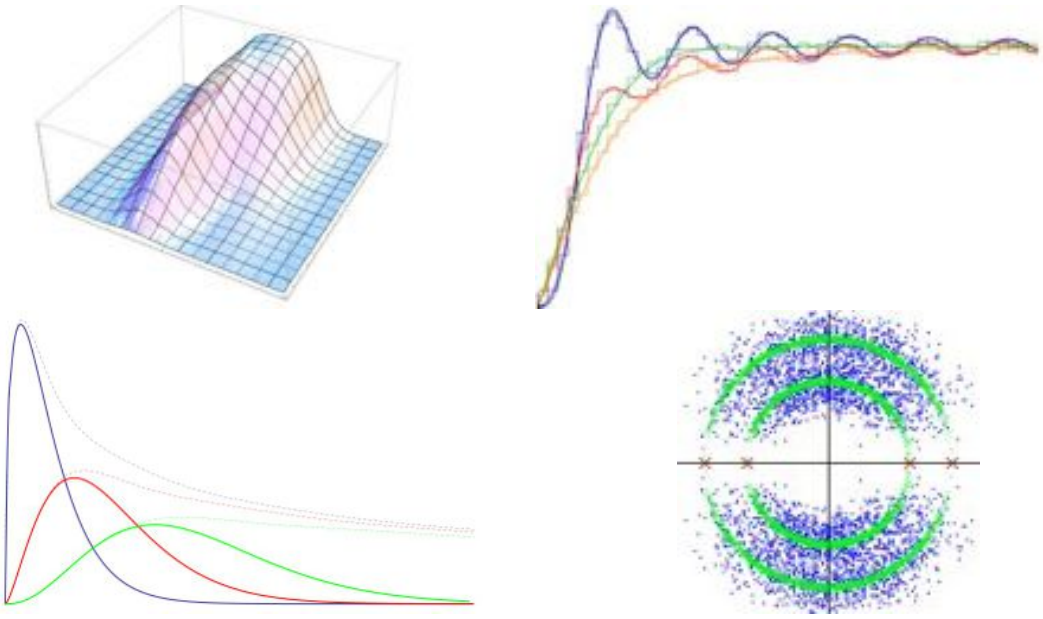
Universität Bielefeld
Fakultät für Physik

Habilitationsschrift
(kumulativ)

Chiral Random Matrix Theory: Generalizations and Applications



von

Dr. rer. nat. Mario Kieburg

geboren am 23.06.1981 in Zossen

Der stete Blick auf das Ziel gerichtet.
Verschleiert wirkt dieser,
müde und immer noch suchend.

Taumelnd wankt das Ziel hin und her,
wie eine Boje auf dem Meer,
flüchtend vor dem gierenden Blick.

Wie blind die Augen doch sind.
Getrieben vom eignen Stolz
erkennen sie den Weg nicht,
auf dem sie gehen.

Mario Kieburg

Acknowledgements

First of all I want to thank my wife Janine. She had a lot of patience with me especially when I calculated at home for hours and hours without noticing anything except the numbers, letters and symbols I am writing on almost everything which serves as a sufficient sheet of paper. Sometimes it happened that I lived in a different world which RMT indeed offers with its aesthetic, analytical structures such that one can only muse in awe. My wife reminded me in those moments that there is also another universe which is as beautiful if not even much more wonderful as the world of matrices and spectra.

Moreover I thank all my colleagues and collaborates. Without them all the results presented in this thesis would have never been possible. Especially I am very grateful to Jac Verbaarschot. I do not only appreciate the exciting scientific discussions with him which sometimes have lasted for hours but also his opinion as a very good friend of mine and the time he always had for me to talk about anything. I want to particularly thank Gernot Akemann and my former Ph.D. supervisor Thomas Guhr, too. They gave me many advises on my way to an independent researcher and supervisor of students.

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Contents

1 Introduction

Guhr, Müller-Groeling, and Weidenmüller wrote once in a very nice review [1]: *"The enormous development of RMT* [random matrix theory] *during the last decade signals the birth of a new kind of statistical mechanics."* They were right and since their review much more applications have been found and new mathematical developments have taken place. RMT can be indeed understood as a part of statistical physics. It embodies the fundamental idea of reducing a vast amount of informations to only a few observables which cannot only be measured but also analytically predicted.

What is the focus of RMT? In classical statistical mechanics we have the phase space distribution as the fundamental object from which we can derive the average position of particles or their average momenta as well as global observables like temperature and pressure. In quantum mechanics the fundamental object is the density operator which also encodes the statistics of quantum states. In RMT we make predictions about spectra and their corresponding eigenvector statistics. Those spectra can be found in physical systems like the spectrum of the Hamilton operator in quantum mechanics, of the Dirac operator in QCD, or of the d'Alembert operator with boundary conditions for classical wave equations. Yet, spectra can be found in other sciences as well, e.g. in mathematics, sociology, or economics. The observables which were formerly the pressure, the chemical potential, and the temperature in thermodynamics are now replaced by the level spacing distribution, the local and the global macroscopic level density, and the inverse participation ratio of eigenvectors. RMT can make predictions about these observables and many experiments confirm that this approach is ideal to understand generic spectral statistics of systems.

Chiral random matrix theory (χRMT) is the field of RMT where one studies random matrices which have a chiral symmetry like the one of the Dirac operator, see [2]. Due to new applications, e.g. in lattice QCD, time series analysis and telecommunications, generalizations of this particular kind of random matrices were recently considered. Those generalization may involve a weak breaking of chiral symmetry [3, 4, A], the breaking of rotation invariance of the matrices [5, 6, G], or the generalization to products and sums of matrices [7, 8, L]. Breakthroughs in the analytical computations of those random matrix ensembles made it possible to deduce new predictions and discover new kinds of spectral statistics.

In section 2, a brief historical overview about RMT, especially χRMT, is given. In particular, the various motivations and goals of some of the topics in RMT are explained. The main ideas of a statistical theory of spectra are outlined, especially how RMT has historically arisen from experimental and empirical observations and why the state of the art of RMT is as it is now. Thereby the section starts with the statistics of covariance matrices which were the first matrices modelled by RMT, see subsection 2.1. Then the importance and the richness of the local statistics in general systems will be explained in subsection 2.2. In particular the local statistics at critical points of spectra, see subsection 2.3, invite to classify systems. Those points are very sensitive with respect to the global symmetries of systems such that they seem to be universal and apply to large classes of spectra. One particular way to express these universal spectral statistics is in terms of non-linear σ-models, see subsection 2.4. Those non-linear σ-models are also the link between quantum field theory and RMT. Due to this link it is well understood why RMT, particularly χRMT, applies so well to QCD

and condensed matter physics.

Connections to my works, which are attached in the three appendices A, B and C and summarized in section 3, are only roughly sketched in section 2. They are discussed in more detail in section 3. My works are presented in three separate parts. The main topics of these parts are about applications of RMT to lattice QCD (subsection 3.1), about Wishart random matrix models 3.2, and about product matrices 3.3. The particular motivations and applications of the specific random matrix ensembles are explained in section 3. Moreover the main ideas of the calculations performed in the works attached are outlined and a brief overview and discussion of the results is given.

The three big parts of my works slightly overlap in their context due to universality of the statistics and the similar techniques in deriving the results. Therefore the different topics should not be understood as parallel studies but as different pieces of a big mosaic. To underline this impression a short outlook is given in section 4.

Finally, I want to point out one particularity of the citation system in this thesis. The works attached in the appendices are denoted by capital Latin letters while all other citations are given in Arabic numbers.

2 From Experiments to RMT

Before we come to the particular applications of RMT let us consider some historical problems which were encountered on the way to the current state of the art of RMT. Then we will understand that completely different systems regardless whether they are found in physics, mathematics, engineering, economics, sociology or other types of sciences may share common statistical features. This fact is on the hand quite surprising since the mechanisms working in these particular systems are certainly completely different. Yet, it is a common experience that, on the other hand, the statistical fluctuations in a chaotic or disordered system, which might be even open or in a non-equilibrium, can be quite dominating such that almost all system specific details are washed out.

This behavior is from a mathematical point of view quite natural due to the central limit theorem. You require only very weak conditions on the fluctuations and the system tends to a class of systems sharing the same universal properties. In the case of the original central limit theorem you only need to assume that all moments of a real random variable have to exist and it is immediate that the limiting distribution, after infinitely many convolutions of the random variable with itself, yields a Gaussian [9, 10]. Even when such conditions are not fulfilled the central limit theorem still applies. However the system may approach another attractor which is not a Gaussian but as universal as the Gaussian [11, 12]. These attractors are called stable distributions and exhibit Lévy tails which algebraically drop off at infinity. Such distributions are usually found in open, non-equilibrium systems like financial markets and biology. We come to this point again in subsection 3.3.

2.1 The Problem of big Covariance Matrices

In the middle of the 20s of the last century statisticians became aware of the problem that the amount of data becomes bigger and bigger. At some point one will not be able to see which correlations are system specific and relevant and which are generic not to mention the problem to find any correlations at all.

Let us assume we have measured p time series $v_1, \ldots, v_p$ at n times implying that each time series $v_j = (v_{j1}, \ldots, v_{jn})$ can be interpreted as an n-dimensional vector. Such time series can be either climate data like pressure, temperature, and humidity [13], they can result from financial markets like the prices of stocks [14], they can be signals measured at different positions in wave guides [15, 16], they can be EEG data [17], etc. In the next step we normalize all vectors such that their mean vanishes and their variance is unity,

$$\widehat{v}_j = \frac{n(v_{j1}, \ldots, v_{jn}) - \sum_{l=1}^{n} v_{jl}(1, \ldots, 1)}{\sqrt{n^2 \sum_{l=1}^{n} |v_{jl}|^2 - n \left|\sum_{l=1}^{n} v_{jl}\right|^2}}. \tag{2.1}$$

Then the covariances of the times series can be arranged in a $p \times p$ matrix

$$C_{ji} = \widehat{v}_j \widehat{v}_i^{\dagger} = \sum_{l=1}^{n} \widehat{v}_{jp} \widehat{v}_{ip}^{*} \tag{2.2}$$

with "†" the Hermitian adjoint. One can interpret the normalized time series $\widehat{v}_j$ as a

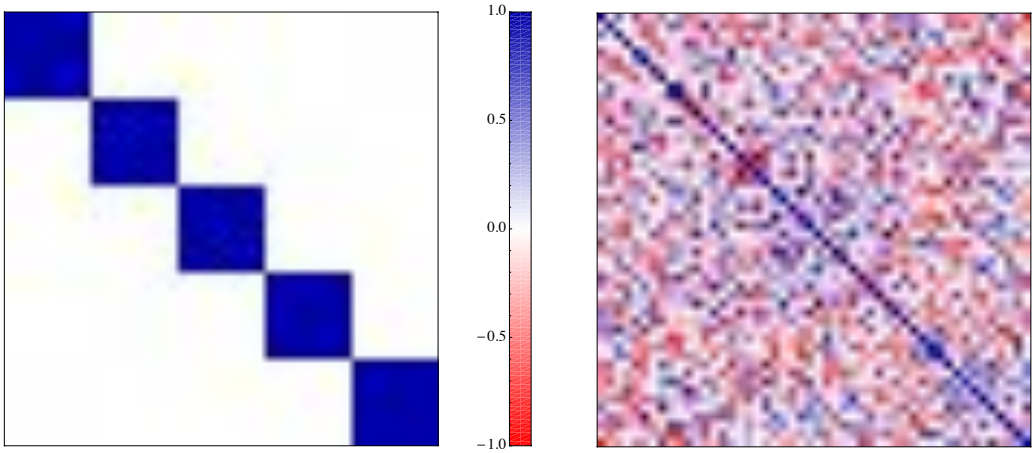

Figure 1: Visualization of the 50×50 covariance matrices C and C' corresponding to the time series V in Eq. (2.4) and to $W = OV$, respectively, with O an orthogonal matrix randomly chosen. Their matrix entries vary between $[-1, 1]$ and are encoded with a color code. Although both matrices differ only via an orthogonal rotation of the time series the obvious separation of the correlations of C (left plot) are completely obscured for C'. Those correlations are still present in the eigenvalues of both matrices. For both C and C' we find five eigenvalues of about $\lambda \approx 10$ reflecting the size of the strongly correlated subsystems while the other eigenvalues almost vanish.

row vector of a $p \times n$ matrix $\widehat{V}$. Then the matrix C is the matrix product

$$C = \widehat{V}\widehat{V}^{\dagger} \tag{2.3}$$

which has the unity on the diagonal and the off-diagonal elements are either inside the complex unit disc or in the interval $[-1, 1]$ depending on whether we have complex or real time series. The task is to extract system specific correlations from C and interpret them. For a small number of time series this can be readily done. But what is when p is large? The situation in the 20s has been not that drastic as it is the case today in the time of "big data" where p can easily reach the values of $p \propto 10^2$ and upwards. Nowadays it is almost impossible to identify correlations by only looking at the matrix C.

To illustrate this problem we construct an example. We choose fifty real time series as follows

$$v_{ij} = \begin{cases} \sin j, & i = 1, \ldots, 10, \\ \sin 2j, & i = 11, \ldots, 20, \\ \sin 3j, & i = 21, \ldots, 30, \\ \sin 4j, & i = 31, \ldots, 40, \\ \sin 5j, & i = 41, \ldots, 50, \end{cases} \qquad \text{with } j = 1, \ldots, 200. \tag{2.4}$$

The corresponding covariance matrix is visualized in the left plot of Fig. 1. The separation into five strongly correlated sub-blocks are clearly visible. This separation has to be expected since we made ten copies of five time series with a specific frequency in the sine function and the sum $\sum_{j=1}^{n} \sin(aj)\sin(bj)$ tends to zero if $|a| \neq |b|$ and $n \to \infty$.

