G \to P$ is smooth by Proposition 2.7. Furthermore, Theorem 3.10 implies that the map σ is free. Since σ is additionally assumed to be proper, the claim now follows from the Quotient Theorem (cp. [21, Kapitel VIII, Abschnitt 21]), which states that each free and proper smooth action $\sigma : P \times G \to P$ defines a principal bundle of the form $(P, P/G, G, \mathrm{pr}, \sigma)$.

(b) For the second statement we first choose a smooth point separating representation of G. According to Lemma 4.1 such a representation always exists in this setting. In order to prove the freeness of the smooth dynamical system $(C^\infty(P), G, \alpha)$, it would therefore be enough to show that the map

$$\mathrm{ev}_p : C^\infty(P, V)^G \to V, \; f \mapsto f(p) \tag{3}$$

is surjective for all $p \in P$ (cp. Remark 3.11 if V is infinite-dimensional). We proceed as follows:

(i) We first observe that the surjectivity of the maps (3) is a local condition. Further, we (again) recall that according to Remark 3.6 the map

$$\Psi_\pi : C^\infty(P, V)^G \to \Gamma\mathbb{V}, \; \Psi_\pi(f)(q(p)) := [p, f(p)],$$

where $\mathbb{V}$ denotes the vector bundle over M associated to (P, M, G, q, σ) via the representation (π, V) of G, is a (topological) isomorphism of $C^\infty(M)$-modules.

(ii) Now, we choose $p \in P$, $v \in V$ and construct a smooth section $s \in \Gamma\mathbb{V}$ with $s(q(p)) = [p, v]$. Indeed, such a section can always be constructed locally and then extended to the whole of M by multiplying with a smooth bump function. The construction of s implies that the function

$$f_s := \Psi_\pi^{-1}(s) \in C^\infty(P, V)^G$$

satisfies $f_s(p) = v$. As $p \in P$, $v \in V$ were arbitrary, this completes the proof. $\square$

Remark 4.4. Note that Theorem 4.3 (a) means that it is possible to test the freeness of a (smooth) group action $\sigma : P \times G \to P$ in terms of surjective maps defined on spaces of sections of associated (singular) vector bundles.

Remark 4.5. (Infinite-dimensional Lie groups)

(a) Let (P, M, G, q, σ) be an infinite-dimensional principal bundle in the sense of [14, Def. 37.1]. Further, assume that M is modeled on a locally convex space admitting smooth bump functions (locally convex spaces which are nuclear admit smooth bump functions (cp. [14, Prop. 14.4])). Then a similar argument as in the proof of Theorem 4.3 (b) shows that the corresponding smooth dynamical system $(C^\infty(P), G, \alpha)$ is free.

(b) Let (P, M, G, q, σ) be a principal bundle with compact base manifold M. Then we conclude from [25] that the group $\mathrm{Aut}(P)$ of bundle automorphism is an extension of (Fréchet-) Lie groups:

$$1 \longrightarrow \mathrm{Gau}(P) \longrightarrow \mathrm{Aut}(P) \xrightarrow{\overline{q}} \mathrm{Diff}(M)_{[P]} \longrightarrow 1.$$

Here,

$$\mathrm{Gau}(P) = \{\varphi \in \mathrm{Aut}(P) \mid q \circ \varphi = \varphi\}$$

denotes the gauge group of P and $\mathrm{Diff}(M)_{[P]}$ is the open subgroup of $\mathrm{Diff}(M)$ consisting of all diffeomorphism preserving the bundle class $[P]$ under pull backs, i.e.,

$$\mathrm{Diff}(M)_{[P]} := \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*(P) \cong P\}.$$

Furthermore, the assignment $\overline{q}(\varphi) = \varphi_M$ is defined through $q \circ \varphi = \varphi_M \circ q$. If G is abelian, then

$$\mathrm{Gau}(P) \cong C^\infty(M, G)$$

and we have an abelian extension of Lie groups. Summarizing, we have just seen that each principal bundle (P, M, G, q, σ) with compact base manifold M induces an infinite-dimensional principal bundle of the form

$$(\mathrm{Aut}(P), \mathrm{Diff}(M)_{[P]}, \mathrm{Gau}(P), \overline{q}, \overline{\sigma}),$$

where $\overline{\sigma}$ denotes the natural subgroup action of $\mathrm{Gau}(P)$ on $\mathrm{Aut}(P)$. We finally recall that $\mathrm{Diff}(M)_{[P]}$ is modeled on the nuclear space $\mathcal{V}(M)$ of smooth vector fields on M.

