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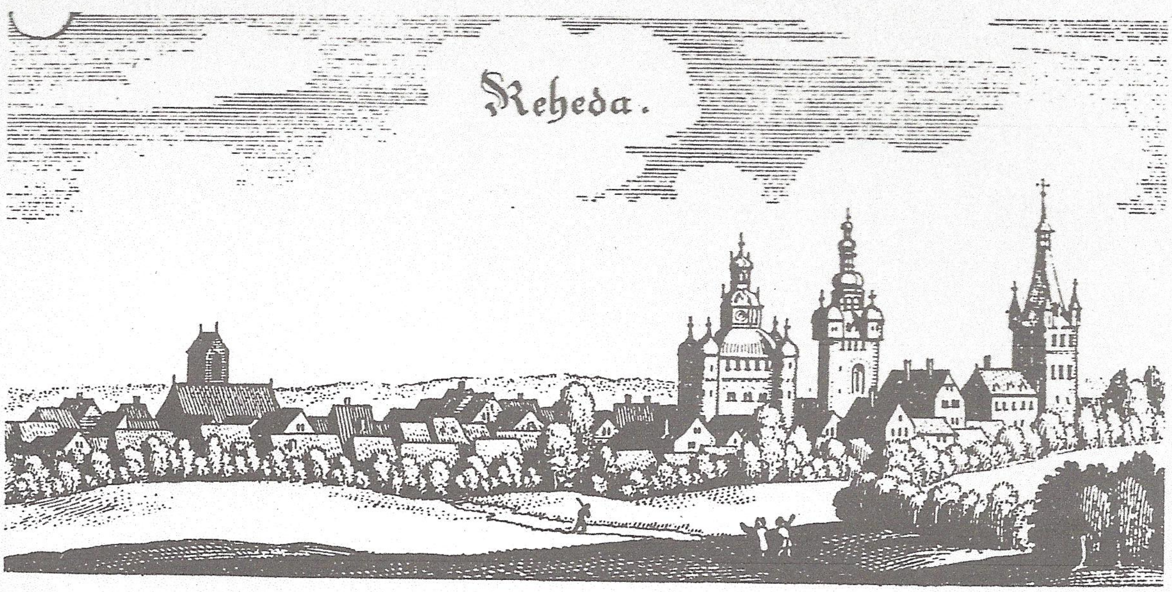

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SOME APPLICATIONS OF A RESULT IN CONTROL THEORY TO ECONOMIC PLANNING MODELS[+]

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ABSTRACT

SOME APPLICATIONS OF A RESULT IN CONTROL
THEORY TO ECONOMIC PLANNING MODELS.

H.W. Gottinger

Recent results on necessary conditions for optimal control problems with inequality constraints on the state variables have shown that, for a particular class of problems, the optimal trajectory does not stay on the constraint boundary for non-zero intervals of time.

These results are used in connection with discussing three models of economic planning taken from different areas. The first two examples consider the problem of optimal employment in a planning context, they reveal the property that the optimal solution is such that full employment is achieved at most instantaneously and is not maintained for non-zero intervals of time.

The third example is concerned with the optimal investment and foreign aid policies for a developing economy. An institutional constraint on the stock of foreign debts may be derived from the 'debt-service' ratio. Contrary to earlier results obtained from simpler models this constraint is binding at most instantaneously along the optimal trajectory.

1. Introduction

Optimal control problems with inequality constraints on the state variables have been the subject of much research in the past twenty years. Necessary conditions of optimality for the first order case,[1] i.e. where the first time derivative of the constraint contained the control variable explicitly, were obtained by Gamkrelidze [1] in 1960. Later researchers extended these to the general case where the order of the constraint, p, was greater than one. Recently, Jacobson, Lele and Speyer [2] have obtained sharper results for this general case, than those known heretofore. This reference also contains a brief history of earlier research. A consequence of the new necessary conditions of optimality is that, under certain conditions, problems with odd-ordered constraints - except the case $p = 1$ - will not, in general, exhibit optimal trajectories with boundary arcs over non-zero intervals of time, i.e. the optimal trajectory will at most only touch the constraint boundary, but will not lie along it.

The significance of this result lies in the fact that, inasmuch as the conditions under which it is valid are not overly stringent, it reveals, *a priori*, considerable information as regards the structure of the optimal solution. Such information would be of particular qualitative value in problems of economic policy. To this end, after presenting a brief summary of the relevant control-theoretic results, we will consider three examples from economic planning. Two of these examples deal with the problem of optimal employment; the third is drawn from development planning.

2. Restatement of Results from Control Theory

We briefly summarize the results which we will be using from control theory. The details are given in [2], or [3].

For simplicity, the basic optimization problem which we consider is in the form of Mayer-Bliss. There is, however, no loss of generality, so that we are at liberty to apply these results to other formulations.

The basic problem is

(1) $\underset{u}{\text{Maximize}}\ \Phi(x(T))$

subject to

(2) $\dot{x}(t) = f(x(t),u(t));$ $\qquad x(0) = x_o$

and the scalar state variable inequality constraint

(3) $S(x(t)) \leq 0$ $\qquad$ for all $t \in [0,T]$.