Now we multiply a generic orthogonal matrix $O \in \mathrm{O}(50)$ to the matrix $V = (v_1, \ldots, v_{50})^T$ containing the non-normalized times series, i.e. we obtain a new matrix $W = OV$ which mixes the time series. Indeed the rows of the new matrix W can

be also understood as time series such that we can perform the same construction (2.1) to find a new matrix $\widetilde{W}$ of normalized time series. The matrix entries of its corresponding covariance matrix $C' = \widetilde{W}\widetilde{W}^T$ does not reflect the separation into five sub-blocks anymore, cf. right plot of Fig. 1. Does this mean that we have lost the splitting into strongly correlated subsystems? This cannot be true otherwise the experimentalist has a problem because he does not know in which basis of his time series he will see the correlations he is looking for. Usually he does not have the simple situation as we have with our artificial problem where we constructed the correlation in a given basis. He gets some time series which when they are plotted match more the right plot of Fig. 1 than the left one. Therefore the correlations are hidden. We emphasize that we have not introduced any fluctuations, yet. The problem of identifying correlations with statistical fluctuations, which is more realistic, is even more involved and is briefly discussed below.

Assuming that we have not lost the separation into subsystems by a simple change of basis in the time series, where are they? Considering the eigenvalue spectrum of C and C' we immediately notice where they can be still found. For both covariance matrices we find five large eigenvalues which are about $\lambda \approx 10$ while all other eigenvalues are almost zero. Those largest eigenvalues are also called outliers since they can be separately studied from the gross of the spectrum.

How can we interpret the spectra of C and C'? For a perfectly correlated system, i.e. $C_{ij} = 1$ for all $i, j = 1, \ldots, p$, the covariance matrix C has one eigenvalue which is equal to p and all other eigenvalues vanish. In the opposite case where all time series are uncorrelated, $C_{ij} = \delta_{ij}$ for all $i, j = 1, \ldots, p$, all eigenvalues of C are equal to $\lambda = 1$. In the mixed case where we have perfectly correlated subsystems and each subsystem is uncorrelated to the other the eigenvalues of the covariance matrix C either match the size of the subsystems or they vanish. The conclusion of this gedankenexperiment is that the largest eigenvalues of C can be understood as the effective dimension of the strongly correlated subsystems and the corresponding eigenvectors point out which time series are involved in this subsystem. In the case of our artificial example (2.4) this means that we always find five strongly correlated subsystems of dimension 10 which barely interact with each other. This conclusion can be found regardless what basis of the time series is chosen. Hence the matrix elements of C can look quite inconclusive as in the right plot of Fig. 1 while the spectrum of C still exhibits the correlations.

The question how statistical fluctuations affect the spectrum of covariance matrices was first studied by Wishart [18] with the help of RMT in the 20s when he considered biological systems. He introduced a random matrix model consisting of a $p \times n$ rectangular real matrix W which is distributed via the Gaussian

$$P(W) = \frac{\exp[-\operatorname{tr} WW^T]}{\pi^{np/2}} = \prod_{j=1}^{p}\prod_{l=1}^{n} \frac{\exp[-W_{jl}^2]}{\sqrt{\pi}}. \tag{2.5}$$

This model is also known as the real Laguerre ensemble [19] or the chiral Gaussian orthogonal ensemble (χGOE) [20]. Forty years later Marčenko and Pastur [21] computed the macroscopic level density of $WW^\dagger$ in the limit of large matrix dimensions $n, p \to \infty$ with $n/p = \text{const}$. The result of their calculation is

$$\rho_{\mathrm{MP}}(\lambda) = \frac{2\sqrt{(\lambda_{\max} - \lambda)(\lambda - \lambda_{\min})}}{\pi(\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}})^2\lambda}\Theta[(\lambda_{\max} - \lambda)(\lambda - \lambda_{min})], \tag{2.6}$$

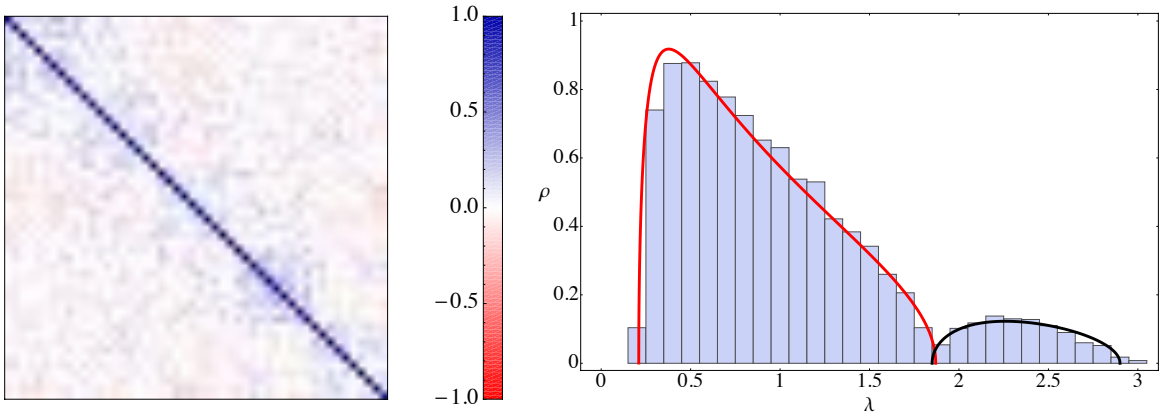

Figure 2: Visualization of the structure of the covariance matrix C'' by color encoding its matrix elements (left plot) and its level density (histogram in the right plot). The covariance matrix C'' corresponds to the times series V, see Eq. (2.4), which are perturbed by independent, identical Gaussian fluctuations δv_{ij} of each matrix entry v_{ij}. The strong correlations are almost washed out. The former almost zero eigenvalues are broadened to a spectrum which is on the same scale as the outliers which are still manifested in a separate spectrum. Both parts of the spectrum can be independently approximated by the Marčenko-Pastur distribution (2.6) (solid curves in the right plot).

which is a distribution on the finite interval $[\lambda_{\min}, \lambda_{\max}]$ reflected by the Heaviside step function Θ. Since Marčenko's and Pastur's derivation Wishart random matrices serve as benchmark models for almost any covariance matrices, e.g. see [13, 14, 16, 22, 23]. In particular the application of the distribution $\rho_{\mathrm{MP}}(\lambda)$ is great success in time series analysis.

Let us consider our example again to understand why RMT is helping in time series analysis. We add to the time series (2.4) white noise with a variance which is of the order of the data. Then the strong correlations in the five subsystems are extremely weakened, see left plot of Fig. 2. However the reminiscent correlations still survive. Despite the fact that almost all eigenvalues which were formerly almost zero are now of the order $\mathcal{O}(1)$ the five outliers are still visible, see right plot of Fig. 2. They are the deviations of the level spacing distribution of the empirical data from the Marčenko-Pastur distribution (2.6). The gross of the spectrum is indeed approximately described by $\rho_{\mathrm{MP}}(\lambda)$ but the remaining part is not. Since the five outliers were almost the same they separately follow an own Marčenko-Pastur distribution. This is rarely the case in a realistic situation. Usually the outliers are so far apart such that they cannot affect each other, e.g. see [13, 14, 22, 23].

This simple example shows the power of RMT as a tool to separate the system specific correlations and the generic influence of statistical fluctuations on the data. Indeed in realistic situations this procedure is not always that simple. Some of the eigenvalues which belong to the system specific correlations may touch or even enter the gross of eigenvalues which represent the fluctuations, cf. the right plot of Fig. 2. This may influence and deform the Marčenko Pastur distribution. To understand and model this situation the Wishart random matrix model is modified to [5]

$$P(W) = \frac{\exp[-\mathrm{tr}\, WW^T C^{-1}]}{\pi^{np/2} \det{}^{n/2} C} \tag{2.7}$$

Figure 3: Unfolded spectra of various systems comprising each 100 levels. From left to right: quantum harmonic oscillator; randomly and independently chosen levels; harmonic oscillator with small white noise fluctuations; resonance levels of erbium 166 with same spin and parity excited by neutrons; 100 eigenvalues around the origin of a 300×300 real symmetric Gaussian random matrix; positions of the $1023, \ldots, 1122$ zero of the Riemann zeta function $\zeta(z) = \sum_{j=1}^{\infty} j^{-z}$ along the line $\mathrm{Re}\, z = 1/2$; the $103\,613, \ldots, 103\,712$ prime numbers; locations of the 100 northernmost overpasses and underpasses along Interstate 85; positions of cross ties near a railroad siding; growth rings from 1884 through 1983 in a fir tree on Mount Saint Helens (Washington); temporal occurrence of earthquakes in California with a Richter magnitude larger or equal to 5.0 (from 1969 to 2001); lengths of 100 consecutive bike rides. Arrows assign two or more levels which are very close to each other. Taken from [24].

with C the empirical covariance matrix. This model is briefly reviewed in subsection 3.2 together with some results on the distribution of the smallest and largest eigenvalue. The latter two distributions are also sensitive to the situation if an outlier touches the generic part of the spectrum given by ρ_{MP}. Both distributions are given for an uncorrelated Wishart matrix by the Tracy-Widom distribution [25] when in the double scaling limit the ratio n/p is fixed. This behavior drastically changes when the difference $n - p$ is fixed instead of the ratio. This situation becomes important in QCD [2, 20]. We discuss these critical points in more detail in subsection 2.3.

2.2 Local Spectral Statistics in various Systems

In the previous subsection we have seen that the spectra of covariance matrices crucially encode the correlations between their corresponding time series. This is not the only way how spectral statistics appear to be important. Indeed spectra surround us any time and everywhere even when we do not recognize them as such. In Fig. 3 we enlist such examples which range from physical systems and models to number theoretical

observations and everyday phenomena.

When looking at the spectra shown in Fig. 3 we may ask what do they have in common. Some may share the same law of their distributions others obviously do not. For example the levels of the quantum harmonic oscillator are equidistant which are adequately shared by railroad cross ties while the resonance spectrum of erbium and the zeros of the Riemann zeta function are alike since their levels slightly repel each other, cf. Fig. 3.

We have two choices to judge whether two spectra are alike or not. In the first choice, we can consider the macroscopic spectrum as it is done with the Marčenko-Pastur distribution (2.6) which is compared to real data to separate the statistical fluctuations and the system specific correlations. However on the macroscopic scale the level spacing may vary over many scales. Then we cannot accurately compare two different spectra with their local spectral statistics. For example the distance of two consecutive tree rings is about 1mm while the distance of two neighboring resonance levels of Erbium is about 0.1keV. How do we compare these two different units of observables completely differently measured? Even the comparison of the statistics at two different positions of a single spectrum is not possible on a macroscopic scale. For example the spectrum of a d-dimensional QCD Dirac operator or a quantum harmonic oscillator exhibits an eigenvalue density $\rho(\lambda)$ which behaves as λ^{d-1} while it is $\rho(\lambda) \propto \lambda^{d/2-1}$ for a free particle in a d-dimensional box. Then the spacing between two consecutive level has certainly to change, too. Thus we come to the second choice that we compare spectra on the local scale.

The problem discussed above already appeared in the beginning of the early 50s when experimentalists started to catalog nuclear resonance spectra of heavy nuclei. They solved the problem in the following way. First we have to take an arbitrary spectrum of some measurements. Before analyzing this spectrum we have to split it into sub-spectra which correspond to exact discrete symmetries of the system. For the nuclei these sub-spectra are characterized by their spin and their parity which are both conserved. We have to split the full spectrum since the sub-spectra can be identified with subsystems which do not interact with each other and have thus their own independent spectrum.

Choosing one of the sub-spectra, say $\{\lambda_1 \leq \ldots \leq \lambda_N\}$, we have to zoom onto the scale of the local mean level spacing in the next step. For this purpose we define the counting function

$$N(E) = \sum_{j=1}^{N} \Theta(E - \lambda_j) = \int_{-\infty}^{E} d\lambda \sum_{j=1}^{N} \delta(\lambda - \lambda_j). \tag{2.8}$$

This counting function has to be approximated by a monotonously increasing, differentiable function $N_{\text{smooth}}(E)$. Then the unfolded level density,

$$\rho_{\text{unfolded}}(\lambda) = \sum_{j=1}^{N} \delta(\lambda - N_{\text{smooth}}(\lambda_j)), \tag{2.9}$$

is in average constant implying that the unfolded level spacing distribution,

$$p_{\text{unfolded}}(s) = \frac{1}{N-1} \sum_{j=1}^{N-1} \delta(s - N_{\text{smooth}}(\lambda_{j+1}) + N_{\text{smooth}}(\lambda_j)), \tag{2.10}$$

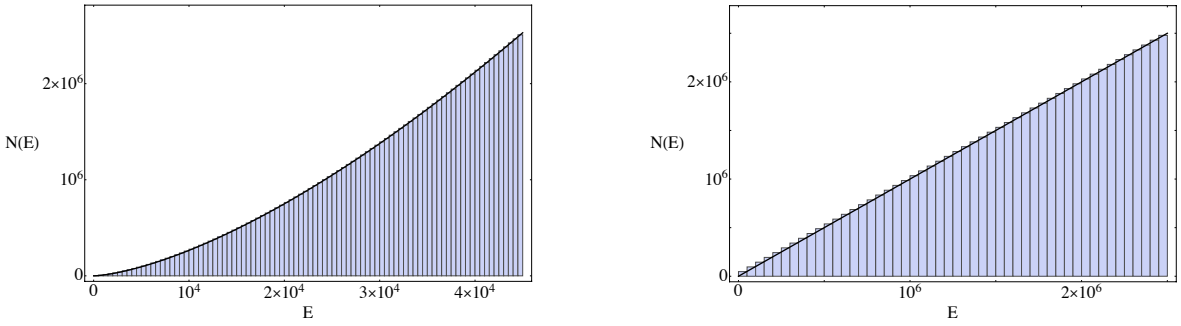

Figure 4: Counting function $N(E)$ of a free quantum particle without spin in a box of size $[-1,1]\times[-\sqrt{\pi},\sqrt{\pi}]\times[-\pi,\pi]$ and Dirichlet boundary conditions. The system is invariant under each of the reflections $x\to -x$, $y\to -y$, and $z\to -z$ such that its energy spectrum splits into $2^3=8$ sub-spectra. We have chosen the subspace of wave functions $\psi(x,y,z)\propto \sin(\pi n_x x)\sin(\sqrt{\pi}n_y y)\sin(n_z z)$ with the energies $E=(\pi^2 n_x^2+\pi n_y^2+n_z^2)/2$, $n_x,n_y,n_z=1,2,\ldots$ The left plot shows the counting function (histogram) before unfolding up to the eigenvalue $E=45000$. The black solid curve is a fit for the smooth part of the counting function which is $N_{\rm smooth}(E)=0.265E^{3/2}$. The right plot is the counting function after unfolding with respect to the described procedure. The unfolded counting function is linear by construction.