(c) As a more concrete example we consider the trivial principal $\mathbb{S}^1$-bundle $(M \times \mathbb{S}^1, M, \mathbb{S}^1, \mathrm{pr}, \sigma_{\mathbb{S}^1})$ for some compact manifold M (e.g. $M = \mathbb{S}^1$). Since the bundle is trivial, so is any pull back and therefore we obtain

$$\mathrm{Diff}(M)_{[M\times\mathbb{S}^1]} = \mathrm{Diff}(M).$$

Moreover, a short observation shows that

$$\mathrm{Aut}(M\times\mathbb{S}^1) = C^\infty(M,\mathbb{S}^1) \rtimes_\gamma \mathrm{Diff}(M),$$

where

$$\gamma : \mathrm{Diff}(M) \to \mathrm{Aut}(C^\infty(M,\mathbb{S}^1)),\ \gamma(\varphi).f := f\circ\varphi^{-1}.$$

The following corollary gives a one-to-one correspondence between free dynamical systems and free group action in the case where the structure group G is a compact Lie group:

Corollary 4.6. (Characterization of free group actions) *Let P be a manifold, G a compact Lie group and $(C^\infty(P), G, \alpha)$ a smooth dynamical system. Then the following statements are equivalent:*

(a) The smooth dynamical system $(C^\infty(P), G, \alpha)$ is free.
(b) The induced smooth group action $\sigma : P\times G \to P$ is free.

In particular, in this situation the two concepts of freeness coincide.

Proof. Since the group G is compact, the properness of the action σ is automatic. Hence, the equivalence follows from Theorem 4.3. The last statement is now a consequence of Remark 2.8. □

5. Free dynamical systems with compact abelian structure group

In this section we rewrite the freeness condition for a dynamical system (A, G, α) with compact abelian structure group G. In particular, we present natural conditions which ensure the freeness of such a dynamical system. These conditions do not depend on the commutativity of the algebra A and may therefore be transferred to the context of noncommutative geometry. Given a dynamical system (A, G, α) with compact abelian structure group G we write $\widehat{G}$ for the character group of G and

$$A_\varphi := \{a \in A \mid (\forall g\in G)\ \alpha(g).a = \varphi(g)\cdot a\}$$

for the isotypic component corresponding to the character $\varphi \in \widehat{G}$.

Lemma 5.1. *Let A be a unital locally convex algebra, G a compact abelian group and (A, G, α) a dynamical system. Further let $\varphi : G \to \mathbb{C}^\times$ be a character and $\pi_\varphi : G \to \mathrm{GL}_1(\mathbb{C})$ be the corresponding representation given by $\pi_\varphi(g).z = \varphi(g)\cdot z$ for all $z \in \mathbb{C}$. Then the map*

$$\Gamma_A\mathbb{C} \to A_{\varphi^{-1}},\ a\otimes 1 \mapsto a$$

is an isomorphism of locally convex spaces.

Proof. We just note that $a\otimes 1 \in \Gamma_A\mathbb{C}$ implies $\alpha(g)(a)\otimes 1 = \varphi^{-1}(g)\cdot a\otimes 1$. □

Remark 5.2. (The characters separate the points) At this stage we recall that the characters of a compact abelian group separate the points. Indeed, this statement is a consequence of Theorem 3.3 and Schur's Lemma (cp. [15, Thm. 4.2.7]).

Proposition 5.3. (The freeness condition for compact abelian groups) *Let A be a commutative unital locally convex algebra and G a compact abelian group. A dynamical system (A, G, α) is free in the sense of Definition 3.9 if the map*

$$\mathrm{ev}_\chi^\varphi : A_\varphi \to \mathbb{C}, \ a \mapsto \chi(a)$$

is surjective for all $\varphi \in \widehat{G}$ and all $\chi \in \Gamma_A$.

Proof. The claim is a consequence of Proposition 5.2 and Lemma 5.1. □

Remark 5.4. We recall from [11, Prop. 2.42] that each compact abelian Lie group G is isomorphic to $\mathbb{T}^n \times \Lambda$ for some natural number n and a finite abelian group Λ. In particular, the character group $\widehat{G}$ of a compact abelian Lie group is finitely generated.

Proposition 5.5. *Let A be a commutative unital locally convex algebra, G a compact abelian Lie group and (A, G, α) a dynamical system. Further, let $(\varphi_i)_{i \in I}$ be a finite set of generators of $\widehat{G}$. Then the following two conditions are equivalent:*

(a) The map

$$\mathrm{ev}_\chi^\varphi : A_\varphi \to \mathbb{C}, \ a \mapsto \chi(a)$$

is surjective for all $\varphi \in \widehat{G}$ and all $\chi \in \Gamma_A$.

(b) The maps

$$\mathrm{ev}_\chi^{\varphi_i^{\pm 1}} : A_{\varphi_i^{\pm 1}} \to \mathbb{C}, \ a \mapsto \chi(a)$$

are surjective for all $i \in I$ and all $\chi \in \Gamma_A$.

In particular, if one of the statements holds, then the dynamical system (A, G, α) is free.

Proof. (a) ⇒ (b): This direction is trivial.