Here

x - n-dimensional state-vector

u - scalar control variable

f - n-dimensional vector function

S - pth order state variable constraint

Φ - scalar function of the terminal state

$(\dot{\ }) \equiv \frac{d}{dt}$

The necessary conditions of optimality for the above problem are given by the following theorem, the proofs of which and of the following theorems are now quite standard (see [2] or [3]).

<u>Theorem 1.</u> <u>The necessary conditions of optimality for the basic problem are</u>

(4) $$\frac{\partial H}{\partial u} = 0 = f_u^T \lambda$$

(5) $$-\dot{\lambda} = \left\{ \begin{array}{l} \frac{\partial H}{\partial x} \\ \\ f_x^T \lambda + \eta S_x \end{array} \right\} \qquad \lambda(T) = \left.\frac{\partial \Phi}{\partial x}\right|_T$$

where

(6) $$\eta(t) = \begin{cases} \leq 0, & S(x(t)) = 0 \\ 0, & S(x(t)) < 0 \end{cases}$$

is a bounded function for $t \in [0,T]$.

At junction points t_i of boundary and interior arcs, the influence functions $\lambda(t)$ may be discontinuous. The boundary conditions are

(7) $$\lambda(t_i^+) = \lambda(t_i^-) - \nu(t_i)(\frac{\partial S}{\partial x})_{t_i}$$

and

(8) $$H(t_i^+) = H(t_i^-)$$

where $\nu(t_i) \leq 0$. The Hamiltonian H used above, is defined by

(9) $$H \equiv H(x,u,\lambda) = \eta S + \lambda^T f.$$

The assumptions made in the derivation of these necessary conditions are given in [3]; the only noteworthy assumption is that

(10) $$\frac{\partial}{\partial u} [\frac{d^p}{dt^p} (S(x(t))] \neq 0$$

i.e. that along the constraint boundary small changes in control can be related uniquely to small changes in the state variables; this is a common assumption [3].

Before proceeding further, let us make the

Definition: The Hamiltonian H, is said to be regular along an extremal if it possesses a unique maximum with respect to the control at every instant in time. Note that the Hamiltonian H, is really $H(x,u,\lambda)$. The above definition requires that for all $t \in [0,T]$, given the extremal trajectories (say) $\bar{x}(t)$, $\bar{\lambda}(t)$, $H(\bar{x},\bar{\lambda}, u)$ have a unique maximum with respect to $u(t)$. This does not exclude the possibility of multiple, non-neighboring, extremals, along each of which the Hamiltonian may still be regular i.e. does not exclude problems having multi-maxima. So the imposition of regularity on the Hamiltonian is not a very stringent requirement on the control problem.[2]

If now, we impose the additional restriction that the Hamiltonian be regular, it can be shown [2] that the optimal control variable u, as well as its p - 1 time derivatives must be continuous. Furthermore, a rather simple expression can be obtained for the entry-point multiplier, ν.

We have

<u>Theorem 2.</u> <u>If the Hamiltonian is regular,</u> $S \in B_{2p-1}[0,T]$[3] <u>and the extremal path has a boundary arc of non-zero length, then</u>

(11) $$\nu(t_1) = \frac{(-1)^p H^-_{uu} [\overset{p-1}{(u)}{}^- - \overset{p-1}{(u)}{}^+]^2}{\overset{2p-1}{(S)}{}^-}$$

<u>where</u> $(\cdot)^-$ <u>denotes</u> ( ) <u>on the interior arc at the junction time</u> t_1.

This expression for the entry point multiplier is very significant. Note that $H^-_{uu} < 0$ (strengthened necessary condition for H to have a maximum), $[\overset{p-1}{(u)}{}^- - \overset{p-1}{(u)}{}^+]^2 \geq 0$, and as S and its time derivatives up to $d^{2p-1}/dt^{2p-2}(S)$ are continuous (therefore zero)

(12) $$\overset{2p-1}{(S)}{}^- > 0$$

in order for the trajectory to reach the boundary. This implies that

(13) $$\nu(t_1) \geq 0$$

for p odd. But $\nu(t_1) \leq 0$ (necessary condition of optimality) and hence (13) implies that for odd-order constraints, the optimal trajectory will, at most, only touch the constraint boundary, provided $\overset{p-1}{(u)}{}^- \neq \overset{p-1}{(u)}{}^+$.[4] Note that for p = 1, it can be shown that $\nu(t_1) = 0$; thus for first order constraints boundary arcs <u>are possible</u>.

3. Employment Model A

The first example deals with the imposition of a subsidy for employing additional laborers where the marginal product of labor is already equal to the minimum wage. Assume that the neo-classical theory of the firm holds; in particular, that the long-run ratio of labor demanded to capital demanded may be determined by differentiating the production function and setting the ratio of the partial derivatives equal to the factor price ratio. The production function in this case is Cobb-Douglas with constant returns to scale (homogeneous of degree one). Suppose further that the demand for capital is equated to the capital actually used but that labor demanded and labor employed are not necessarily identical, then we may express the ratio of labor demanded to capital in use by the equation

$$(14) \qquad \left(\frac{D}{K}\right)_t = \left(\frac{1 - \alpha}{\alpha}\right) \left(\frac{r}{w}\right)$$

where $(\frac{r}{w})$ is the interest wage ratio, D is the labor demanded, K is the value of the capital stack in use, and α is the capital elasticity of the output.