is not only normalized but also its first moment is unity, i.e. $\int dssp_{\rm unfolded}(s)=1$. Apart from the level spacing distribution $p_{\rm unfolded}(s)$ one can also consider other correlation functions. The most prominent one is the k-point correlation function [19, 26, 27],

$$R^{(k)}_{\rm unfolded}(x_1,\ldots,x_k)=\frac{(N-k)!}{N!}\sum_{j_1\neq j_2\neq\ldots\neq j_k=1}^{N}\prod_{l=1}^{k}\delta(x_l-N_{\rm smooth}(\lambda_{j_l})), \tag{2.11}$$

which is the density of the probability that we find one eigenvalue in each of the intervals $[x_j,x_j+dx_j]$ for $j=1,\ldots,k$. The case $k=1$ yields the normalized level density $R^{(1)}_{\rm unfolded}(\lambda)=\rho_{\rm unfolded}(\lambda)/N$. The normalization of R_k is chosen such that $\int dx_k R^{(k)}_{\rm unfolded}(x_1,\ldots,x_k)=R^{(k-1)}_{\rm unfolded}(x_1,\ldots,x_{k-1})$.

The procedure described above is called unfolding [1]. It is demonstrated with the help of the spectrum of a free particle in a three dimensional box in Fig. 4. This example is a very simple system compared to heavy nuclei. The unfolding of most spectra is quite non-trivial in practise. For example spectra may statistically behave completely different at two different positions in the spectra, e.g. see QCD [28]. The spectrum can also comprise more than only the macroscopic scale and the scale of the local mean level spacing. For example in [29] the authors found three different statistics due to a separation of four scales in their spectra (the forth is the macroscopic scale).

The spectral statistics of simple systems like a quantum particle in a box were studied since the 70s. The reason of such a development in studying spectra was the empirical observation that the level spacing distribution of heavy nuclei follow the distributions calculated by very simple random matrices. Already a 2×2 real symmetric random matrix H whose entries are distributed by $\exp[-\mathrm{tr}\,H^2]$ yields a level spacing distribution $p_{\rm GOE}(s)=\pi s e^{-\pi s^2/4}/2$ which accurately agrees with many spectra, see Fig. 5. This distribution is known as the Wigner surmise [30]. The random matrix

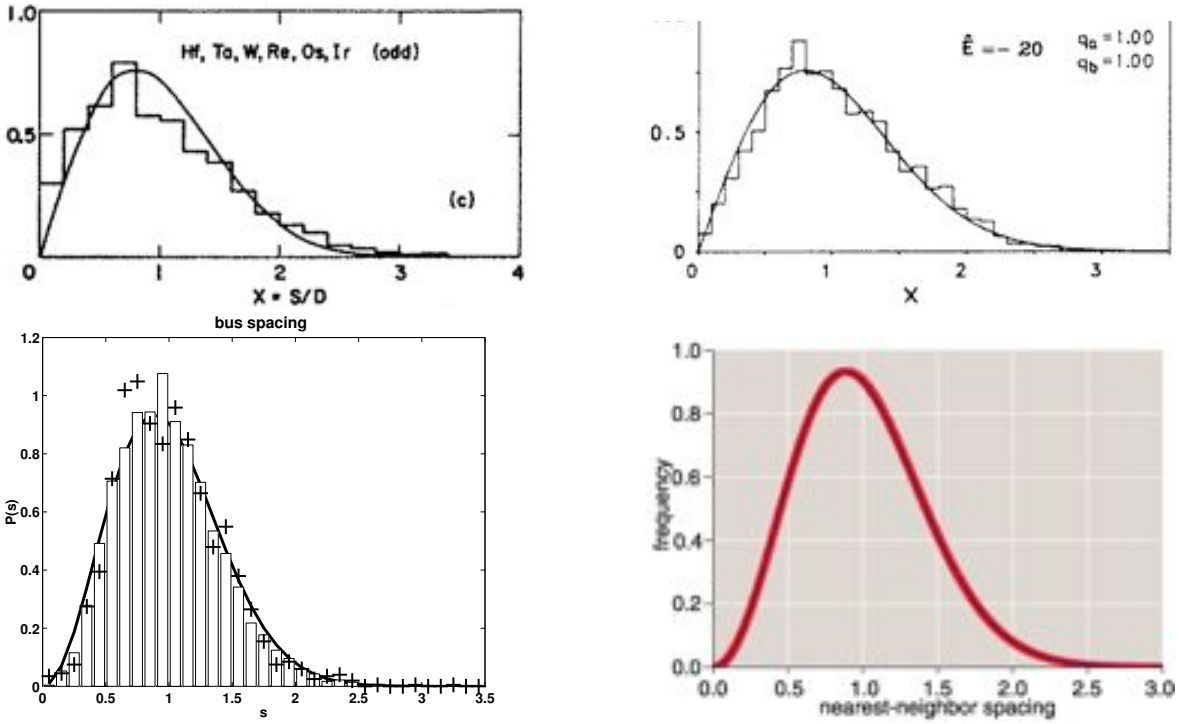

Figure 5: Level spacing distributions of various systems. **Upper left plot:** Level spacing distribution of resonance spectra of heavy nuclei (histogram) compared with $p_{\mathrm{GOE}}(s)$ of GOE (solid curve). Taken from [31]. **Upper right plot:** The level spacing distribution extracted from the spectrum of a hydrogen atom in a strong magnetic field (histogram) compared to $p_{\mathrm{GOE}}(s)$. Taken from [32]. **Lower left plot:** Distribution of temporal intervals between buses of the city line No. 4 in Cuernavaca (Mexico) (crosses) compared to the Wigner surmise $p_{\mathrm{GUE}}(s) = 32s^2e^{-4s^2/\pi}/\pi^2$ of GUE. The histogram corresponds to the Wigner surmise where 0.8% of the generated random matrices are neglected. Taken from [33]. **Lower right plot:** Level spacing distribution of a billion zeros of the Riemann zeta function along the line $\mathrm{Re}\, z = 1/2$ starting from the 10^{23}rd zero (red dots). The blue curve is $p_{\mathrm{GUE}}(s)$. Taken from [24].

ensemble where H is drawn from is the Gaussian orthogonal ensemble (GOE) [34]. When the random matrix H is only Hermitian but not real the ensemble is called Gaussian unitary ensemble (GUE).

First it was thought that the agreement between RMT and spectra is due to the size of the system because heavy nuclei are many body systems and thus many individual properties are washed out. However in the early 70's it was discovered that the non-trivial zeros of the Riemann zeta function $\zeta(z) = \sum_{j=1}^{\infty} j^{-z}$ along the line $\mathrm{Re}\, z = 1/2$ satisfies the same spectral statistics [35] as a GUE, cf. lower right plot of Fig. 5. Since this accidental observation it was clear that the statistics of spectra for completely different systems can agree although the systems seem to have nothing in common.

Let us get some idea why at least physical operators like the Hamilton operator H or the Dirac operator D exhibit generic spectral properties. For this purpose we consider integrable systems of the type of a one-dimensional quantum harmonic oscillator which can be very briefly proven. The energies $E_n = f(n)$ of such a system are given by a strictly increasing, differentiable function f which depends on a single quantum number

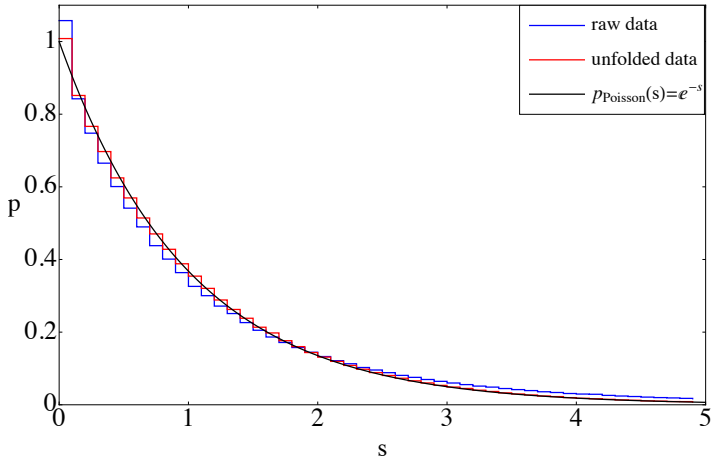

Figure 6: The level spacing distribution of the raw (blue histogram) and the unfolded (red histogram) data of the quantum system in Fig. 4. Both distributions are compared with $p_{\text{Poisson}}(s)$ (black curve). We considered the eigenvalues for $10^4 < N(E) < 2.5 \times 10^6$. The raw data is only rescaled by a global prefactor such that the mean level spacing is unity. Though the deviations between the raw data and the Poisson distribution are only small it is nonetheless clearly visible that unfolding of the data increases the agreement with the theory. The reason for the small size of the deviations of the raw data is the scaling behaviour $N_{\text{unfolded}}(E) \propto E^{3/2}$ which is near to the linear behavior.

$n = 1, 2, 3 \ldots$. The d-dimensional quantum harmonic oscillator with the same frequency in each direction is of such a kind as well as the non-relativistic hydrogen atom and any local one-dimensional system (including the one-dimensional quantum particle in a box while its higher dimensional counterpart is not). Then the smooth counting function is given by $N(E) = f^{-1}(E)$ such that the unfolded spectrum is indeed equidistant, i.e. $E_n^{(\text{unfolded})} = f^{-1}(E_n) = n$. In particular the level spacing distribution is given by a Dirac delta function $p_{\text{harm}}(s) = \delta(s)$ which is also the result of a one-dimensional quantum harmonic oscillator.

Another class of spectral statistics corresponding to quantum systems is covered by the Berry-Tabor conjecture [36]. This conjecture claims that generic integrable systems exhibit the Poisson statistics, in particular the level spacing distribution is $p_{\text{Poisson}}(s) = e^{-s}$. Indeed not all integrable systems can follow this conjecture since some integrable systems may share the statistics of a harmonic oscillator, see above. Rigorous proofs of this conjecture were performed for some classes of integrable systems, see references in the review [37]. For example a free particle in a $(d > 1)$-dimensional box follows the Berry-tabor conjecture. In Fig. 6 we compare the raw spectrum of a free particle in a three dimensional box without unfolding and the Poisson distribution and find slight deviations as expected. These deviations become even stronger when the smooth counting function N_{smooth} differs much stronger from a linear behavior as it is the case for a free particle in a higher dimensional box. Yet, the unfolded spectrum perfectly agrees with $p_{\text{Poisson}}(s)$ as predicted by the Berry-Tabor conjecture.

To get an idea what the mechanism of the Poisson statistics is let us consider N randomly chosen levels $x_1, \ldots, x_N$ which are independently drawn from the probability densities $p_1(x_1), \ldots, p_N(x_N)$. The number of levels N will be eventually sent to infinity.

The level spacing distribution $p(s)$ can be calculated via the gap probability,

$$P(a,b) = \prod_j \Theta[(E'_j - b)(E'_j - a)] = \lim_{N\to\infty} \prod_{j=1}^{N} \left(\int_{-\infty}^{\infty} - \int_a^b \right) dx_j R^{(N)}(x_1, \ldots, x_N), \quad (2.12)$$

which is the probability that all energies lie outside the interval $[a, b]$. We assume that the limit $\lim_{N\to\infty} \sum_{j=1}^{N} p_j(x)/N = \bar{p}(x)$ exists. Then the distribution $p(s)$ is explicitly given by

$$p(s) = \frac{\partial^2}{\partial s^2} \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} ds' P\left(s' - \frac{s}{2N\bar{p}(s')}, s' + \frac{s}{2N\bar{p}(s')} \right) \quad (2.13)$$

after unfolding $b - a \to (b-a)/\bar{p}[(b+a)/2N] = s$. Since the gap probability $P(a,b) = \lim_{N\to\infty} \prod_{j=1}^{N} P_j(a,b)$ of the full spectrum is the product of the gap probabilities $P_j(a,b)$ of each single eigenvalue we have

$$\begin{aligned} \lim_{N\to\infty} P\left(s' - \frac{s}{2N\bar{p}(s')}, s' + \frac{s}{2N\bar{p}(s')} \right) &= \lim_{N\to\infty} \prod_{j=1}^{N} \left[1 - \int_{-s}^{s} \frac{dx_j}{2N\bar{p}(s')} p_j\left(\frac{x_j}{2N\bar{p}(s')} + s' \right) \right] \\ &= \exp\left[-s \right], \end{aligned} \quad (2.14)$$

which is indeed the Poisson distribution. Note that the Poisson statistics relies on two facts. First, we have infinitely many levels and, second, the levels are independently distributed. In a real system we do not have these requirements since the levels are fixed such that we have to replace the infinite number of levels by an infinite number of subsequences (each having infinitely many levels itself) of the spectra which are pairwise uncorrelated. Thereby we notice that the splitting of spectra into sub-spectra due to a finite symmetry group is highly important. Those finite symmetry groups like the spin or the parity usually yield correlated series. This is also the reason why we have chosen incommensurate lengths in our example in Fig. 4.