(b) ⇒ (a): For the second direction we fix $\chi \in \Gamma_A$. Further, we choose for each $i \in I$ elements $a_{\varphi_i^{\pm 1}} \in A_{\varphi_i^{\pm 1}}$ with $\chi(a_{\varphi_i^{\pm 1}}) \neq 0$. Now, if $\varphi \in \widehat{G}$, then there exist $k \in \mathbb{N}$ and integers $n_1, \ldots, n_k \in \mathbb{Z}$ such that

$$\varphi = \varphi_{i_1}^{n_1} \cdots \varphi_{i_k}^{n_k} \text{ for some } i_1, \ldots, i_k \in I.$$

Hence, the element

$$a_\varphi := (a_{\varphi_{i_1}^{\mathrm{sgn}(n_1)}})^{|n_1|} \cdots (a_{\varphi_{i_k}^{\mathrm{sgn}(n_k)}})^{|n_k|} \in A_\varphi$$

satisfies $\chi(a_\varphi) \neq 0$.

The last assertion is a direct consequence of Proposition 5.3. □

Proposition 5.6. (Invertible elements in isotypic components) *Let A be a commutative unital locally convex algebra, G a compact abelian group and (A, G, α) a dynamical system. If each isotypic component A_φ contains an invertible element, then the dynamical system (A, G, α) is free.*

Proof. The assertion easily follows from Proposition 5.3. Indeed, if $a_\varphi \in A_\varphi$ is invertible, then $\chi(a) \neq 0$ for all $\chi \in \Gamma_A$. □

Remark 5.7. Note that it is possible to ask for invertible elements in the isotypic components even if the algebra A is noncommutative.

Proposition 5.8. *Let A be a unital locally convex algebra, G a compact abelian group and (A, G, α) a dynamical system. Further, let $(\varphi_i)_{i\in I}$ be a finite set of generators of $\widehat{G}$. Then the following two statements are equivalent:*

(a) A_φ contains invertible elements for all $\varphi \in \widehat{G}$.
(b) A_{φ_i} contains invertible elements for all $i \in I$.

In particular, if A is commutative and one of the statements holds, then the dynamical system (A, G, α) is free.

Proof. (a) ⇒ (b): This direction is trivial.

(b) ⇒ (a): For each $i \in I$ we choose an invertible element $a_{\varphi_i} \in A_{\varphi_i}$. Next, if $\varphi \in \widehat{G}$, then there exist $k \in \mathbb{N}$ and integers $n_1, \ldots, n_k \in \mathbb{Z}$ such that

$$\varphi = \varphi_{i_1}^{n_1} \cdots \varphi_{i_k}^{n_k} \text{ for some } i_1, \ldots, i_k \in I.$$

Hence, $a_\varphi := a_{\varphi_{i_1}}^{n_1} \cdots a_{\varphi_{i_k}}^{n_k}$ is an invertible element in A_φ.

The last assertion is a direct consequence of Proposition 5.6. □

The following proposition shows that if all isotypic components of a dynamical system contain invertible elements, then they are "mutually" isomorphic to each other as modules of the fixed point algebra:

Proposition 5.9. *Let A be a commutative unital locally convex algebra and G a compact abelian group. Further, let (A, G, α) be a dynamical system and A^G the corresponding fixed point algebra. If each isotypic component A_φ contains an invertible element, then the map*

$$\Psi_\varphi : A^G \to A_\varphi, \ a \mapsto a_\varphi a,$$

where a_φ denotes some fixed invertible element in A_φ, is an isomorphism of locally convex A^G-modules for each $\varphi \in \widehat{G}$. In particular, each isotypic component A_φ is a free A^G-module.

Proof. An easy calculation shows that Ψ_φ is a morphism of locally convex A^G-modules, and therefore the assertion follows from the fact that $a_\varphi \in A_\varphi$ is invertible. □

We finally apply the results of this section to dynamical systems arising from classical geometry. The following theorem may be viewed as a first answer to Remark 2.9. A more detailed analysis can be found in [23].

Theorem 5.10. *Let P be a manifold and G be a compact abelian Lie group. Further, let $(C^\infty(P), G, \alpha)$ be a smooth dynamical system. If $\mathrm{pr}: P \to P/G$ denotes the orbit map corresponding to the action of G on P (cp. Proposition 2.7), then the following assertions hold:*

(a) If each isotypic component $C^\infty(P)_\varphi$ contains an invertible element, then we obtain a principal bundle $(P, P/G, G, \mathrm{pr}, \sigma)$.

(b) If $(\varphi_i)_{i\in I}$ is a finite set of generators of $\widehat{G}$ and each isotypic component $C^\infty(P)_{\varphi_i}$ contains an invertible element, then we obtain a principal bundle $(P, P/G, G, \mathrm{pr}, \sigma)$.

Proof. (a) Since G is compact, the induced action σ is automatically proper. Therefore, the first assertion follows from Theorem 4.3 and Proposition 5.6.

(b) The second assertion follows from Proposition 5.8 (b) and part (a). □

Remark 5.11. If G is a compact abelian Lie group and P a trivial principal G-bundle, then each isotypic component $C^\infty(P)_\varphi$ of the corresponding smooth dynamical system $(C^\infty(P), G, \alpha)$ contains invertible elements. Indeed, such elements can be found as the components of a trivialization map of P.