The term on the r.h.s. of (14) must be complicated a bit when we allow for the presence of government tax policies and subsidies, and the effect of lagged responses to changes in the effective factor price ratio. Consider first the tax adjustment coefficient for the interest rate. This coefficient is defined by the equation

$$(15) \qquad h = \frac{(1 - \beta)}{(1 - u_c)}$$

where β is the amount of the investment tax credits and depreciation deductions expressed as a percentage of the gross investment and u_c is the tax rate on capital income. A similar relationship may be defined for the wage adjustment coefficient v. Here we assume that the firm does not pay a tax on the labor it employs and that the coefficient represents a subsidy for the employment of additional laborers expressed as a percentage of the increment in the wage bill. The effective interest-wage ratio g may then be expressed as

$$(16) \qquad g = (\frac{r}{w})(\frac{h}{v})$$

We wish to avoid specifying how factor prices are determined within the model. Therefore, we shall define the change in the effective factor- price ratio between time periods, $\dot{g}$, as our control variable rather than $(\frac{\dot{h}}{v})$, i.e.

$$(17) \qquad \dot{g} = u \ .$$

Brown and de Cani [4] and David and van de Klundert [5] have given reasons why the response to changes in factor prices may be considerably greater in the long-run, than in the short-run. The main point is that to introduce a new technology with different factor proportions may require a new machine; on existing machines, proportions may be fixed. Given that the lag structure is one with geometrically declining weights,[5] the equation for desired factor proportions may be written as

$$(18) \qquad (\frac{D}{K})_t = (\frac{1 - \alpha}{\alpha}) \sum_{i=0}^{\infty} (1 - \lambda)\lambda^i g_{t-i}$$

where $\lambda (0 < \lambda < 1)$ is the initial weight. As Koyck [6] has demonstrated, a reduced form of this relationship may be obtained. Lag the equation one period, multiply it through by λ and then subtract the lagged equation from (18). The procedure yields,

$$(19) \qquad (\frac{D}{K})_t = (1 - \lambda)(\frac{1 - \alpha}{\alpha}) g_t + \lambda(\frac{D}{K})_{t-1}$$

This first order difference equation may be approximated by the first order differential equation. (For an alternative procedure, see Griliches [7]).

(20) $$\frac{d}{dt}(\frac{D}{K}) = m_1 (\frac{1 - \alpha}{\alpha})g - m_2(\frac{D}{K})$$

($0 < m_1$, $0 < m_2 < 1$. In what follows all coefficients are positive constants, unless otherwise stated).

There is a distinct difference between the change in the effective interest-wage ratio (Δg) and the final change in that ratio which equates demand and supply for labor, (Δg^*). In the short-run, changes in the technical labor requirements may produce a slow response from the supply side. There may not be enough laborers with the requisite skills in times of excess demand, and training may be necessary; in times of deficit demand, firms may retain their skilled personnel. Also, the empirical evidence suggests that labor force participation rates, while positively correlated with employment, tend to lag fluctuations in employment. Finally, the firms may anticipate a future change in demand opposite to the one that has just occurred. Under these circumstances they may concentrate on meeting their demand for production workers and not overhead personnel.[6]

Rather than determine an equilibrating change in the effective interest-wage ratio, we choose to represent these bottlenecks by an adjustment equation

(21) $$L_t = p[D_t - L_{t-1}] + L_{t-1} \qquad 0 < p < 1$$

where L_t is the level of employment attained in period t. This is approximated by the first order differential equation

(22) $$\frac{d}{dt}(L) = \eta_1 D - \eta_2 L.$$

The remaining dynamic equation concerns the growth of capital stock over time. Given a Cobb-Douglas production function with constant returns to scale, value added is given by

(23) $$v_t = K_t^{\alpha} \zeta_t^{1-\alpha}$$

where

(24) ζ_t = minimum [L(t),D(t)].

Suppose that saving is expressed as a constant proportion of value added, then the capital accumulation equation is

(25) $$\frac{d}{dt}(K) = sK^{\alpha}\zeta^{1-\alpha} - \delta K$$

where s is the constant average propensity to save and δ is the rate of depreciation.

The state variable L(t) is constrained below by zero[7] and above by the full-employment ceiling (given exogenously) $\bar{L}(t)$; i.e.

(26) $$0 \leq L(t) \leq \bar{L}(t)^{8}, \qquad t \in [0,T].$$

Finally, denoting the termination date by T, the rate of social discount by γ, the control variable cost coefficient by σ, we may write the welfare functional which we wish to maximize, in the form

(27) $$J = \int_0^T e^{-\gamma t}[(1 - s)K^{\alpha}\zeta^{1-\alpha} - \sigma u^2]dt.$$

The first term in the welfare functional represents consumption[9]; this is a familiar component of aggregate welfare. But why, one may ask, is the control variable costly? The answer is that changes in the control may be expensive in the sense that risk may be increased and speculative behavior on the part of the firm may result. (Perhaps the reason that "control cost" of this form has not been used earlier is that models with high-order dynamics, where such cost is appropriate, have been rarely considered in the literature on planning).