There are not only integrable quantum systems but also chaotic systems. Bohigas, Giannoni and Schmidt conjectured [38]: *"Spectra of time-reversal-invariant systems whose classical analogs are K systems show the same fluctuation properties as predicted by GOE."* Here "K systems" are the kind of classical chaotic systems which are the strongest mixing. In [39, 40, 41, 42] arguments are given why this conjecture should be true. The time reversal invariance in the conjecture is important otherwise we would find other statistics like GUE or something which interpolates between GUE and GOE, see [34]. The time reversal symmetry is a global symmetry of a system as it is also parity or chirality. Such global symmetries which are related to RMT were classified by Dyson [34, 43], Verbaarschot [20], and Altland and Zirnbauer [44], see also subsection 2.4. They were found in many other systems, too, and served as a basis of classifying these systems as well, for example see [45, 46, F].

2.3 Critical Points in Spectra

Let us come back to the example of a quantum particle in a three dimensional box. When extracting the unfolded level spacing distribution, see Fig. 6, we had to omit the first 10^4 eigenvalues to get a perfect agreement with the Poisson distribution. Why did we do this? When considering the smallest eigenvalues it becomes immediate that there

is a minimal distance between consecutive energies such that the density is less than the smooth fit $\rho(E) \approx 0.398\sqrt{E}$. Such a minimal distance also exists for larger energies. However it is much smaller and becomes even smaller for increasing eigenvalues. Hence its influence is negligible in this regime. Indeed we could also sum over the first 10^4 eigenvalues in the spectral average (2.10), too. The influence would be a slight dip around the origin of the level spacing distribution. This dip eventually vanishes when taken more and more eigenvalues into account. But for smaller data sets it becomes crucial to exclude the smallest eigenvalues.

Excluding levels at the edges of the spectrum is a common procedure. For example also the non-trivial zeros of the Riemann zeta function, see lower right plot of Fig. 5, near the real axis are usually excluded in the level statistics. The reason of their exclusion is again their different statistical behavior. Hence one can ask two questions. First, is this mechanism generic? And second, are the spectral statistics at these edges also as universal as the bulk statistics?

To get an idea what statistics may usually appear we consider the particular example of a two-dimensional QCD Dirac operator on a 10×10 lattice with three colors and the fermions in the fundamental representation. Since we want to keep the example as simple as possible, the Dirac operator shall be given in the naive discretization and the gauge action is set to zero such that we are in the strong coupling limit, see [E] for the details of constructing this operator. Moreover we rotate the spectrum of D by the imaginary unit $\imath$ since $D = -D^\dagger$ is anti-Hermitian. We show the structure of this Dirac operator in the left plot of Fig. 7 for a generic gauge field configuration. Although the structure of the Dirac operator looks quite simple the system is nonetheless strongly mixing as can be seen in the inverse participation ratios of its eigenvectors, see also the right plot of Fig. 7. It shows that the eigenvectors are delocalized which usually results in a generic non-Poissonian spectrum. The corresponding averaged level density of its positive eigenvalues is shown in the upper left plot of Fig. 8. Note that any positive eigenvalue λ of D has a negative counterpart $-\lambda$ which is also an eigenvalue of D due to chiral symmetry.

The averaged level density exhibits three scaling regimes where the spectral statistics are completely different. In the bulk of the spectrum (plot (II) of Fig. 8) we obtain the level statistics of the Wigner surmise for GUE [30]. It is also known as the sine kernel [34, 19]. The unfolded level density is constant while its fluctuations are given by oscillations, in particular the two point correlation function behaves as [34, 19]

$$R^{(2)}_{\text{sine}}(x_1, x_2) = 1 - \left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)} \right)^2 . \tag{2.15}$$

The behavior $R^{(2)}_{\text{sine}}(x_1, x_2) \propto (x_1 - x_2)^2$ for small distances $|x_1 - x_2| \ll 1$ reflects the level repulsion which is quadratic for the corresponding Wigner surmise $p_{\text{GUE}}(s)$.

Note that the bulk statistics (2.15) is translation invariant, i.e. $x_1 \to x_1 + a$ and $x_2 \to x_2 + a$ for all $a \in \mathbb{R}$. This property is found in all bulk statistics including the spectral statics of real and quaternion matrices and the statistics of complex eigenvalues for non-Hermitian matrices, see [47].

The translation invariance is broken at the upper edge (plot (III) of Fig. 8) where the largest eigenvalue has only on one side a neighbouring level. Thus the largest eigenvalues can develop exponential tails outside the support of the macroscopic level

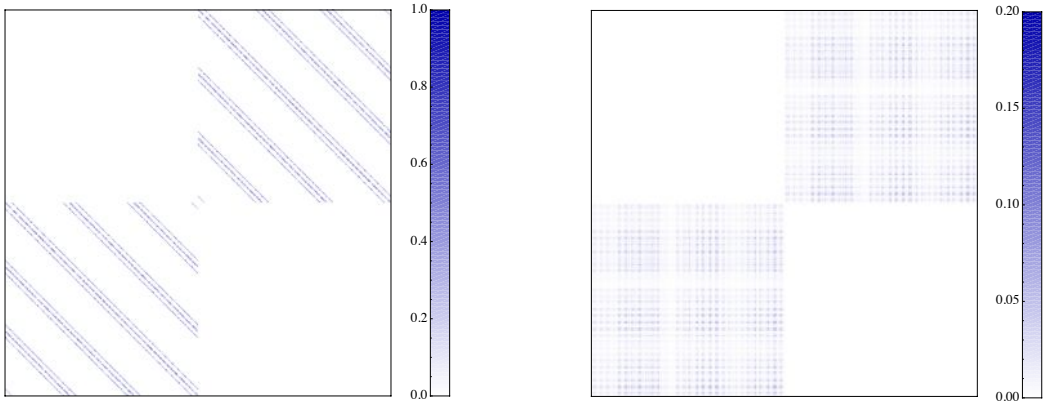

Figure 7: A realization of the naive Dirac operator D for a 10×10 two-dimensional lattice which simulates QCD with three colors (gauge group is SU(3)) and the fermions in the fundamental representation of the gauge group (left plot). The absolute values of the matrix entries are color encoded. The Dirac operator has a chiral structure which yields the splitting into two blocks. Moreover this kind of Dirac operator is doubly degenerated, see the classification in [F]. Despite the simple lattice structure, see left plot, the system is still strongly mixing which is reflected in the contribution of the largest singular value $\lambda_{\max} > 0$ to D (right plot), which is the product of the singular value $\lambda_{\max}$ (the largest eigenvalue of $\sqrt{DD^\dagger}$) and the projector $P_{\max}$ to its four dimensional eigenspace. Note the different scaling of the color code. The inverse participation ratio of an arbitrary eigenvector v of D is $\mathrm{IPR}(v) = \sum_j |v_j|^4 \approx 0.003 - 0.017$. Since the inverse participation ratio is approximately the inverse of the effective size in the Hilbert space, the eigenvectors occupy approximately $10 - 54\%$ of the lattice and are thus delocalized.

density. This tail lies on a different scale than the bulk statistics. In the case of our example the largest eigenvalues are described by the Airy kernel [25]. The two point-correlation is

$$R^{(2)}_{\mathrm{Airy}}(x_1, x_2) = R^{(1)}_{\mathrm{Airy}}(x_1) R^{(1)}_{\mathrm{Airy}}(x_2) - \frac{[\mathrm{Ai}(x_1)\mathrm{Ai}'(x_2) - \mathrm{Ai}'(x_1)\mathrm{Ai}(x_2)]^2}{(x_1 - x_2)^2} \tag{2.16}$$

and the microscopic level density is

$$R^{(1)}_{\mathrm{Airy}}(x) = [\mathrm{Ai}'(x)]^2 - x[\mathrm{Ai}(x)]^2. \tag{2.17}$$

The function $\mathrm{Ai}(x)$ is the Airy function and is the Fourier transform of the exponential $e^{\imath x^3/3}$, see [48]. This scaling regime is known as the soft edge because of the freedom of the eigenvalues to invade the region outside the support of the macroscopic level density though it is exponentially suppressed.

Also at the lower bound of the spectrum (plot (I) of Fig. 8) we have a particular scaling which differs from the bulk. One could think that this scaling is of the same kind as the one found in the upper bound. This is however not the case as long as the smallest eigenvalues are close to the origin and the macroscopic level density has no spectral gap. Then the positive eigenvalues feel their negative counterpart. Therefore their is no exponential tail into the negative real axis but a repulsion such that the

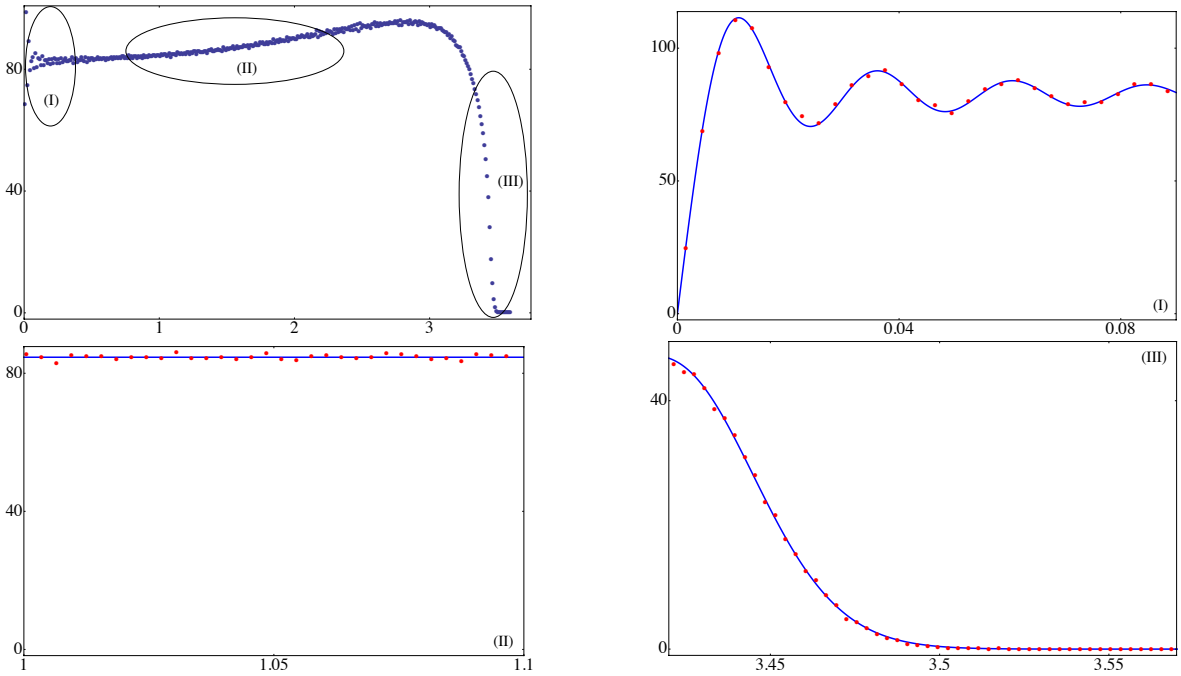

Figure 8: The spectrum of the two-dimensional lattice Dirac operator shown in Fig. 7. The axes are given in arbitrary units. We generated 10^5 gauge field configurations via Monte Carlo simulation (symbols) and compared those with fits of RMT predictions (solid curve). The level density (upper left plot) comprises three different scaling regimes. The eigenvalues close to the origin are in the hard edge scaling regime (I) and are described by the Bessel kernel, i.e. microscopic level density is $R^{(1)}_{\text{Bessel}}(x) = |x|[J_0^2(x) + J_1^2(x)]$. In the bulk of the spectrum (II) the unfolded level density is in average a constant and the two-point correlation function oscillates about this constant yielding the sine kernel. The distribution of the largest eigenvalues (III) follows the Airy kernel, i.e. the microscopic level density is $R^{(1)}_{\text{Airy}}(x) = [\text{Ai}'(x)]^2 - x[\text{Ai}(x)]^2$. The latter scaling limit is known as the soft edge and is due to the finite size of the lattice which is not physical, i.e. continuum QCD has no upper bound of the spectrum.

origin effectively acts as a negatively charged hard wall on electrons (the eigenvalues of D). The corresponding spectral statistics are given by the Bessel kernel [49],

$$R^{(2)}_{\text{Bessel}}(x_1, x_2) = R^{(1)}_{\text{Bessel}}(x_1) R^{(1)}_{\text{Bessel}}(x_2) - \frac{1}{4} \frac{[x_1 J_0'(x_1) J_0(x_2) - x_2 J_0(x_1) J_0'(x_2)]^2}{(x_1^2 - x_2^2)^2} \quad (2.18)$$

for the two-point correlation function and

$$R^{(1)}_{\text{Bessel}}(x) = \frac{|x|}{2} \left(J_0^2(x) + J_1^2(x) \right) \quad (2.19)$$

for the microscopic level density with $J_\nu(x)$ the Bessel function of the first kind [48]. The strong oscillations of the Bessel function are the reason for the strong fluctuations of the macroscopic level density near the origin, cf. Fig. 8. Someone who does not know anything about the local spectral statistics could think that these fluctuations are due to statistical or systematic errors. But the statistical errors are about 1% and

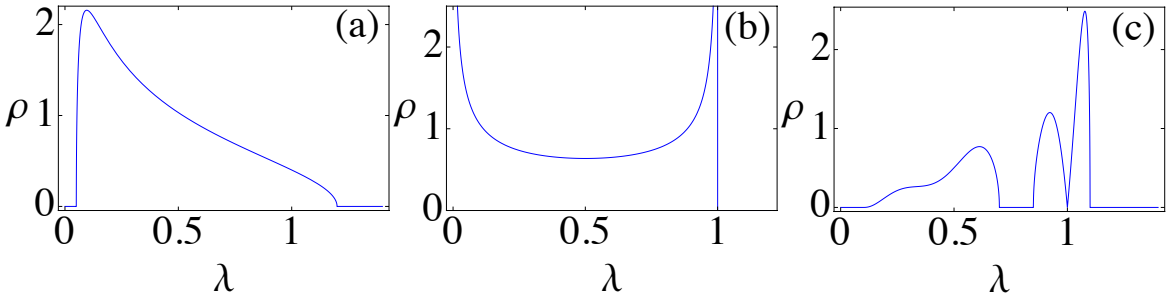

Figure 9: Various examples of level densities with different critical points. **(a)** Marčenko-Pastur distribution 2.6 with two soft edges at $\lambda = 0.05$ and $\lambda = 1.2$. **(b)** The distribution $\rho(\lambda) \propto 1/\sqrt{1-\lambda^2}$ which naturally appears for the real eigenvalues of the Wilson Dirac operator [E] and for signal transmissions where the size of the environment is approximately the same size as the one of the receiving and emitting channels. This distribution has two hard edges at $\lambda = 0$ and $\lambda = 1$. **(c)** A generic density with three disjoint supports. Two supports merge at $\lambda = 1$ yielding a critical point in the interior of the spectrum. The edges at $\lambda = 0.1, 0.7, 0.85, 1.1$ are soft edges. The soft edge at $\lambda = 0.1$ is a multi-critical point where the density behaves as $\rho(\lambda) \propto (\lambda - 0.1)^{5/2}$ instead of a square root.

in the continuum limit the fluctuations still remain though its scale may change, see the discussion of [F].