Remark 5.12. (An application to noncommutative geometry: towards a geometric approach to noncommutative principal torus bundles)

(a) Suppose we are in the situation of Theorem 5.10 with $G = \mathbb{T}^n$. Then $\widehat{G}$ is isomorphic to $\mathbb{Z}^n$ and it turns out that the induced principal $\mathbb{T}$-bundle $(P, P/\mathbb{T}^n, \mathbb{T}^n, \mathrm{pr}, \sigma)$ is trivial. In fact, the invertible elements in the isotypic components $C^\infty(P)_{\mathbf{k}}$ can be used to construct a trivialization map. Here, we write $\mathbf{k} = (k_1, \ldots, k_n)$ for elements of $\mathbb{Z}^n$ and think of them as multi-indices. The crucial point is to note that a trivialization map consists basically of n smooth functions $f_i : P \to \mathbb{T}$ satisfying $f_i(\sigma(p, z)) = f_i(p) \cdot z_i$ for all $p \in P$ and $z \in \mathbb{T}^n$. In view of the previous remark, we conclude that a principal $\mathbb{T}^n$-bundle $(P, M, \mathbb{T}^n, q, \sigma)$ is trivial if and only if each isotypic component $C^\infty(P)_{\mathbf{k}}$ of the corresponding smooth dynamical system $(C^\infty(P), G, \alpha)$ contains invertible elements.

(b) Part (a) justifies to call a dynamical system $(A, \mathbb{T}^n, \alpha)$ a *trivial noncommutative principal $\mathbb{T}^n$-bundle* if the isotypic components contain invertible elements. While in classical (commutative) differential geometry there exists up to isomorphy only one trivial principal $\mathbb{T}^n$-bundle over a given manifold M, the situation completely changes in the noncommutative world. In particular, we provide a complete classification of all possible trivial noncommutative principal torus up to completion in terms of a suitable cohomology theory. An important class of examples of trivial noncommutative principal torus bundles is provided by the so-called noncommutative tori. For a more detailed background of the previous discussion we refer to [24].

(c) In a forthcoming paper we present a new, geometrically oriented approach to the noncommutative geometry of nontrivial principal torus bundles. Our approach is inspired by the classical setting: In fact, we first introduce a convenient (smooth) localization method for noncommutative algebras and say that a dynamical system $(A, \mathbb{T}^n, \alpha)$ is called a *noncommutative principal* $\mathbb{T}^n$-*bundle*, if localization leads to a trivial noncommutative principal $\mathbb{T}^n$-bundle. In particular, we show that this approach covers the classical theory of principal torus bundles and present a bunch of nontrivial noncommutative examples (cp. [23] for a general overview).

6. Strongly free dynamical systems

We introduce a stronger version of freeness for dynamical systems than the one given in Section 3 (cp. Definition 3.9). In fact, instead of considering arbitrary families $(\pi_j, V_j)_{j\in J}$ of (continuous) point separating representations of a topological group G, we restrict our attention to families $(\pi_j, \mathcal{H}_j)_{j\in J}$ of *unitary irreducible* point separating representations. At this point, we recall that each locally compact group G admits a family of continuous unitary irreducible point separating representations (cp. Theorem 3.3). We show that Theorem 3.10 and Theorem 4.6 stay true in this context and that Proposition 5.3 actually turns into a definition for strongly free dynamical systems with compact abelian structure group.

Definition 6.1. (Strongly free dynamical systems) Let A be a commutative unital locally convex algebra and G a topological group. A dynamical system (A, G, α) is called *strongly free* if there exists a family $(\pi_j, \mathcal{H}_j)_{j\in J}$ of unitary irreducible point separating representations of G such that the map

$$\mathrm{ev}_\chi^j := \mathrm{ev}_\chi^{\mathcal{H}_j} : \Gamma_A \mathcal{H}_j \to \mathcal{H}_j,\ a \otimes v \mapsto \chi(a) \cdot v$$

is surjective for all $j \in J$ and all $\chi \in \Gamma_A$.

Proposition 6.2. (Freeness of the induced action again) *If (A, G, α) is a strongly free dynamical system, then the induced action*

$$\sigma : \Gamma_A \times G \to \Gamma_A,\ (\chi, g) \mapsto \chi \circ \alpha(g)$$

of G on the spectrum Γ_A of A is free.

Proof. This assertion immediately follows from Theorem 3.10, because each strongly free dynamical system is clearly free. □

Proposition 6.3. (Characterization of free group actions again) *Let P be a manifold, G a compact Lie group and $(C^\infty(P), G, \alpha)$ a smooth dynamical system. Then the following statements are equivalent:*

(a) The smooth dynamical system $(C^\infty(P), G, \alpha)$ is strongly free.

(b) The induced smooth group action $\sigma : P \times G \to P$ is free.

In particular, in this situation the concepts of freeness coincide.