Gathering together the dynamics, along with the welfare functional (27) and the state-variable inequality constraint (26), optimal employment is given by the following maximization problem:

(28) $$\text{Max } J = \int_0^T e^{-\gamma t}[(1 - s)K^{\alpha}\zeta^{1-\alpha} - \sigma u^2]dt$$

subject to

(29) $$\dot{g} = u$$

(30) $$\dot{\left(\frac{D}{K}\right)} = m_1\left(\frac{1 - \alpha}{\alpha}\right)g - m_2\left(\frac{D}{K}\right)$$

(31) $$\dot{L} = \eta_1 D - \eta_2 L \qquad (1 > \eta_1 > \eta_2 > 0)$$

(32) $$\dot{K} = sK^{\alpha}\zeta^{1-\alpha}$$

where ζ[10] is given by (24), and initial conditions on the state variables are specified (T is fixed); and subject to the third order state constraint

(33) $$0 \leq L(t) \leq \bar{L}(t).$$

4. Employment Model B

Let us now turn to a situation in which the obstacle to full employment is not a floor on the real wage but insufficient training of workers. Dobell and Ho [9] have formulated a model which deals with this set of circumstances. It centers around a well-behaved production function with constant returns to scale

(34) $$V = F(K,W)$$

where K is the stock of machinery and W is the stock of trained workers. Denote the flow of educational services by E, gross additions to machinery by M, and consumption by C. Then according to Dobell and Ho [9] we may write

(35) $$V = M + C + E$$

as an accounting identity.

Dobell and Ho go on to argue that this identity may be used as a transformation surface. It implies that resources may be shifted instantaneously from one form of production to another with no loss in the total value of services rendered. However, time may be required for retraining of workers and for the restructuring or retirement of machines. Even in the case of an increment in V, the planning authority would have to exert considerable control over vocational choice and the bill of new machines to achieve a relationship independent of past production.

Let us remedy this lack of realism with a simple modification of the Dobell-Ho model. Denote the ratio of educational services to total value added by $z = \frac{E}{V}$. Then postulate a relationship involving a delayed adjustment

(36) $z_t = p[z_t^* - z_{t-1}] + z_{t-1} \qquad 0 < p < 1$

where z_t^* is the desired ratio of educational services to value added. We will write z_t^* as

(37) $z_t^* = u_1$

where $u_1(t)$ is a control variable. Then, approximating (36) by a differential equation, we have

(38) $\dot{z} = \pi_1 u_1 - \pi_2 z$

where $0 < \pi_1 < 1$, $0 < \pi_2 < 1$.

Delays in the shifting of resources between machinery and consumption goods production only complicate the problem without affecting the final result. Therefore, in the case of machinery, neglecting depreciation, we may write

(39) $\dot{K} = M.$

Dobell and Ho specifically mention that they neglect gestation lags in the training of workers. Assuming that the rate of worker retirement is zero (for simplicity), their equation for the level of educational services is

(40) $E_t = d(W_{t+1} - W_t)$

where W is the level of employment. The ooefficient d they assume to be an increasing function of the employment rate.[11] Let us make d a constant independent of the employment rate, and denote the desired change in the trained labor force between periods t and t + 1 by

(41) $\Delta W^*_t = \frac{1}{d} E_t.$

Then, given a lag structure with geometrically declining weights, the change in the trained labor force may be expressed as

(42) $\Delta W_t = (1 - \lambda_1) \sum_{i=0}^{\infty} \lambda_1^i \Delta W^*_{t-i}$

Again, following [6], [7], this equation reduces to

(43) $\Delta W_t = \frac{1}{d}(1 - \lambda_1)E_t + \lambda_1 \Delta W_{t-1}$

We approximate this second order difference equation with the differential equation

(44) $\ddot{W} = \mu_1 E - \mu_2 \dot{W}$ $\qquad 0 < \mu_1, \mu_2 < 1$

The dynamic equations for our system are (38), (39) and (44). For convenience, we will re-write these equations in a normalized form, assuming that the total labor force grows at an exponential rate n (0 < n < 1) i.e.

(45) $\bar{L} = L_o e^{nt}$ $\qquad L_o =$ initial value

Define

(46) $\frac{W}{\bar{L}} = \omega$

(47) $\frac{K}{\bar{L}} = k$

(48) $\frac{M}{V} = u_2$ $\qquad$ control variable

(49) $\frac{C}{V} = c$

and

(50) $F(\frac{K}{\bar{L}}, \frac{W}{\bar{L}}) = \frac{V}{\bar{L}} = f(k,\omega).$

Then (38), (39) and (44) may be written as

(51) $\dot{z} = \pi_1 u_1 - \pi_2 z$

(52) $\dot{k} = -nk + u_2 f(k,\omega)$

and

(53) $\ddot{\omega} = \mu_1 z f(k,\omega) - (\mu_2 + 2n)\dot{\omega} - n(n+1)\omega$

In addition to the full employment ceiling on ω,

(54) $0 \le \omega \le 1$,

the identity (35) gives the following constraints on z and u_2;

(55) $0 \le z \le 1$

(56) $0 \le u_2 \le 1$

(57) $0 \le u_2 + z \le 1$.