The three spectral statistics found in our example are not the only ones which exist. First of all, the global symmetries of the operator may change from one system to another such that the level repulsion can be different [34, 44]. But even when keeping the symmetries the spectrum may develop non-trivial structures. The macroscopic level density may exhibit two soft edges as it is the case for most of the covariance matrices which are described by the Marcenko-Pastur distribution (2.6), see plot (a) of Fig. 9. A very large difference between the number of time series and the number of time steps is the reason for the spectral gap between the origin and the smallest eigenvalue. Instead of two soft edges we may have also two hard edges, see plot (b) of Fig. 9. Such a behavior was found for the real eigenvalues of the Wilson Dirac operators for large lattice spacings in [E]. But we expect that it can be also found when transmitting a signal through a chaotic and open cavity. The number of channels to the environment has to be comparable to the number of channels in the transmitter and the receiver.

When considering a general spectrum we may see more than one connected spectrum also referred as cuts [50, 51], see plot (c) of Fig. 9 which is an example. Indeed this can be the case in condensed matter systems where the spectrum may exhibit a gap between different bands. Then we may have more than two soft edges. It can also happen that two cuts are starting to merge and touch each other at one point. One particular kernel corresponding to such a scenario is known as Pearcey kernel [50] which depends on the Fourier transform of the exponential $e^{-x^4/4}$ named after Pearcey [52].

Additionally the soft edges may develop multi-critical behaviors, e.g. see [53, 54], meaning that the macroscopic level density does not vanish with a square root but with another critical exponent. For example for one-matrix models the multi-critical exponent can be any positive half-integer [51, 54]. Quite recently also new kernels for the hard edges were derived for product matrices [55, 56, 57, 58, 59, O, T] and

other ensembles [60, R] which we review in subsection 3.3. The corresponding kernel is known as Meijer-G kernel, see the review [61], because it essentially depends on Meijer-G functions [48]. This kernel is a generalization of the Bessel kernel (2.18). Like multi-critical points, the level density at hard edges belonging to Meijer-G kernels have a different algebraic power than the square root singularity of the Marčenko Pastur distribution. However, the density has a pole at these points instead of a zero, see [55, L, M].

2.4 Symmetries, Universalities and Non-linear σ-Models

The universality of the local spectral statistics is based on two crucial ingredients. First, the scales of different spectral statistics have to separate, meaning the macroscopic level statistics does not know anything of the local spectral statistics. If there are mesoscopic scales as found in [29] also those scales have to split from the other scales. If this separation does not happen the universality of spectral statistics may be destroyed.

The second ingredient concerns the global symmetries which classify the systems in certain universality classes. Those global symmetries are usually a product of discrete and continuous symmetry groups. A particular kind of the discrete symmetries is based on anti-unitary operators T satisfying $T^4 = \mathbb{1}$. They can be always written as a unitary operator times the complex conjugation, i.e. $T = UK$ with U unitary and K the complex conjugation operator. If the considered system does not have such an anti-unitary symmetry the representation of the Hilbert space is generically complex. However if the system satisfies a symmetry under an operator T then the system can be expressed either in a real basis for $T^2 = +\mathbb{1}$ or in a quaternion basis for $T^2 = -\mathbb{1}$. Dyson discovered that the agreement of RMT with physical systems relies on exact global symmetries like anti-unitary symmetries as time reversion for particle systems with integer spins or half-integer spins [34]. This classification became famous under the name Dyson's three-fold way.

Verbaarschot extended the classification by Dyson to chiral systems [20] where he had in mind RMT applications to QCD. Only two years later, Altland and Zirnbauer found all in all ten symmetry classes to cover normal-superconducting hybrid systems in condensed matter theory [44]. All these symmetry classes correspond to particular universality classes exhibiting unique spectral statistical properties. The classification of random matrix models did not only help in classifying chaotic quantum systems [34, 38, 1], but also in classifying topological insulators [46] and the classification of two-dimensional lattice QCD with naive fermions [62, F], see for the latter application also subsection 3.1.

All those classifications only consider Hermitian operators. Thereby the classification by Altland and Zirnbauer is complete in the sense that there is no other symmetry class than these ten classes which correspond to Hermitian random matrix ensembles. We underline that this does not imply that all Hermitian ensembles of the same symmetry class have always the same kernels and thus the same universality. In particular the statistics at the critical points may change from one ensemble to another, e.g. see [53, 51, 54] for one matrix models with different multi-criticality at the edges but the same global symmetries. Yet, the global symmetries exclude certain universality classes. For example a generic Hermitian operator without any symmetries can never show a linear or quartic level repulsion.

Non-Hermitian operators and their corresponding complex eigenvalues and real positive singular values have also many applications. For example let us consider the real matrix W drawn from the Wishart random matrix ensemble (2.5) and setting $n = p$. Then W is a square matrix. The statistics of its singular values (eigenvalues of $WW^\dagger$) are considered in QCD and in time series analysis. Applications of the statistics of its complex eigenvalues which are given by the roots of $\det(W - z\mathbb{1}_n)$ can be found in quantum chaos [63], neural networks [64], and financial markets [65]. When studying the eigenvalues of W the ensemble distributed by Eq. (2.5) is usually called real Ginibre ensemble. The eigenvalue statistics of complex square matrices, the complex Ginibre ensemble, was applied in dissipative quantum maps [66, 67], superconductors [68, 63], and scattering matrices of open chaotic systems [69]. Ginibre was the first who tried to carry over Dyson's three-fold way [34] to non-Hermitian ensembles [47] already in 1965 shortly after Dyson's works. Much later in the beginning of the 2000s the chiral version of Ginibre's ensembles were applied to QCD with non-zero chemical potential [70, 71, 72, 73]. These kinds of random matrices helped to study the famous sign problem which is the problem that complex or even indefinite real statistical weights cannot be interpreted as probabilistic weights such that Monte Carlo simulations fail.

Bernard and LeClair [74] were the first who made an attempt to extend Ginibre's classification of non-Hermitian ensembles to a classification which can be considered complete similar to the classification by Altland and Zirnbauer for the Hermitian ensembles [44]. However they missed a few ensembles and Magnea completed the picture of such a classification [75]. This classification is based on the following four symmetries an operator H can fulfill,

$$H = \pm U_\chi H U_\chi^{-1},\ H = \pm U_T H^T U_T,\ H = \pm U_\dagger H^\dagger U_\dagger^{-1},\ H = \pm U_* H^* U_*^{-1} \tag{2.20}$$

with the four fixed unitary operators

$$U_\chi U_\chi^\dagger = U_T U_T^\dagger = U_\dagger U_\dagger^\dagger = U_* U_*^\dagger = \mathbb{1}. \tag{2.21}$$

All four symmetries can exist either with $+$ or $-$ or they are not present at all. The symmetries (2.20) can be easily interpreted as a kind of chirality ("$-$") or a factorization into sub-systems ("$+$"), of (anti-)symmetry under transposition, of (anti-)Hermiticity, and of a reality condition, respectively. For the classification by Altland and Zirnbauer [44] we have $H = H^\dagger$.

Not only discrete symmetries are important for the classification of the local spectral statistics. Also the continuous symmetries play a crucial role. In QCD of the standard model this continuous symmetry is $\mathrm{SU}(N_\mathrm{f}) \times \mathrm{SU}(N_\mathrm{f}) \times \mathrm{U}(1)$ where N_f is the number of the flavors of quarks (in the standard model: $N_\mathrm{f} = 6$). As pointed out by Zirnbauer [76] the good agreement of RMT with many physical systems, especially with quantum field theories, relies on the same symmetry breaking pattern under spontaneous symmetry breaking of these continuous groups. This was already found by Verbaarschot for four-dimensional QCD [20] and three-dimensional QCD [45].

Spontaneous breaking of global symmetries can occur in various ways. For example continuous symmetries can be spontaneously broken when taking the thermodynamical limit $V \to \infty$ in three and more dimensions as long as the long ranged fluctuations are sufficiently suppressed. Another example is the 't Hooft limit where the number of colors N_c in a QCD-like theory goes to infinity [77]. We underline that the latter

limit is very close to random matrix theory. A common property of these limits are a large parameter, e.g. the volume V or the number of colors N_c, and a sufficiently mixing system (mixing in the Hilbert space of wave functions) such that long ranged fluctuations can be neglected. The large parameter in RMT is the matrix size.

The Goldstone theorem [78, 79] tells us that when a continuous symmetry, say a d-dimensional Lie group G, is spontaneously broken to a d'-dimensional subgroup H then the system exhibits $d - d'$ Goldstone bosons which are massless excitations of the ground state. In particular the Goldstone boson fields U are elements of the local coset $(G/H)_{\text{loc}}$ where local means that one considers a fiber bundle over the space time with the coset G/H as a fiber. The fields are distributed by its unique Haar measure $d\mu(U)$ times the exponential of an effective action. In particular the partition function reads

$$Z = \int_{(\mathrm{G/H})_{\text{loc}}} d\mu(U) \exp[-S_{\text{eff}}(U)], \tag{2.22}$$

which is called non-linear σ-model [80, 81]. Unfortunately, a direct derivation of the effective action $S_{\text{eff}}(U)$ starting from the full quantum field theory is only possible for very few systems like random matrix models. Usually one has to expand the action in its global symmetries, i.e. $S_{\text{eff}}(U) = S_0(U) + S_1(U) + S_2(U) + \ldots$

In QCD this approach is called chiral perturbation theory, see [82]. The spontaneous symmetry breaking pattern is $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1) \to \mathrm{SU}(N_f) \times \mathrm{U}(1)$ such that the Goldstone manifold is $\mathrm{SU}(N_f)$. In the p-regime the leading order term of the effective action is [83, 84]

$$S_0(U) = \int_V d^4x \left(\frac{F_\pi^2}{4} \mathrm{tr}\, \partial_\mu U \partial^\mu U^\dagger - \frac{\Sigma}{2} \mathrm{tr}\, M(U + U^{-1}) \right) \tag{2.23}$$

with $M = \mathrm{diag}\,(m_1, \ldots, m_{N_f})$ the quark masses. The low energy constants F_π and Σ are the pion decay constant and the chiral condensate, respectively. The goal is to fix these constants. Then one can derive and predict non-perturbative effects of QCD at small energies. The task is to circumvent the problem of directly calculating the low energy constants from the full theory. This can be achieved when another, much simpler theory is found yielding exactly the same effective action where observables can be calculated in a closed form. Then the comparison with lattice QCD yields the low energy constants, e.g. see [85, 86, 87, 88] for such comparisons. In realistic situations one has more than the two low energies F_π and Σ and thus the whole programme is more involved. For example non-zero chemical potentials are introduced or the lattice artefacts shall be understood, see also subsection 3.1.

The solution of this problem can be found by using the fact that the space-time volume V should be large. Then one can expand the Goldstone fields $U = U_0 \exp[\imath \Pi / F_\pi]$ around the saddlepoint U_0. In the p-regime the quark masses M, the momenta p_k, energies E and the amplitudes $|\Pi|$ of the Goldstone fields are of order

$$|p_k|^4 \propto E^4 \propto |\Pi|^4 \propto m^2 \propto 1/V. \tag{2.24}$$

Then the effective action reads

$$\begin{aligned} S_0(U) \;\approx\; & \sum_{p,E} \mathrm{tr} \left[\frac{V(p^2 - E^2)}{4} \mathbb{1}_{N_f} + \frac{V\Sigma}{4F_\pi^2} (MU_0 + U_0^\dagger M) \right] \Pi(p,E) \Pi^\dagger(p,E) \\ & - \imath \frac{V\Sigma}{2F_\pi} \mathrm{tr}\, \Pi(p=0, E=0)(MU_0 + U_0^{-1} M) - \frac{V\Sigma}{2} \mathrm{tr}\, M(U_0 + U_0^{-1}) \end{aligned} \tag{2.25}$$

in the first three orders. Note that the meson fields Π are in the momentum space not Hermitian but fulfill the condition $\Pi^\dagger(p,E) = \Pi(-p,-E)$. The second to last and the last term are of order $\mathcal{O}(V^{1/2})$ and $\mathcal{O}(V)$, respectively, while the sum is of order $\mathcal{O}(1)$.