Proof. This assertion can be proved similarly to Corollary 4.6 (cp. Theorem 4.3): Indeed, we first recall that each compact Lie group G admits a faithful finite-dimensional representation (cp. [10, Thm. 11.3.9]). Thus, given such a representation (π, V) of G, it remains to note that it is possible to find an inner product on V such that G acts by unitary transformations ("Weyl's trick") and that each unitary finite-dimensional representation of G can be decomposed into the (finite) sum of irreducible representations. □

Proposition 6.4. (The strong freeness condition for compact abelian groups) *Let A be a commutative unital locally convex algebra and G a compact abelian group. A dynamical system (A, G, α) is strongly free in the sense of Definition 6.1 if and only if the map*

$$\mathrm{ev}_\chi^\varphi : A_\varphi \to \mathbb{C},\ a \mapsto \chi(a)$$

is surjective for all $\varphi \in \widehat{G}$ and all $\chi \in \Gamma_A$.

Proof. ("$\Leftarrow$") This direction is obvious, since the characters of the group G induce a family of unitary irreducible representations that separate the points of G (cp. Lemma 5.1 and Proposition 5.2).

("$\Rightarrow$") For the other direction we first note that each unitary irreducible representation $(\pi, \mathcal{H})$ of G is one-dimensional by Schur's Lemma (cp. [15, Thm. 4.2.7]), i.e., $\pi(g).v = \varphi(g) \cdot v$ for all $g \in G$, $v \in \mathcal{H}$ and some character φ of $\widehat{G}$. Thus, if the dynamical system (A, G, α) is strongly free and $(\pi_j, \mathcal{H}_j)_{j \in J}$ is a family of unitary irreducible point separating representations of G satisfying the conditions of Definition 6.1, then [11, Cor. 2.3.3.(i)] implies that the corresponding characters φ_j generate $\widehat{G}$ and from this we easily conclude that the map

$$\mathrm{ev}_\chi^\varphi : A_\varphi \to \mathbb{C},\ a \mapsto \chi(a)$$

is surjective for all $\varphi \in \widehat{G}$ and all $\chi \in \Gamma_A$ (cp. Lemma 5.1). □

Example 6.5. We now want to use Proposition 6.4 to show that the action of the group $C_2 := \{-1, +1\}$ on $\mathbb{R}$ defined by

$$\sigma : \mathbb{R} \times C_2 \to \mathbb{R},\ r.(-1) := \sigma(r, -1) := -r$$

is not free: Indeed, we first note that the map

$$\Psi : C_2 \to \mathrm{Hom}_{\mathrm{gr}}(C_2, \mathbb{T}),\ \Psi(-1)(-1) := -1$$

is an isomorphism of abelian groups. From this we easily conclude that the isotypic component of the associated smooth dynamical system $(C^\infty(\mathbb{R}), C_2, \alpha)$ (cp. Proposition 2.1) corresponding to the generator $-1 \in C_2$ is given by

$$C^\infty(\mathbb{R})_{-1} = \{f : \mathbb{R} \to \mathbb{C} \mid (\forall r \in \mathbb{R})\ f(-r) = -f(r)\}.$$

Since $f(0) = 0$ for each $f \in C^\infty(\mathbb{R})_{-1}$, the map

$$\mathrm{ev}_0^{-1} : C^\infty(\mathbb{R})_{-1} \to \mathbb{C},\ f \mapsto f(0) = 0$$

is not surjective showing that $(C^\infty(\mathbb{R}), C_2, \alpha)$ is not strongly free (cp. Proposition 6.4). Therefore, Proposition 6.3 implies that the action σ is not free.

Proposition 6.6. (Strongly graded algebras) *Let A be a commutative unital locally convex algebra, G a compact abelian Lie group and (A, G, α) a dynamical system. If A is strongly graded in the sense that $A_\varphi \cdot A_\psi = A_{\varphi \cdot \psi}$ hold for all $\varphi, \psi \in \widehat{G}$, then the dynamical system (A, G, α) is strongly free.*

Proof. To prove the claim, let us assume the converse, i.e., that the dynamical system (A, G, α) is not strongly free. Then there exist $\varphi \in \widehat{G}$ and $\chi \in \Gamma_A$ such that $\chi(a) = 0$ for all $a \in A_\varphi$. Since we have $A_\varphi \cdot A_{\varphi^{-1}} = A_{\mathbf{1}}$ by assumption, we can find elements $a^i_\varphi \in A_\varphi$ and $a^i_{\varphi^{-1}} A_{\varphi^{-1}}$, labeled by some finite index set I, such that $\sum_{i\in I} a^i_\varphi \cdot a^i_{\varphi^{-1}} = 1_A$. Now,

$$1 = \chi(1_A) = \sum_{i\in I} \chi(a^i_\varphi) \cdot \chi(a^i_{\varphi^{-1}}) = 0$$

leads to the desired contradiction. □

Remark 6.7. (Hopf-Galois extensions vs. strongly free dynamical systems)

(a) Let G be a group. An algebra A is a $\mathbb{C}[G]$-comodule algebra if and only if A is a G-graded algebra (cp. [3, Lemma 4.8]). Moreover, we conclude from [19, Ex. 2.1.4] that a G-graded algebra $A = \bigoplus_{g\in G} A_g$ is a Hopf-Galois extension (of A_{1_G}) if and only if A is strongly graded, i.e., if $A_g A_{g'} = A_{gg'}$ for all $g, g' \in G$.