The welfare functional which we wish to maximize consists of the discounted per capita consumption stream less penalty terms (which reflect our desire for some stability in policy). Summarizing, optimal employment is obtained by maximizing the welfare functional

(58) $$J = \int_0^T e^{-\gamma t}[(1 - z - u_2)f(k,\omega) - \sigma_1(\pi_1 u_1 - \pi_2 z)^2 - \sigma_2(u_2 - b(t))^2]dt$$

with the dynamics

(59) $\dot{z} = \pi_1 u_1 - \pi_2 z$

(60) $\dot{k} = -nk + u_2 f(k,\omega)$

(61) $\dot{\omega} = y$

and

(62) $\dot{y} = \mu_1 z f(k,\omega) - (\mu_2 + 2n)y - n(n+1)\omega$

and subject to the constraints

(63) $0 \le \omega \le 1$

(64) $0 \le z \le 1$

(65) $0 \le u_2 \le 1$

(66) $0 \leq u_2 + z \leq 1$.

Before going on to discuss the implications of this model along with those of model A, we note the following:

(i) As the proportion of resources devoted to education changes there may be increasing opportunity costs; thus a quadratic penalty is applied to $\dot{z}$.

(ii) The penalty term $-\sigma_2(u_2 - b(t))^2$ reflects our desire for a stable policy of investment. Absence of such a term would lead to 'bang-bang' solutions; such solutions are not economically very meaningful. The presence of the penalty reflects the desire for some mean rate of machinery investment, b(t), and the possibility of increasing opportunity costs when resources are shifted into machines.

(iii) Some consideration should also be given to the terminal conditions in both models A and B. We have not done so for this would not change our qualitative results. This is, however, certainly a point to remember in computing solutions.

5. <u>Discussion of Models A and B</u>

The constraints on the state variables representing the employment level in both models A and B are third order (odd order). Before applying Theorem 2, however, we must verify if the problems satisfy the conditions imposed therein.

In both models A and B, the Hamiltonians are regular. In model A, the term $\zeta(t) = \min(D,L)$, does not prevent the application of Theorem 2, as shown in Appendix I; we assume $\bar{L}(t)$ to be sufficiently differentiable. In model B, the presence of constraints on u_2 and z apart from that on ω, makes for added complexity. We show in Appendix II that these constraints do not affect our conclusions provided $\mu_2^2 > 4n(1 - \mu_2)$.[12]

With these preliminaries, we can now state our central result for both problems: From Theorem 2 (and Appendix II) the characteristic of the optimal employment path is that full employment is attained, at most, instantaneously, i.e. full employment is not maintained for non-zero intervals of time.

Note that in model B, the cost of training has been assumed to remain constant as the employment level rises. By contrast, the example discussed by Dobell and Ho in [9] does not exhibit full employment along the optimal trajectory due to increasing training costs; whereas their constant training cost model of [10] exhibits sustained full employment. In both our models A and B it is the lag structure (i.e. the order of the constraint) alone that causes sustained full employment to be non-optimal.

What are the policy implications of our two employment models? If the problems were re-formulated so that $\dot{L}$ or $\dot{\omega}$ was controlled directly, the employment level would move to its upper bound and stay there. Clearly, the integral of the discounted consumption stream would be larger in the case of model A (provided there is no control variable cost) than it is in the original case with the third order constraint; for value added, and hence consumption, is an everywhere increasing function of employment.

How effective the planning authorities can be in eliminating delays remains unclear. But some possibilities do exist. One is subsidized on-the-job training. Employment model B may depend too heavily on the assumption that a threshold level of skill must be achieved before hiring can be justified. On-the-job training may be less efficient from a production standpoint, but it still affords the possibility of optimal full employment. Training is conducted as part of the education process; thus an instantaneous transfer of resources into education may be plausible. Further, there is flexibility in the skill level required for employment. Whether or not it is optimal to substitute this scheme for slower training procedures is still open to question.

6. A Model of Foreign Debt

In the study of developing economies, it is interesting to evaluate the optimal accumulation path of foreign debt over time. Here mounting interest costs must be balanced against the increased capital accumulation and imports which an inflow of foreign loans permits.

Our model centers around four dynamic equations. Let us first consider the investment-demand equation.

Assume again that total value added is given by a Cobb-Douglas production function with constant returns to scale. The price of traded and domestically produced goods is the same. The level of employment is constant, and the units in which labor is expressed may be normalized to equal one. Then, following the principles of optimality for the firm, the desired level of capital stock K_t^* may be derived from the condition that the value of the marginal product of capital should be equal to the rental price of capital; for a Cobb-Douglas production function this gives:

$$K_t^* = \frac{\alpha K^\alpha}{z_t} \qquad (67)$$

where z_t is the real rate of interest multiplied by a tax adjustment coefficient like the one in (15), and α is the capital elasticity of output.[13]

On the basis of a 'putty-clay' model of the firm, Bischoff [12] has shown that the response of investment to changes in relative prices is more delayed than its response to output. Therefore, let us divide the demand for investment, I^*, into two parts: that due to output changes I_v^*, and that due to interest changes I_z^*. Assume that there is no demand lag in reaction to variations in output. Then we may approximate I_v^* as

(68) $$I_v^* = \frac{\alpha^2 K^{\alpha-1}}{z} \cdot \frac{d}{dt}(K)$$

The capital accumulation equation is

(69) $$\dot{K} = I - \delta K$$

where I is gross investment and δ is the rate of depreciation.