From the expression (2.25) we can read off three facts. First, the masses of the lightest pseudo-scalar mesons M_π is given by the celebrated Gell-Mann-Oakes-Renner relation [89]

$$M_\pi^2 \propto \frac{\Sigma|M|}{F_\pi^2}. \tag{2.26}$$

Second, the saddle point is only determined by the modes with vanishing momentum. Hence we only need to find an effective theory which reproduces the last term $S_0(U_0)$ which is a zero-dimensional effective theory. This leads us to the third point, namely χRMT solves this problem. To see this we have to slightly prepare the partition function. We recall that we originally integrate over $\mathrm{SU}(N_\mathrm{f})$. However any integral over $\mathrm{SU}(N_\mathrm{f})$ can be traced back by the relation,

$$\int_{\mathrm{SU}(N_\mathrm{f})} d\mu(U_0) \exp\left[-S_0(U_0)\right] = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} \int_{\mathrm{U}(N_\mathrm{f})} d\mu(U) \exp\left[-S_0(U_0)\right] \det{}^\nu U_0. \tag{2.27}$$

The summation index ν can be identified with the topological charge which is the difference of the right handed and left handed zero modes and is gauge independent. The integral over $\mathrm{U}(N_\mathrm{f})$ is the integral which is shared with RMT. This observation was first made by Verbaarschot [2, 20].

The infra-red limit of QCD performed above is nothing else than an unfolding at the hard edge of the spectrum of the Dirac operator D. This hard edge is at the origin. The corresponding k-point correlation function is given by [71]

$$R_{\mathrm{Bessel}}^{(k)}(x) \propto \Delta_k^2(x^2) \int_{\mathrm{U}(2k)} d\mu(U) \exp\left[-\imath \frac{V\Sigma}{2} \mathrm{tr}\left(\mathrm{diag}\left(x_1,\ldots,x_k\right) \otimes \mathbb{1}_2\right)(U + U^{-1})\right] \det{}^\nu U \tag{2.28}$$

and agrees with the Bessel kernel, i.e. Eqs. (2.18) and (2.19) for $\nu = 0$ and $k = 2$ and $k = 1$, respectively. We employed the Vandermonde determinant $\Delta_k(x^2) = \prod_{1\le a<b\le k}(x_a^2 - x_b^2)$.

One can also ask if other kernels have a representation in terms of non-linear σ-models. Indeed such an effective theory was found for the sine kernel, too, i.e.

$$R_{\mathrm{sine}}^{(k)}(x) \propto \Delta_k^2(x) \int_{\mathrm{U}(2k)/\mathrm{U}^2(k)} d\mu(U) \exp\left[-\imath \mathrm{tr}\left(\mathrm{diag}\left(x_1,\ldots,x_k\right) \otimes \mathbb{1}_2\right) U \mathrm{diag}\left(\mathbb{1}_k, -\mathbb{1}_k\right) U^{-1}\right]. \tag{2.29}$$

For example three dimensional QCD with three colors and the fermions in the fundamental representation share this effective theory [45] since they have the symmetry breaking pattern $\mathrm{U}(2k) \to \mathrm{U}(k) \times \mathrm{U}(k)$. Also for other kernels such non-linear σ-models can be found, for example the one for the Meijer G-kernel is briefly reviewed in subsection 3.3. Even the Airy kernel satisfies a non-linear σ-model,

$$R_{\mathrm{Airy}}^{(k)}(x) \propto \Delta_k^2(x) \int_{\mathrm{u}(2k)} d\mu(U) \exp\left[\mathrm{tr}\left(\mathrm{diag}\left(x_1,\ldots,x_k\right) \otimes \mathbb{1}_2\right) U + \frac{1}{3}\mathrm{tr}\, U^3\right]. \tag{2.30}$$

Here, the set $\mathrm{u}(2k)$ is the Lie algebra of the unitary group $\mathrm{U}(2k)$ such that U is anti-Hermitian. The induced group action on the Lie algebra from the group $\mathrm{U}(2k)$ is the

addition. Hence the Haar measure $d\mu(U)$ is flat in this case meaning it is the product of all independent differentials of U. But we have to be careful to interpret U as Goldstone bosons since they are the massive particles for the Airy kernel which do not factorize with the saddlepoint manifold anymore. Note, that we have no spontaneous breaking of symmetry at the soft edge of the spectrum, in particular the pattern is $\mathrm{U}(2k) \to \mathrm{U}(2k)$ implying that the Goldstone manifold has to be a point set.

Two questions regarding the relationship of non-linear σ-models and local spectral statistics are still open. First, do all kernels correspond to non-linear σ-models? We would expect such a one-to-one correspondence on the level of the effective potentials (zero-dimensional theories). This is confirmed for many examples, e.g. see [44, 76, P, S] and subsections 3.1 and 3.3. The co-sets can be generated in one way or another. Furthermore, the potentials can be arbitrarily "engineered" with RMT weights. We underline that the question whether the random matrix model can be analytically computed in the end is a completely different issue, e.g. see [90] for the quite complicated random matrix model for the staggered Dirac operator in lattice QCD.

The second point is about the interpretation of the non-linear σ-models. For example the sine-kernel can be easily interpreted with Goldstone modes if we consider the Dirac spectra of three dimensional QCD at the origin [45]. However the sine kernel also applies to the bulk of the Dirac spectrum of four-dimensional QCD where it is not as easy to understand the spectral statistics from this point of view. What are the Goldstone bosons corresponding to this part of the spectrum? It becomes even more complicated when considering non-quantum field systems like the spectrum of covariance matrices or the non-trivial zeros of the Riemann zeta function. Then we have to be careful with such an interpretation and understand non-linear σ-models as another representation of the spectral statistics. Not more and not less. Especially we suggest to use this representation for proving universality of random matrix ensembles where the spectral statistics at finite matrix dimension are not that easy to compute, e.g. the singular value statistics of a product of real or quaternion matrices, see subsection 3.3.

3 Applications and Generalizations of χRMT

Here we summarize the ideas and results of the works collected in the appendices. In subsection 3.1 we discuss some random matrix models and their relations to lattice QCD, see the works in appendix A. In particular we consider the lattice realizations of naive and Wilson fermions. Subsection 3.2 deals with the works in appendix B which are about the uncorrelated and the correlated real Wishart random matrix ensemble, see Eqs. (2.5) and (2.7). In particular we briefly explain the idea of the supersymmetry method which was applied in this topic. The supersymmetry method was also applied to the Meijer G-ensembles discussed in subsection 3.3 and studied in the works collected in appendix C. Those Meijer G-ensembles are products of a certain class of random matrices.

3.1 RMT Models in Lattice QCD

Some generalizations of χRMT also apply to lattice QCD at finite lattice spacing a. Thereby we recall that many discretizations of QCD exist. The most prominent discretizations are with staggered fermions, Wilson fermions, overlap fermions, and domain wall fermions, see [91, 92] for some introductions. Each of these fermions try to cure the doubler problem which exists for the naive Dirac operator. But what is the doubler problem? To answer this question we consider the naive Dirac operator. In this operator the covariant derivatives $D_\mu = \partial_\mu + \imath A_\mu$ are simply replaced by the differences of the transfer matrices and their adjoint, $(T_\mu - T_\mu^\dagger)/a$ with gauge group elements $U \in \mathrm{SU}(N_\mathrm{c})$ on links connecting neighbouring sites and zero else. Recall that such a transfer matrix is unitary, $T_\mu^{-1} = T_\mu^\dagger$, as long as the number of sites is in each direction at least three.

The doubler problem can be briefly explained when considering the situations where all gauge group elements are equal to the identity matrix $\mathbb{1}$. Then the system is integrable and the Dirac operator can be diagonalized by plane waves. Let $\vec{k} = \{k_\mu\}$ be a wave vector. Then the Dirac operator reads in terms of this wave

$$D_\mathrm{naive}|\vec{k}\rangle = \sum_{\mu=1}^{d} \frac{2\imath\gamma_\mu \sin ak_\mu}{a}|\vec{k}\rangle. \tag{3.1}$$

We recall that the first Brillouin zone is the set $\vec{k} \in [-\pi/a, \pi/a]^d$ with d the dimension of the lattice. Therefore we find for a fixed $\sin ak_\mu$ two wave vectors, namely k_μ and $\pi/a - k_\mu$, with the same eigenvalue for each direction. This implies $2^4 = 16$ eigenmodes in four dimensions although only one of these eigenmodes is physical namely the one in the vicinity of the origin. All other modes are called doublers. They are not physical and have to be excluded. However the selection of the physical modes is even more difficult when switching on the gauge group elements since they may mix with the physical ones.

In [F] we studied the global symmetries of the naive Dirac operator for two-dimensional QCD-like theories, including QCD with two and more colors, $N_\mathrm{c} \geq 2$, and the fermions in the fundamental and in the adjoint represention of the gauge group $\mathrm{SU}(N_\mathrm{c})$. Thereby we varied the size $L_x \times L_y$ of the two dimensional lattice to classify the exact global symmetries of the naive Dirac operator. We underline that

additionally to the doubler problem one can also change the global symmetries of the Dirac operator only by discretizing the lattice as described above only by switching L_x and/or L_y from odd to even. Say L_x is even, then the corresponding transfer matrix T_x anti-commutes with the constant matrix Γ_x which assigns to an odd lattice site a plus sign and to an even lattice site a minus sign. This symmetry carries over to the Dirac operator which now does not only anti-commute with σ_3 but also with $T_x\sigma_1$. Recall that the Dirac matrices are given by the Pauli matrices in two dimensions. The understanding of the mechanism of this change of symmetries was the goal in [F].

As briefly discussed in section 2 any additional symmetry may influence the local spectrum which is indeed the case for the classification of the naive Dirac operator. We found eight of the ten symmetry classes enlisted by Altland and Zirnbauer [44] by varying the number of colors, the gauge group representations of the fermions, and the lattice size, see table II of [F]. This shows how severe a disretization affects the theory since almost always the discrete symmetries did not agree with the continuum symmetry. Only the choice L_x and L_y even for the fundamental representation of $\mathrm{SU}(N_\mathrm{c} \geq 3)$ as well as the choice L_x and L_y odd with $\mathrm{SU}(N_\mathrm{c} \geq 2)$ fundamental or adjoint have the correct discrete symmetries. These symmetries are particularly important in the infra-red limit of QCD. Then the physics is governed by the lightest pseudo-scalar mesons. With the wrong discrete symmetries we do not only obtain the wrong number of particles in the continuum limit via the doubler problem but also the Goldstone manifold is completely different. This effect was also found for staggered fermions in three- [93] and four-dimensional [85] QCD-like theories, cf. Fig. 10. The spectrum of the naive Dirac operator and the staggered Dirac operator is up to a trivial degeneracy exactly the same in the quenched (no dynamical quarks) theory. Fortunately naive and staggered fermions in four-dimensional QCD of the standard model do not exhibit this problem [94]. The reason is the same as in two dimensions for QCD with the gauge group $\mathrm{SU}(N_\mathrm{c} \geq 3)$ and the fermions in the fundamental representation. Then the microscopic spectra of the Dirac operator for lattices where both sizes, L_x and L_y, are either even or odd agree up to a degeneracy of two.

Another kind of solving the doubler problem is via Wilson fermions. The corresponding spectrum of the Wilson Dirac operator at the hard edge is discussed in a series of works [A, B, C, D, E] attached in appendix A. Wilson's idea of curing the problem was by adding the Laplacian $\Delta = \sum_\mu D_\mu^2 \to 4\sum_\mu (T_\mu + T_\mu^\dagger - 2\mathbb{1})/a^2$ to the Dirac operator [95]. Then the counterpart of Eq. (3.1) reads

$$D_\mathrm{W}|\vec{k}\rangle = (D_\mathrm{naive} - a\Delta)|\vec{k}\rangle = \sum_{\mu=1}^{d} \left(\frac{2\imath\gamma_\mu \sin ak_\mu}{a} + \frac{16\sin^2(ak_\mu/2)}{a} \right) |\vec{k}\rangle. \tag{3.2}$$

Thus the Wilson term makes the doublers at the boundary of the first Brillouin zone infinitely heavy in the continuum limit $a \to 0$ while the Laplacian for the modes at the origin have a vanishing contribution. Therefore the doublers become too inertial and decouple from the rest of the system.