(b) Next, let G be a compact abelian Lie group and A be a $\mathbb{C}[\widehat{G}]$-comodule algebra. A short observation shows that the $\mathbb{C}[\widehat{G}]$-comodule algebra A corresponds, up to a suitable completion, to a dynamical system (A, G, α). Thus, if A is a commutative Hopf-Galois extension (of A_{1_G}), then we conclude from part (a) and Proposition 6.6 that the corresponding dynamical system (A, G, α) is strongly free. Unfortunately, it is not clear to us if the converse is also true, i.e., if each strongly free dynamical system (A, G, α) is strongly $\widehat{G}$-graded.

(c) Again, let G be a compact abelian Lie group. If (P, M, G, q, σ) is a principal G-bundle and $(C^\infty(P), G, \alpha)$ the corresponding smooth dynamical system, then a partition-of-unity argument can be used to show that $C^\infty(P)$ is strongly $\widehat{G}$-graded. In particular, we conclude that a smooth dynamical system of the form $(C^\infty(P), G, \alpha)$ is strongly free if and only if $C^\infty(P)$ is strongly graded.

7. Some topological aspects of free dynamical systems

In this section we discuss some topological aspects of (free) dynamical systems. Our main goal is to provide conditions which ensure that a dynamical system induces a topological principal bundle. Again, all groups are assumed to act continuously by morphisms of algebras. We start with the following lemma:

Lemma 7.1 (Continuity of the evaluation map)**.** *Let A be a commutative unital locally convex algebra. If Γ_A is locally equicontinuous, then the evaluation map*

$$\mathrm{ev}_A : \Gamma_A \times A \to \mathbb{C}, \ (\chi, a) \mapsto \chi(a)$$

is continuous.

Proof. To prove the continuity of ev_A, we pick $(\chi_0, a_0) \in \Gamma_A \times A$, $\epsilon > 0$ and choose an equicontinuous neighborhood V of χ_0 in Γ_A such that

$$V \subseteq \left\{\chi \in \Gamma_A \;\middle|\; |(\chi - \chi_0)(a_0)| < \frac{\epsilon}{2}\right\}.$$

Further, we choose a neighborhood W of a_0 in A such that $|\chi(a - a_0)| < \frac{\epsilon}{2}$ for all $a \in W$ and $\chi \in V$. We thus obtain

$$|\chi(a) - \chi_0(a_0)| \leq |\chi(a) - \chi(a_0)| + |\chi(a_0) - \chi_0(a_0)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $\chi \in V$ and $a \in W$. □

Remark 7.2. (Sources of algebras with equicontinuous spectrum)

(a) A unital locally convex algebra A is called a *continuous inverse algebra*, or CIA for short, if its group of units $A^\times$ is open in A and the inversion

$$\iota : A^\times \to A^\times, \; a \mapsto a^{-1}$$

is continuous at 1_A. The spectrum Γ_A of each CIA A is compact and Hausdorff. Moreover, Γ_A is equicontinuous. In fact, let U be a balanced 0-neighborhood such that $U \subseteq 1_A - A^\times$. Then $|\Gamma_A(U)| < 1$ (cp. [23, Appendix C] for a detailed discussion on CIA's).

(b) Moreover, if A is a *ρ-seminormed* algebra, then [1, Cor. 7.3.9] implies that Γ_A^{cont} is equicontinuous.

Proposition 7.3. (Continuity of the induced action map) *Let A be a commutative unital locally convex algebra, G a topological group and (A, G, α) a dynamical system. If the evaluation map ev_A is continuous, then the induced action*

$$\sigma : \Gamma_A \times G \to \Gamma_A, \; (\chi, g) \mapsto \chi \circ \alpha(g)$$

of G on Γ_A is continuous.

Proof. The topology (of pointwise convergence) on Γ_A implies that the map σ is continuous if and only if the maps

$$\sigma_a : \Gamma_A \times G \to \mathbb{C}, \; (\chi, g) \mapsto \chi(\alpha(g)(a))$$

are continuous for all $a \in A$. Therefore, we fix an element $a \in A$ and note that $\sigma_a = \text{ev}_A \circ (\text{id}_{\Gamma_A} \times \alpha_a)$, where

$$\alpha_a : G \to A, \; g \mapsto \alpha(g, a)$$

denotes the continuous orbit map of a. In view of the assumption, the map σ_a is continuous as a composition of continuous maps. Since a was arbitrary, this proves the proposition. □

Remark 7.4. Recall that if $\sigma : X \times G \to X$ is an action of a topological group G on a topological space X, then the orbit map

$$\text{pr} : X \to X/G, \; x \mapsto x.G := \sigma(x, G)$$

is surjective, continuous and open.