On the other hand, I_z^* depends on past as well as present changes in the real rate of interest, z. With a lag structure involving geometrically declining weights, this can be expressed as

(70) $$(I_z^*)_t = (1 - \lambda')(\Delta K^*_z)_t + \lambda'(I_z^*)_{t-1}$$

where $(\Delta K_z^*)_t$ is the change in the desired capital stock between time periods t - 1 and t due to a change in z. This is approximated by the first order differential equation

(71) $$\dot{I}_z^* = \eta_1 \frac{\alpha K^\alpha}{z^2} \dot{z} - \eta_2 I_z^*$$

The control variable is defined by the equation

(72) $$\dot{z} = u$$

Thus far, nothing has been said about the lag in supply; this brings us to the third dynamic equation. Generally there is a considerable length of time between the ordering of machines and structures and their availability. Specifying this gestation lag as one with geometrically declining weights, we determine current gross investment with the function

(73) $$I_t = (1 - \lambda_2)I_t^* + \lambda_2 I_{t-1}^* + \delta_1 K_t$$

which we approximate by the differential equation

(74) $$\dot{I} = \mu_1 I^* - \mu_2 I + \mu_3 K .$$

Recognizing that $I^* \equiv I_z^* + I_v^*$ and using (68) we have for I

(75) $$\dot{I} = \mu_1[I_z^* + \frac{\alpha^2 K^{\alpha-1}}{z}(I - \delta K)] - \mu_2 I + \mu_3 K .$$

The control u enters through $\dot{I}^*_z$; hence we control not $\dot{I}$ but $\ddot{I}$.

The last dynamic equation concerns the growth in foreign debt. Net foreign capital inflow is the difference between gross investment and gross domestic saving. Let V represent gross domestic product, and assume that gross domestic savings S may be expressed as a linear function of V $(= K^{\alpha})$

(76) $$S = sK^{\alpha} - b$$

where s is the aggregate (constant) savings rate. Substituting this into the identity for net foreign capital inflow and subtracting off interest payments from savings yields

(77) $$\dot{D} = I - sK^{\alpha} + b + fD$$

where D is foreign debt, and f represents interest payments as a proportion of foreign debt.

An explicit balance of payments equation is neglected. Imports, it is presumed, are allowed to vary so as to always equal the sum of exports and net capital inflow. We note that creditor nations tend to scrutinize the ratio of total debt service payments to export earnings, the 'debt-service ratio'. Suppose that debt-service payments may be represented as a fixed proportion, θ, of total debt and that export earnings $\bar{E}$ are taken as exogenous. Then, we assert, no more loans will be forthcoming once debt-service payments reach a certain percentage, β, of export earnings. Symbolically

(78) $$\theta D(t) \leq \beta \bar{E}(t)$$

i.e. a constraint on the debt level.

In this model the welfare functional which is to be maximized is

(79) $$J = \int_0^T e^{-\gamma t}[\{(1 - s)K^{\alpha} + b\} - \sigma u^2]dt$$

subject to (69), (71), (72), (75), (77) and (78). Here T is the planning horizon and γ is the rate of social discount.[14] The main component of the integrand is consumption which is determined by subtracting savings from gross domestic product. For the reasons cited earlier, we penalize changes in the control variable with the coefficient σ. Some allowance should also be made for the effect of tax policy on income distribution. This effect is not accounted for as it is difficult to do so without explicitly determining the rate of interest.

In sum, the optimal debt policy is given by the following optimization problem.

$$\text{(80)} \qquad \underset{u}{\text{Maximize}}\ J = \int_0^T e^{-\gamma t}[\{(1 - s)K^{\alpha} + b\} - \sigma u^2]dt$$

subject to

$$\text{(81)} \qquad \dot{z} = u$$

$$\text{(82)} \qquad (\dot{I}^*_z) = \eta_1 \frac{\alpha K^{\alpha}}{z^2} u - \eta_2 I^*_z$$

$$\text{(83)} \qquad \dot{I} = \mu_1[I^*_z + \frac{\alpha^2 K^{\alpha-1}}{z}(I - \delta K)] - \mu_2 I + \mu_3 K$$

$$\text{(84)} \qquad \dot{K} = I - \delta K$$

and

$$\text{(85)} \qquad \dot{D} = I - sK^{\alpha} + b + fD$$

and subject to the third order state constraint

$$\text{(86)} \qquad \theta D(t) \leq \beta \bar{E}(t)$$

with specified initial and terminal conditions on the five state variables.

As the Hamiltonian for the above problem is regular, we have by Theorem 2 that the optimal debt level touches the debt-service ratio ceiling at most instantaneously i.e. optimal debt level is not maintained equal to the debt-service ratio ceiling for a non-zero time interval.

There is a significant contrast between the optimal path in this model and the 'bang-bang' solutions of Chenery and MacEwan [13] and Cetin and Manne [14]. The latter models are linear programs; upper bounds are placed either on the change in investment or the net change in the level of debt. These constraints are continuously binding in the early periods of the solutions. Considering the general structure of both models, this is equivalent to saying that the constraints on maximum debt are binding in each of the initial periods. Yet, we know from the order of the constraint in our model that the debt level touches its upper bound at most instantaneously. Consequently, the nature of the optimum debt path for higher order models (which are perhaps more realistic) is considerably more complex than past programming results suggest.

Given a lower order constraint on D(t), i.e. more direct control of the debt level, and no cost on the control, the economy is better off growing along the upper bound on debt than it is in meeting the optimality conditions in the third order constrained model. In calculating the social cost of discarding capital, the possible social benefits of eliminating the lagged demand must be singled out. Thus, because firms do not consider this effect in their cost calculations, subsidized discarding of capital may be justified under some circumstances. For the delay in the response of investment demand to changes in the effective interest rate may be reduced. Obviously, such tradeoffs can only be examined in the context of larger models.