The advantage of the Wilson Dirac operator D_W is that it does not change the Goldstone manifold such that we have from the start the correct lightest pseudo-scalar mesons. Therefore the physics is preserved from this point of view. Yet, there is a price to pay. The chiral symmetry is explicitly broken by the Laplacian. This breaking is weak because it is of the order $\mathcal{O}(a)$ and is thus similar to the study of weak non-Hermiticity introduced by Fyodorov, Khoruzhenko, and Sommers [96]. Nevertheless the

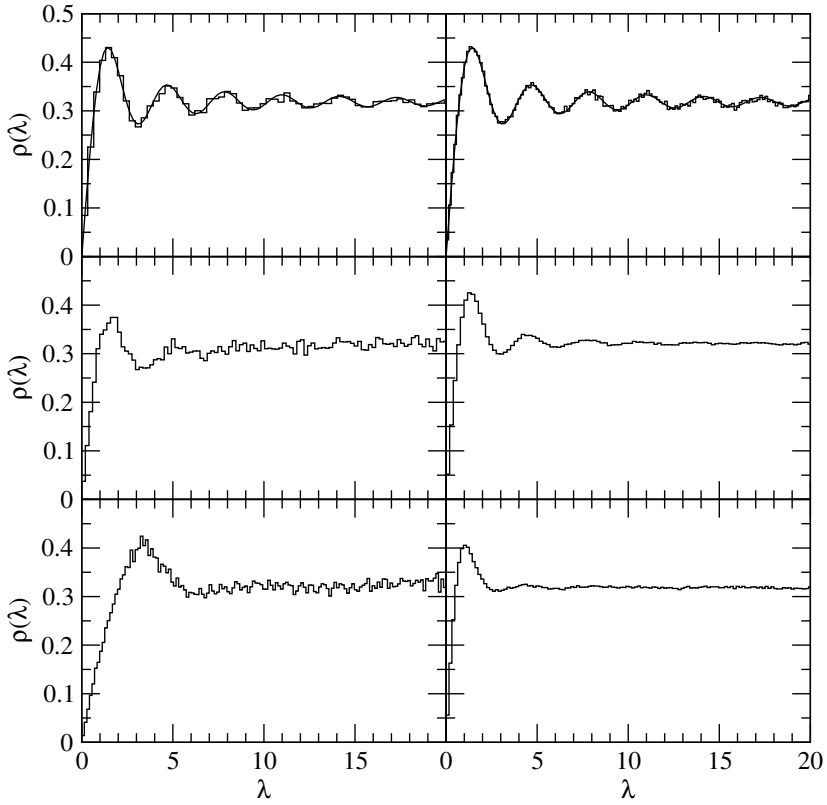

Figure 10: Local spectrum at the hard edge of the quenched staggered Dirac operator for an SU(3) gauge theory and the fermions in the fundamental representation (left column) for different lattice spacing and of a random matrix model interpolating between χGUE and GUE (right column) for different coupling parameter. The change of the spectrum is due to a slight breaking of chiral symmetry. Note that in the continuum theory three color QCD in three dimensions has a flat spectrum around the origin [93] (lowest plot). However the naive and the staggered Dirac operator exhibit a spectrum of QCD in four dimensions in the strong coupling limit (upper plot). Taken from [93].

physics is still strongly effected. The spectrum of the Dirac operator becomes complex which was formerly imaginary. Moreover the phase diagram exhibits new artificial phases. In the case were the squared lattice spacing is of the order of the quark mass, $a^2 \propto m$, we may have two situations in the thermodynamical limit $V \to \infty$. Either the mass dependent chiral condensate,

$$\Sigma(m) \propto \mathrm{Re}\frac{1}{V}\left\langle \mathrm{tr}\,\frac{1}{D_{\mathrm{W}} - m\mathbb{1}}\right\rangle \tag{3.3}$$

with $\langle . \rangle$ the average over all configurations, jumps from a positive to a negative value when the quark mass m crosses the origin implying a first order phase transition. Then the system is in the Sharpe-Singleton scenario where the pions are massive even in the chiral limit [97]. It may also happen that the chiral condensate (3.3) changes its constant behavior to a linear slope at a second order phase transition. Then all pions become massless at the transition point. The massive chiral condensate $\Sigma(m)$ does not jump anymore since the origin is in the Aoki phase [98] where the flavour symmetry and the parity are spontaneously broken.

All these lattice artefacts of the Wilson Dirac operator are non-physical and have to be understood. Interestingly the smallest eigenvalues of the Dirac operator encode the major part of these effects. Therefore understanding the smallest eigenvalues helps in measuring the quality of lattice simulations, especially to judge how far away continuum QCD is.

In the works [A, B, C, D, E] we used the idea sketched in subsection 2.4 and started with the potential of Wilson chiral perturbation theory, e.g. see [97, 99, 100],

$$\begin{aligned} S_0(U) &= -\frac{V\Sigma}{2}\mathrm{tr}\, M(U+U^{-1}) \\ &\quad + VW_6a^2\mathrm{tr}^{\,2}(U+U^{-1}) + VW_7a^2\mathrm{tr}^{\,2}(U-U^{-1}) + VW_8a^2\mathrm{tr}\,(U^2+U^{-2}). \end{aligned} \tag{3.4}$$

The corrections of a finite lattice spacing a enter via the last three terms at leading order. Each term comes with a low energy constant, W_6, W_7, and W_8. These low energy constants fulfill some conditions. For example the γ_5-Hermiticity of D_{W}, i.e. $D_{\mathrm{W}}^\dagger = \gamma_5 D_{\mathrm{W}} \gamma_5$, allows to consider the Hermitian Wilson Dirac operator $D_5 = \gamma_5 D_{\mathrm{W}}$. Then we can conclude from the positivity of the partition function of D_5,

$$Z_{N_{\mathrm{f}}}^{N_{\mathrm{f}}} = \left\langle \det{}^{N_{\mathrm{f}}}(D_5 - m\gamma_5 - \lambda\mathbb{1}) \right\rangle > 0, \text{ for } N_{\mathrm{f}} \text{ even}, \tag{3.5}$$

and from the integrability of the partially quenched partition function

$$Z_{N_{\mathrm{f}}}^{N_{\mathrm{f}}+1|1} = \left\langle \det{}^{N_{\mathrm{f}}}(D_5 - m\gamma_5 - \lambda\mathbb{1}) \frac{\det(D_5 - m_2\gamma_5 - \lambda_2\mathbb{1})}{\det(D_5 - m_1\gamma_5 - \lambda_1\mathbb{1})} \right\rangle, \tag{3.6}$$

that the inequalities $W_8 - W_6 - W_7 > 0$ and $W_8 > 0$ hold, see [4, 101, 102, C]. In [C] we also argued that $W_6, W_7 < 0$. The reason for the latter inequalities is the interpretation of the low energy constants W_6 and W_7 as collective motions of the eigenvalues of D_{W} and D_5. Both terms represent the squared traces in the action and can be linearized by Gaussian convolutions in the quark mass m and the axial mass λ. In the case that one of the low energy constants W_6 and W_7 is positive we have a convolution with an imaginary variable, for example an imaginary mass when $W_6 > 0$. This is an average over non-physical configurations which can be excluded because their contribution enters via the Laplacian whose eigenvalues are real.

The random matrix model corresponding to the Wilson Dirac operator is given by

$$D_{\mathrm{W}} = \begin{pmatrix} a(A + [m_6 + \lambda_7]\mathbb{1}) & W \\ -W^\dagger & a(B + [m_6 - \lambda_7]\mathbb{1}) \end{pmatrix}, \tag{3.7}$$

where all matrix entries of the two Hermitian matrices A and B, of the complex rectangular matrix W, and the two scalar variables m_6 and λ_7 are independently distributed by Gaussians, see [3, 4, 103, 104, 105, 106, 107, A, E] for details. In the limit of a large matrix dimension we find at the hard edge around the origin the effective action (3.4). The diagonal blocks model the Laplacian of D_{W} and the off-diagonal blocks already appeared in χRMT for continuum QCD [2, 20]. The Hermitian matrices A and B generate the W_8-term while the scalar random variables m_6 and λ_7 correspond to the low energy constants W_6 and W_7, respectively.

The joint probability density of the real eigenvalues of the Hermitian Wilson Dirac random matrix $D_5 = \gamma_5 D_{\mathrm{W}}$ was solved by Akemann and Nagao [108]. In [E] we derived

the joint probability density of all eigenvalues of $D_{\rm W}$ which is mathematically more involved. Note that the model (3.7) is in the class 6Q of the symmetry classification by Magnea [75]. Hence the non-compact unitary group $\mathrm{U}(n, n+\nu)$ keeps the random matrix $D_{\rm W}$ form invariant and can be used to quasi-diagonalize the matrix but is also the reason for the difficulty in the computation. We emphasize that $D_{\rm W}$ cannot be always diagonalized while keeping the γ_5-Hermiticity intact. An eigenvalue of $D_{\rm W}$ is either real or has a complex conjugate partner. Thus when assuming that $D_{\rm W}$ has l complex conjugate pairs $X^{(2)} \pm \imath Y^{(2)} = \mathrm{diag}\,(x_1^{(2)} \pm \imath y_1^{(2)}, \ldots, x_l^{(2)} \pm \imath y_l^{(2)})$ and $2n+\nu-l$ real eigenvalues $X^{(1)} = \mathrm{diag}\,(x_1^{(1)}, \ldots, x_{n-l}^{(1)})$ and $X^{(3)} = \mathrm{diag}\,(x_1^{(3)}, \ldots, x_{n+\nu-l}^{(3)})$, the Wilson Dirac matrix can brought into the form

$$D_{\rm W} = U \left(\begin{array}{c|cc|c} X^{(1)} & \multicolumn{2}{c|}{0} & 0 \\ \hline & X^{(2)} & Y^{(2)} & \\ 0 & -Y^{(2)} & X^{(2)} & 0 \\ \hline 0 & \multicolumn{2}{c|}{0} & X^{(3)} \end{array} \right) U^\dagger \quad \text{with } U \in \mathrm{U}(n, n+\nu). \tag{3.8}$$

When calculating the joint probability density of the eigenvalues of $D_{\rm W}$ one has to perform the integral over U eventually. This integral is the crucial, non-trivial step since it is an Itzykson-Zuber-Harish-Chandra-like integral of the form

$$\Phi_{l,l'}(X,K) = \int_{\mathcal{M}_{l,l'}} d\mu(U) \exp[\mathrm{tr}\, UXU^\dagger K], \tag{3.9}$$

where $\mathcal{M}_{l,l'}$ is a coset of $\mathrm{U}(n, n+\nu)$ depending on the number of complex pairs of the two matrices X and K. The coset might be non-compact, too, such that one needs a mixture of group theory, saddlepoint analysis, and partial differential equations to compute this integral, see appendix A in [E]. It is a generalization of a derivation by Fyodorov and Strahov [109] of the integral for the case $l = l' = 0$ which is essentially based on the idea of the Duistermaat-Heckman localization theorem [110]. However the situation is not exactly as in the localization theorem where the stationary phase approximation is exact. Only the algebraic structure is exact in the saddle point approximation for the integral (3.8) while the functions in the resulting determinant are not.

Despite the complicated structure of such a quasi-diagonalization it allows us some insight into the eigenvector statistics. In particular we can say that an eigenmode $|\psi^{(1)}\rangle$ corresponding to the eigenvalue $x_j^{(1)}$ is right-handed, i.e. $\langle\psi^{(1)}|\gamma_5|\psi^{(1)}\rangle > 0$, and the real eigenvalues $x_j^{(3)}$ have left-handed eigenmodes (negative chirality). Hence we were able to study different level densities introduced in [A, D, E]. The distributions are the ones of the left- and the right-handed modes over the real eigenvalues, $\rho_{\rm r}$ and $\rho_{\rm l}$, respectively, and of the complex eigenvalues $\rho_{\rm c}$. This completely stands in contrast to the level statistics of D_5 whose level density is given by only a single distribution ρ_5. The two distributions of the real eigenvalues of $D_{\rm W}$ can be combined to two other distributions proposed in [4] which are the level density over all real eigenvalues $\rho_{\rm real} = \rho_{\rm r} + \rho_{\rm l}$ and the distribution of chirality over the real eigenvalues $\rho_\chi = \rho_{\rm r} - \rho_{\rm l}$. In the case of a small lattice spacing ρ_χ is equal to the broadening of the former zero modes of the continuum Dirac operator into the real axis. If we also assume $\nu > 0$ then $\rho_{\rm r}$ is equal to the distribution of the additional real modes which enter the real

axis. In particular its integral yields the average number of additional real modes $N_{\rm add} = 2\int dx \rho_{\rm r}(x)$ which is a good measure of the strength of the lattice artefacts.

The quantities introduced above were calculated in a series of works. In [D] we traced back both level statistics of D_5 and $D_{\rm W}$ to partition functions which are an average over two characteristic polynomials of $D_{\rm W}$ and of D_5, only. Thereby we calculated the orthogonal polynomials corresponding to this ensemble. Interestingly the polynomials are neither completely orthogonal nor skew-orthogonal to each other as it is usually the case, e.g. see [19, 27, 26] and references therein. The polynomials of order $0, \ldots, \nu - 1$ are orthogonal to each other while the polynomials of order $l > \nu$ are skew-orthogonal. This observation was found for the Hermitian as well as for the non-Hermitian Wilson Dirac operator. Such a mixture is quite unique and is expected to be found for other ensembles, as well, if their joint probability densities are either of the form

$$p_5(x) = \Delta_{2n+\nu}(x) \mathrm{Pf} \left[\begin{array}{cc} g_5(x_a, x_b) & p_c(x_a) \\ -p_d(x_b) & 0 \end{array} \right]_{\substack{1 \le a,b \le 2n+\nu \\ 1 \le c,d \le \nu}} \tag{3.10}$$

or of the form

$$p_{\rm W}(z) = \Delta_{2n+\nu}(z^{(\rm r)}, z^{(\rm l)}) \det \left[\begin{array}{cc} g_{\rm W}(z_a^{(\rm l)}, z_b^{(\rm r)}) & p_c(z_a^{(\rm l)}) \end{array} \right]_{\substack{1 \le a \le n+\nu \\ 1 \le b \le n \\ 1 \le c \le \nu}}, \tag{3.11}$$

where $g_5(x_a, x_b)$, $g_{\rm W}(z_a^{(\rm l)}, z_b^{(\rm r)})$, and $p_b(z_a)$ are certain functions. We used the Pfaffian Pf[.] which is essentially the square root of a determinant of an even dimensional, antisymmetric matrix. Indeed the joint probability density functions of D_5 and $D_{\rm W}$ are of the form p_5 and $p_{\rm W}$, see [108, D, E] for the explicit expressions.

The point process corresponding to the joint probability densities p_5 and $p_{\rm W}$ is Pfaffian [D]. This is quite puzzling since the continuum limit $a \to 0$ should yield a determinantal point process [111, 112]. This apparent contradiction was resolved in [B] where we have shown that any determinantal point process can be written as a Pfaffian one whose structure resembles exactly the one of GOE and GSE, see [113, 114] for general derivations of Pfaffian point processes. This was up to this work a missing piece in the understanding that all point processes can be dealt on equal footing. With respect to the problem of the Wilson Dirac operator it has shown how the continuum limit has to be interpreted after the kernels of the Pfaffian point process were calculated in [D].