Proposition 7.5. *Let A be a commutative unital locally convex algebra, G a compact group and (A, G, α) a dynamical system. If the induced action $\sigma : \Gamma_A \times G \to \Gamma_A$ is free and continuous, then the following assertions hold:*

(a) For each $\chi \in \Gamma_A$ the map $\sigma_\chi : G \to \Gamma_A,\ g \mapsto \chi.g := \sigma(\chi, g)$ is a homeomorphism of G onto the orbit $\mathcal{O}_\chi$.

(b) If Γ_A is locally compact, then the orbit space Γ_A/G is locally compact and Hausdorff.

(c) For each pair $(\chi, \chi') \in \Gamma_A \times \Gamma_A$ with $\mathcal{O}_\chi = \mathcal{O}_{\chi'}$ there is a unique element $\tau(\chi, \chi') \in G$ such that $\chi.\tau(\chi, \chi') = \chi'$, and the map

$$\tau : \Gamma_A \times_{\Gamma_A/G} \Gamma_A := \{(\chi, \chi') \in \Gamma_A \times \Gamma_A \mid \mathrm{pr}(\chi) = \mathrm{pr}(\chi')\} \to G$$

is continuous and surjective.

Proof. (a) The continuity of the map σ_χ follows from the continuity of σ. Further, the bijectivitiy of σ_χ follows from the freeness of σ. Since G is compact and Γ_A is Hausdorff, a well-known theorem from topology now implies that σ_χ is a homeomorphism of G onto the orbit $\mathcal{O}_\chi$.

(b) If Γ_A is locally compact, then the orbit space Γ_A/G is locally compact because the orbit map is open and continuous. Moreover, the compactness of G implies that the action σ is proper. Therefore, the image of the map

$$\Gamma_A \times G \to \Gamma_A \times \Gamma_A,\ (\chi, g) \mapsto (\chi, \chi.g)$$

is a closed subset of $\Gamma_A \times \Gamma_A$. Now, the assertion follows from Remark 7.4 and the more general fact that the target space of a surjective, continuous, open map $f : X \to Y$ is Hausdorff if and only if the preimage of the diagonal under $f \times f$ is closed.

(c) Suppose $\chi_i \to \chi$, $\chi_i' \to \chi'$, and $\mathcal{O}_\chi = \mathcal{O}_{\chi'}$ so that by definition, $\chi_i.\tau(\chi_i, \chi_i') = \chi_i'$. Since G is compact, we can assume by passing to a subnet that $\tau(\chi_i, \chi_i')$ converges to g, say. Then we have

$$\chi' = \lim_i \chi_i' = \lim_i (\chi_i \cdot \tau(\chi_i, \chi_i')) = \chi \cdot g,$$

which implies $\tau(\chi, \chi') = g$ and $\tau(\chi_i, \chi_i') \to \tau(\chi, \chi)$. □

Remark 7.6. (Topological principal bundles) The map τ in Proposition 7.5 (c) is called the *translation map* and is part of the definition of principal bundles in [12]. A short observation shows that if a topological group G acts freely, continuously and satisfies (c), then G automatically acts properly; thus the principal bundles in [12] are by definition the free and proper G-spaces. We point out that these principal bundles are, in general not, locally trivial. For this reason, we call a free and proper G-space which is Hausdorff a *topological principal bundle*, if each orbit of the action is homeomorphic to G and the orbit space is Hausdorff.

Theorem 7.7. *Let A be a commutative CIA, G a compact group and (A, G, α) a free dynamical system. Then the induced action $\sigma : \Gamma_A \times G \to \Gamma_A$ is continuous and we obtain a topological principal bundle*

$$(\Gamma_A, \Gamma_A/G, G, \sigma, \mathrm{pr}).$$

Proof. We first recall from Remark 7.2 that Γ_A is equicontinuous. Hence, Lemma 7.1 and Proposition 7.3 imply that the map σ is continuous. Further, we note that the map σ is proper. Indeed, this follows from the compactness of G. In view of Theorem 3.10, we conclude that σ is free. Therefore, Γ_A is a free and proper G-space which is Hausdorff and thus the claim follows from Proposition 7.5 (a) and (b). □

Corollary 7.8. *Let A be a commutative CIA and G a compact abelian group. Furthermore, let (A, G, α) be a dynamical system. If each isotypic component A_φ contains an invertible element, then we obtain a topological principal bundle $(\Gamma_A, \Gamma_A/G, G, \sigma, \mathrm{pr})$.*

Proof. This assertion immediately follows from Proposition 5.6 and Theorem 7.7. □