7. Conclusions

We have shown that there exist economic planning problems, with constraints on the state variables, where some qualitative features of the optimal trajectories may be deduced a priori, if the order of the constraint is odd (but not one). The empirical validity of geometrically declining weights for the lag functions -

an assumption upon which many of the differential equations rely - is perhaps questionable.[15] But few would deny that some form of lag structure exists. This, together with our results, suggests that decision rules based on systems not incorporating these lag structures must be held suspect.

Thus in this sense, this contribution has been negative. Let us now outline some possible areas for future research.

What are the properties of the optimal employment trajectory beyond its touching the full employment bound at most instantaneously?
Clearly, the policy implications differ radically depending upon whether or not the state variable path is cyclical in nature, i.e. bounces on and off the full employment bound. In situations where a certain level of unemployment is optimal, it may be desirable to use income redistribution policy to compensate the unemployed (e.g., "a guaranteed annual income"). But for an optimal trajectory which is cyclical, the timing of such a policy would be critical; thus, in this case redistribution of income may be intractable in view of the institutional inflexibilities of fiscal policy. For an optimal trajectory which touches only in the terminal period, if at all, a slow implementation of policies may be considerably less serious.

Past programming results ([13] and [14]) indicate that it may be extremely costly to employ periodic limits on accumulated foreign debt other than the usual terminal condition. Our analysis throws no light on the expense inherent in the imposition of the state variable constraint. Nor do we have any indication of the number of times, if at all, the service payments strike their upper bound. These are crucial issues. Take, for instance, the case where the upper bound on service payments coincides with the terminal requirement at that point in time. If it could be established that the state variable constraint is binding only in the terminal period, then considerable support would be given to the institutionalized system of debt allocation where a given debt-service ratio is not exceeded. Insight into these problems may well come from numerical solutions.

8. Acknowledgements

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Appendix I

Lemma: The Hamiltonian of employment model A is continuous and piecewise differentiable, except at points $t_i \in [0,T] \ni D(t_i) = L(t_i)$.

Proof: The only (possibly) non-differentiable function is

$$\zeta(t) = \min[D(t), L(t)]$$

But D(t), L(t) are continuous and differentiable on [0,T]. Hence $\zeta(t)$ is continuous and piecewise differentiable except (possibly) at $t_i \in [0,T] \ni D(t_i) = L(t_i)$ where the derivative $\dot{\zeta}(t)$ may not be continuous.

The discontinuities in $\dot{\zeta}(t)$ will not affect the application of Theorem 2 provided they do not occur at $t_j \in [0,T]$ such that $L(t_j) = \bar{L}(t_j)$. That this is the case is seen if we re-write (31) in the form

$$\dot{L} = \eta_1(D - L) + (\eta_1 - \eta_2)L.$$

Then, assuming that the initial value of L is less than $\bar{L}$, $L(t_j) = \bar{L}(t_j)$ if and only if $D(t_j) > \bar{L}(t_j)$ (as $\eta_1 > \eta_2$ and $\eta_1 - \eta_2 < 1$). So the changeover $\zeta(t) = L(t)[D(t)]$ from $\zeta(t) = D(t)[L(t)]$ does not occur at t_j. The case when the initial value of L, $L(t_o)$, is equal to $\bar{L}(t_o)$ and $D(t_o) = L(t_o)$ $(= \bar{L}(t_o))$ causes no difficulties; for at $t = t_o + \varepsilon$ $(\varepsilon > 0)$

$$L(t_o + \varepsilon) < L(t_o) = \bar{L}(t_o)^{1*}$$

So, if at t_o we set $\zeta(t_o) = L(t_o)$, at time t $\zeta(t)$ is still equal to L(t) [if $D(t) > L(t)$, if not set $\zeta(t_o) = D(t_o) = L(t_o)$]. In either case, there is no changeover when $L(t) = \bar{L}(t)$.

Appendix II

We have the following constraints [(63)-(66)]

(1*) $0 \leq \omega \leq 1$

(2*) $0 \leq z \leq 1$

(3*) $0 \leq u_2 \leq 1$

(4*) $0 \leq u_2 + z \leq 1$

Before we can apply Theorem 2, we must ensure that the presence of constraints (2*) - (4*) does not affect the continuity of u_1, u_2 and their derivatives.

Consider (2*) - (4*). This gives rise to the seven cases:

A. z and u_2 both slack, $0 < u_2 + z < 1$.

B. z and u_2 both slack, $0 < u_2 + z = 1$.

C. $z = 0$, $u_2 = 0$, $u_2 + z = 0$.

D. $z = 0$, $0 < u_2 < 1$, $0 < u_2 + z < 1$.

E. $z = 0$, $u_2 = 1$, $u_2 + z = 1$.

F. $0 < z < 1$, $u_2 = 0$, $0 < u_2 + z < 1$.

G. $z = 1$, $u_2 = 0$, $u_2 + z = 1$.

We will now analyze each case individually.

A. No constraints effective. Then we can apply Theorem 2 directly.

B. u_2 and z slack; $u_2 + z = 1$. As z is slack, u_1 is given by

(5*) $$H_{u_1} = 0 = -2\sigma_1 u_1 + \pi_2 \lambda z$$

Then

(6*) $$u_1 = \frac{\pi_2}{2\sigma_1} \lambda z$$

and, assuming that f_k, f_ω exist - a reasonable assumption - u_1, $\dot{u}_1$ are continuous and $\ddot{u}_1$ exists. From $u_2 + z = 1$, $u_2 = 1 - z$; hence, u_2, $\dot{u}_2$, $\ddot{u}_2$ are continuous. We can now apply the analysis (given in [2] or [3] of optimal control) leading to the equation (11); this gives an expression for the multiplier ν_ω of the form

$$(7^*) \qquad \nu_\omega = (-1)^p \frac{H^-_{u_1u_1}[(\ddot{u}_1)^- - (\ddot{u}_1)^+]^2 + H^-_{u_2u_2}[(\ddot{u}_2)^- - (\ddot{u}_2)^+]^2}{(S)^{2p-1}_-} .$$

The second term on the r.h.s. is zero, but the first is not (except under very unusual circumstances). Thus we can apply a modified version of Theorem 2, and the state that for this case $\omega = 1$ at most instantaneously.

Cases C, D and E can be treated together, for from the dynamic equations for ω and y, namely

$$(8^*) \qquad \dot{\omega} = y$$

$$(9^*) \qquad \dot{y} = \mu_1 f(k,\omega)\, z - (\mu_2 + 2n)y - n(n+1)\omega$$

we see that if $z = 0$, ω is governed by a homogeneous second order differential equation. For feasibility, if $\omega = 1$, $\dot{\omega} = y \leq 0$. So, provided $\mu_2^2 > 4n(1 - \mu_2)$ i.e. negative real eigenvalues, the trajectory will not stay on $\omega = 1$ except instantaneously.

F. z slack, $u_2 = 0$, $u_2 + z$ slack. This is further subdivided as follows: Let $\omega = 1$ at $t = t_1$ where condition F holds. Then either

i) u_2 has saturated prior to time t_1. In this case $\dot{u}_2$, $\ddot{u}_2$ exist and are continuous and the modification of Theorem 2 used in case B holds, and $\omega = 1$ only instantaneously.

ii) u_2 saturates at the same time, i.e. the transition from any of the other cases to this is made at time t_1. In this case as $\omega < 1$ prior to t_1 (we anticipate some of our analysis) we cannot unequivocally rule out the possibility of a sustained boundary arc in ω, i.e. $\omega = 1$ for a non-zero interval of time. Were this to be the case, however, the optimal solution would exhibit simultaneously

a. decreasing per capita capital stock

b. increasing per capita investment in education

c. decreasing per capita consumption.

Economically, this makes little sense; further an optimal solution exhibiting such a sustained arc would be (economically) indefensible. We thus conclude that we can safely rule out this possibility on purely economic grounds.

G. $z = 1$, $u_2 = 0$, $u_2 + z = 1$. Then u_2 must saturate prior to the time, t_1, at which $\omega = 1$ - from continuity of $\dot{z}$ and $u_2 + z = 1$. This leaves only two possibilities

i) z saturates at the same instant, t_1

ii) z saturates prior to t_1.

In either case, subsequent to the time t_1, if the solution is to exhibit a sustained boundary arc in ω, the controls u_1, u_2 must satisfy

(10*) $$u_1 = \frac{\pi_2}{\pi_1} = \text{constant}$$

(11*) $$u_2 = 0$$

and

(12*) $$\dddot{\omega} = 0 = \psi(u_1, u_2, t).$$

But (12*) would require u_1 and u_2 to be time-varying, hence a boundary arc cannot be sustained.

Footnotes

1 A state-variable inequality constraint is pth order if its pth time derivative is the first to contain the control variable explicitly.

2 In point of fact, it is rather difficult to construct a non-regular Hamiltonian e.g. see [3] p. 126.

3 $B_{2p-1}[0,T] \equiv$ Class of functions whose (2p - 1)th derivative is bounded in the interval [0,T].

4 The case $(u^{p-1})^{-} = (u^{p-1})^{+}$ is possible, but numerical experience with problems ($p \geq 3$) indicates that this is rare. In fact x_o would have to be carefully chosen for this to occur.

5 Estimates, using this form of the lag structure, have been carried out in [4], [5] for the more general C.E.S. production function.

6 Most of these arguments, along with the empirical evidence supporting them, are given in Kuh [8].

7 The lower ceiling on L(t) is of little economic significance.

8 $\bar{L}(t)$ will be assumed to be monotone increasing.

9 More correctly, this should be modified to include per capita consumption. However, this does not affect our results.

10 The Hamiltonian is still piecewise continuous and differentiable. See Appendix I.

11 In [10] Dobell and Ho consider the case of a constant training cost. For this case, their results unequivocally yield sustained full employment for a significant portion of the optimal trajectory. The case of an increasing training cost gives the solution discussed in [9].

12 This is not an overly stringent condition if education in the model is defined as short-term vocational training. Other educational services may be expressed as an exogenous component of E without affecting the final result.

13 See, for example, Jorgenson [11].

14 Footnotes 7, 8, 9 apply here also.

15 However, see Griliches [7] with regard to other estimates.

1* $\bar{L}(t)$ is assumed to be a monotone increasing function of time.