Example 7.9. The group algebra $\ell^1(\mathbb{Z}^n)$ is a commutative Banach *-algebra. Moreover, the map

$$\widehat{\alpha} : \mathbb{T}^n \times \ell^1(\mathbb{Z}^n) \to \ell^1(\mathbb{Z}^n), \ (\widehat{\alpha}(z, f))(\mathbf{k}) := (z.f)(\mathbf{k}) := z^{\mathbf{k}} \cdot f(\mathbf{k})$$

defines a continuous action of $\mathbb{T}^n$ on $\ell^1(\mathbb{Z}^n)$ by algebra automorphisms. In particular, the triple $(\ell^1(\mathbb{Z}^n), \mathbb{T}^n, \widehat{\alpha})$ defines a dynamical system and a short observation shows that the corresponding isotypic components contain invertible elements ([24, Ex. 2.8]). The induced principal bundle turns out to be the trivial principal $\mathbb{T}^n$-bundle over a single point $\{*\}$, i.e., $(\mathbb{T}^n, \{*\}, \mathbb{T}^n, q, \sigma_{\mathbb{T}^n})$ for $q : \mathbb{T}^n \to \{*\}, \ z \mapsto *$.

8. An open problem

This short section is dedicated to the following interesting open problem and the resulting application to the generalized Effros-Hahn conjecture:

Open Problem 8.1. (Primitive ideals) Theorem 3.10 may be viewed as a first step towards a geometric approach to noncommutative principal bundles. Nevertheless, in order to get a broader picture, it might be helpful to get rid of the characters. This might be done with the help of primitive ideals, i.e., kernels of irreducible representations (ρ, W) of the (locally convex) algebra A, since they can be considered as generalizations of characters (points). To be more precise:

Let (A, G, α) be a dynamical system, consisting of a (not necessarily commutative) unital locally convex algebra A, a topological group G and a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$, which induces a continuous action of G on A. Further, let $\mathrm{Prim}(A)$ denote the set of primitive ideals of A. As already mentioned, note that if A is commutative, then $\mathrm{Prim}(A) \cong \Gamma_A$. Do there exist "geometrically oriented" conditions which ensure that the corresponding action

$$\sigma : \mathrm{Prim}(A) \times G \to \mathrm{Prim}(A), \ (I, g) \mapsto \alpha(g).I$$

of G on the primitive ideals $\mathrm{Prim}(A)$ of A is free? In this context, the paper [18] is of particular interest.

An interesting application to the structure theory of C^*-algebras is given by the "generalized Effros-Hahn Conjecture":

Theorem 8.2. (The generalized Effros-Hahn conjecture) *Suppose G is an amenable group, A a separable C^*-algebra and (A, G, α) a C^*-dynamical system. If G acts freely on* $\operatorname{Prim}(A)$, *then there is one and only one primitive ideal of the crossed product $A \rtimes_\alpha G$ lying over each hull-kernel quasi-orbit in* $\operatorname{Prim}(A)$. *In particular, if every orbit is also hull-kernel dense, then $A \rtimes_\alpha G$ is simple.*

Proof. A proof of this theorem can be found in [6, Cor. 3.3]. □

A. The spectrum of the algebra of smooth function

In this part of the appendix we discuss the spectrum of the algebra of smooth functions.

Lemma A.1. *If M is a manifold, then each character $\chi : C^\infty(M) \to \mathbb{C}$ is an evaluation in some point $m \in M$.*

Proof. A proof of this statement can be found in [23, Cor. 4.3.2]. □

The next proposition shows that the correspondence between M and $\Gamma_{C^\infty(M)}$ is actually a topological isomorphism:

Proposition A.2. *Let M be a manifold. Then the map*

$$\Phi_M : M \to \Gamma_{C^\infty(M)}, \ m \mapsto \delta_m.$$

is a homeomorphism.

Proof. (i) The surjectivity of Φ follows from Lemma A.1. To show that Φ is injective, choose elements $m \neq m'$ of M. Since M is manifold, there exists a function f in $C^\infty(M)$ with $f(m) \neq f(m')$. Then

$$\delta_m(f) = f(m) \neq f(m') = \delta_{m'}(f)$$

implies that $\delta_m \neq \delta_{m'}$, i.e., Φ is injective.

(ii) Next, we show that Φ is continuous: Let $m_n \to m$ be a convergent sequence in M. Then we have

$$\delta_{m_n}(f) = f(m_n) \to f(m) = \delta_m(f) \text{ for all } f \text{ in } C^\infty(M),$$

i.e., $\delta_{m_n} \to \delta_m$ in the topology of pointwise convergence. Hence, Φ is continuous.

(iii) We complete the proof by showing that Φ is an open map: For this let U be an open subset of M, m_0 in U and h a smooth real-valued function with $h(m_0) \neq 0$ and $\operatorname{supp}(h) \subset U$. Since the map

$$\delta_h : \Gamma_{C^\infty(M)} \to \mathbb{C}, \ \delta_m \mapsto h(m)$$

is continuous, a short calculations shows that $\Phi(U)$ is a neighborhood of m_0 containing the open subset $\delta_h^{-1}(\mathbb{C}^\times)$. Hence, Φ is open. □

Acknowledgment

We thank Henrik Seppänen for useful hints while writing this paper. In addition we thank the *Studienstiftung des deutschen Volkes* for a doctoral scholarship for my work.

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Received December 8, 2011; accepted February 19, 2012

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