With the help of the analytical results of the spectrum of $D_{\rm W}$ we could solve a puzzle in [C]. Namely in the quenched theory only the Aoki scenario was observed in lattice simulations [115, 116] while the unquenched theory exhibits the Aoki phase and the Sharpe singleton scenario [117, 118]. The additional determinants are the reason for this observation, cf. Eq. (3.5), when dynamical quarks are present. These determinants strongly repel the smallest eigenvalues from the quark mass and can push the strip of the complex eigenvalues of $D_{\rm W}$ away if $W_8 < -2W_6$, see Fig. 2 in [C]. When the quark mass crosses the origin this strip rips apart and collectively jumps onto the other side of the complex plane which is manifested in a first order phase transition, cf. Fig. 3 in [C].

In [A, E] we presented all exact results for arbitrary lattice spacing and quark mass in the quenched theory and discuss their implications. Let us start summarizing these results with explaining the effect of the low energy constants $W_6 < 0$ and $W_7 < 0$. The

low energy constant W_6 is introduced via a Gaussian convolution in the quark mass. Thus the full spectrum of the continuum Dirac operator is broadened parallel to the real axis into the complex plane with a width proportional to $\sqrt{VW_6}a$, cf. left plot of Fig. 1 in [E]. Thus the whole continuum spectrum can be completely regained by projecting all eigenvalues onto the real axis.

The pure effect of W_7 is more involved. Then the spectrum does not broaden into the complex plane. It pushes the eigenvalues along the imaginary axis to the origin. When an eigenvalue pair reaches the origin it enters the real axis and a new pair of additional real eigenvalues corresponding to one right-handed and to one left-handed mode is created. Thus the spectrum is still real and imaginary but not complex if only W_7 is non-zero. The width of the distribution of the real modes is of the order $\sqrt{VW_7}a$, cf. right plot of Fig. 1 in [E]. Interestingly the distribution ρ_χ is a pure Gaussian despite the completely non-trivial effect on ρ_{real} and ρ_{c}.

The effect of the low energy constant W_8 is completely non-trivial. It is somehow a mixture of the broadening of the spectrum into the complex plane and pushing the complex conjugate pairs into the real axis. It has also a crucial effect on the thermodynamical limit $V \to \infty$. While at $W_8 = 0$ the width of the strip of the complex eigenvalues along the imaginary axis is of order $\mathcal{O}(a)$ and the shape is Gaussian, we have at $W_8 > 0$ a flat strip with a width of the order $\mathcal{O}(a^2)$. The same statement about the width (not the shape) holds for the distributions of the real eigenvalues, ρ_{r} and ρ_{l}, too.

As already pointed out, the average number of the additional real modes N_{add} is a good quantity for measuring the strength of lattice artefacts. In [A, E] we found that its behavior drastically changes when tuning the lattice spacing a. For large lattice spacing (the case which should be avoided) N_{add} is of the order a and thus independent of the index ν. In contrast, at small $a \ll 1$ the number N_{add} is of order $a^{2\nu+1}$. Therefore additional real modes and, hence, lattice artefacts are strongly suppressed for configurations with a non-trivial topological charge. This is a good message for lattice simulations since the massive Wilson Dirac operator, $D_{\text{W}} + m\mathbb{1}$, has to be inverted yielding the matrix Green function. Such an inversion fails if the quark mass is in the cloud of real eigenvalues of D_{W}.

In [E] we also suggested quantities which can be measured in lattice simulations to fix the low energy constants. At small lattice spacing the dependence of these observables on the low energy constants is particularly simple because it is affine linear. For example the average number of the additional real eigenvalues is $N_{\text{add}} \approx 2Va^2(W_8 - 2W_7)$ or the width of the strip of the complex eigenvalues is $\sigma^2/\Delta^2 \approx 4Va^2(W_8 - 2W_6)/\pi^2$ with Δ the mean level spacing at the hard edge of the spectrum. Furthermore, the number of relations derived by us is larger than the constants to be fixed. Hence our results also offer a possibility to estimate the quality of the approximation of chiral perturbation theory by its leading order and measure some bounds of the strength of higher order terms. We underline that the approximation of the suggested observables is accurately fulfilled for $|VW_i a^2| < 0.1$ for all $i = 6, 7, 8$. This is indeed accessible with lattice simulations, see [88].

In the work by Damgaard, Heller, and Splittorff [87] one obviously notices the limit of applicability of chiral perturbation theory in its lowest order. The asymmetry in the spectrum, cf. Fig. 11, is quite persistent though the trend to a symmetric level density of the real eigenvalues in the continuum limit is correct. This asymmetry results from

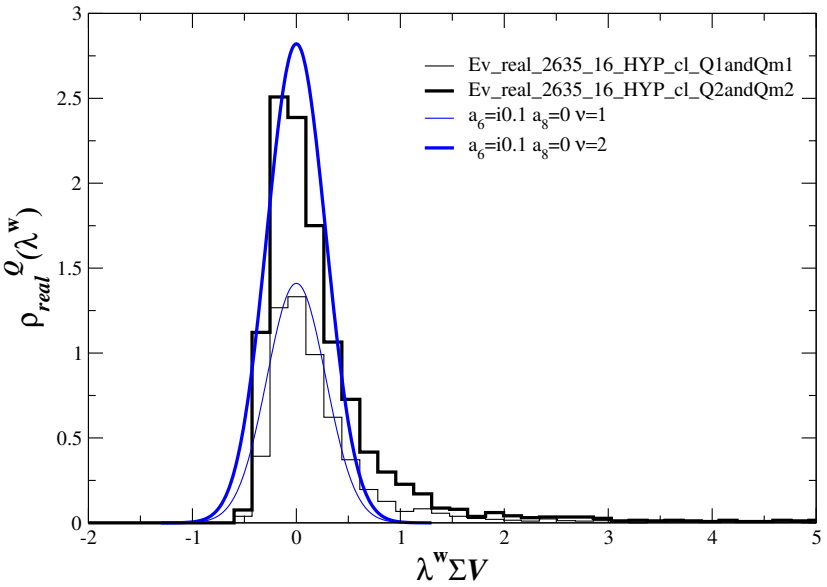

Figure 11: Fit of lattice data with analytical results of the level density ρ_{real} of all real eigenvalues of D_{W} when setting the low energy constants $W_8 = W_7 = 0$. The lattice simulations are for the quenched Wilson Dirac operator with clover improvement. Our critique in [E] of this fit was that additional real modes can only be created if at least one of the two low energy constants W_7 and W_8 does not vanish. Unfortunately the authors of [88] set a priori $W_7 = 0$ such that it might be possible to find a better fit for this data with $W_7, W_8 \neq 0$. The persistent asymmetry results from the positivity of the Wilson term and is not covered by chiral perturbation theory. Taken from [88].

the fact that the Wilson term (minus the Laplacian) is positive definite. In private communications with Splittorff it was mentioned that the modes corresponding to this asymmetry seem to be localized. Hence it is possible that the asymmetry cannot be covered with chiral perturbation theory.

3.2 Uncorrelated and Correlated Wishart Random Matrices

When looking at real time series we notice that they often deviate from the predictions of the Marčenko-Pastur distribution (2.6). For example data of financial markets, see Fig. 12, exhibit a very pronounced tail. There are many ways to interpret such deviations with Wishart RMT, see subsection 2.1. One interpretation is that the deviations correspond to system specific correlations, see our example in Fig. 2. Then we do not only interpret the largest outlier at about $\lambda \approx 58$ as the effective size of a strongly correlated subsystem but also the outliers which are slightly above the largest eigenvalue described by the Marčenko-Pastur distribution (2.6). A second possible interpretation is that the system specific correlations mix with the bulk of eigenvalues distributed by the Marčenko-Pastur distribution. Then the bulk is deformed due to level repulsion and we should fit the data with a level density of the correlated Wishart random matrix model (2.7). The level density as well as the distribution of the largest and the smallest eigenvalue of correlated real Wishart matrices were computed in [119, 120, 121, 122, G, H, J]. There is also a third possibility. Maybe the assumption of a Gaussian distribution was wrong and we have to choose a heavy-tailed distribution

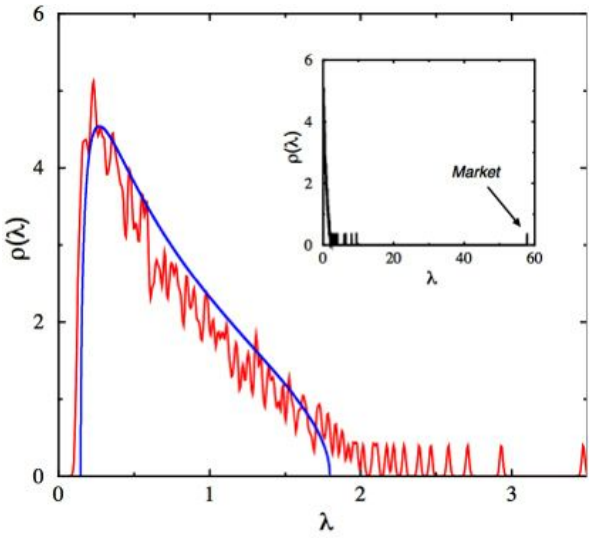

Figure 12: The empirical level density of a covariance matrix constructed from financial market data of 406 stocks from the S&P 500 (red curve) is compared with a fit of the Marčenko-Pastur distribution (2.6) (blue curve). The largest outlier is shown in the inset and is called the market mode. Taken from [14].

like the Cauchy-Lorentz distribution [P],

$$P(W) \propto \frac{1}{\det^{\gamma}(\Gamma^2 \mathbb{1}_p + WW^{\dagger})} \tag{3.12}$$

with γ large enough to guarantee the integrability. The latter possibility was discussed in [123] because financial data indeed exhibit Levy-tails.

Before we come to the results for some observables of the correlated real Wishart random matrix model let us briefly explain the mathematical problem involved in this topic which makes analytical computations so difficult even for RMT. We start with the distribution (2.6) with $\nu = n - p > 0$ meaning that we have less time series than time steps. The index ν also agrees with the one employed in the context of QCD, see subsection 3.1, because it is the generic number of zero modes of the matrix W. To derive the joint probability distribution of the eigenvalues of $WW^{\dagger}$ we have to perform a singular value decomposition of $W = U\Lambda V$ with $U \in \mathrm{O}(p)$, $V \in \mathrm{O}(n)$, and Λ a rectangular $p \times n$ matrix which has on one of its main diagonals positive real numbers and zero everywhere else. Then the distribution of the eigenvalues $S = \Lambda\Lambda^T = \mathrm{diag}\,(s_1, \ldots, s_p)$ is,

$$p(S) \propto \Delta_p^2(S) \det{}^{(\nu-1)/2} S \int_{\mathrm{O}(p)} d\mu(U) \exp[-\mathrm{tr}\, USU^T C^{-1}]. \tag{3.13}$$

The integral over V drops out while the integral over U has to be done. The latter is a group integral over the orthogonal group and is called the real Itzykson-Zuber integral [124]. The complex version (an integral over the unitary group instead over the orthogonal) is equal to a Harish-Chandra integral [125] and thus the Duistermaat-Heckman localization theorem [110] applies. However the real version does not correspond to a Harish-Chandra integral because the matrices S and C^{-1} are not elements

of the Lie algebra of O(p). The applicability of the localization theorem relies on such a property of the matrices.

The Itzykson-Zuber integral in Eq. (3.13) is one of the hardest problems in RMT. Many experts tried to solve this problem but were only partially successful for small dimensions p [126, 127, 128]. In its full generality the Itzykson-Zuber integral is still unsolved. Thus one has to go new ways to circumvent this group integral.

Exactly this was done with the help of the supersymmetry technique, see [129, 130]. We emphasize that in RMT supersymmetry does not appear as a physical theory like in particle physics or quantum gravity. Supersymmetry is purely employed as a mathematical tool to encode complicated differential operators as simple integrals over anti-commuting variables. See [131] for an introduction to superanalysis and superalgebra.

Let us briefly present the main definitions and introduce our notations. A superalgebra consists of commuting variables $x_1, x_2, \ldots$ ($x_i x_j = x_j x_i$) like the real and complex numbers and of anti-commuting variables $\eta_1, \eta_2, \ldots$ ($\eta_i \eta_j = -\eta_j \eta_i$) which are called Grassmann variables. Due to the anti-commutation relations of the Grassmann variables each η_j is nilpotent ($\eta_j^2 = 0$) such that a function f depending on a Grassmann variable is defined by a finite Taylor series ($f(\eta) = f(0) + f'(0)\eta$). An integral over a Grassmann variable is then given by the derivative with respect to this Grassmann variable ($\int f(\eta) d\eta = f'(0)$). One can also introduce a complex conjugation of the Grassmann variables where one has the choice of a conjugation of the first kind ($(\eta^*)^* = \eta$ and $(\eta_1 \eta_2)^* = \eta_2^* \eta_1^*$) and a conjugation of the second kind ($(\eta^*)^* = -\eta$ and $(\eta_1 \eta_2)^* = \eta_1^* \eta_2^*$). It does not matter which one is chosen as long as one sticks to one of the two possibilities during the whole calculation. We employ the conjugation of the second kind in the following.

Now we come to supermatrices which have an important role in the supersymmetry method of RMT. A supermatrix σ can be arranged in such a way that it has the block structure

$$\sigma = \begin{pmatrix} \sigma_{\mathrm{BB}} & \sigma_{BF} \\ \sigma_{\mathrm{FB}} & \sigma_{FF} \end{pmatrix}, \tag{3.14}$$

where σ_{BB} and σ_{FF} only comprise commuting variables which may also include an even product of Grassmann variables because such a product is commuting, too. The matrix elements of the off-diagonal blocks σ_{BF} and σ_{FB} only consist of anti-commuting variables which can be any product of an odd number of Grassmann variables. The adjoint of such a supermatrix σ is then given via the formula

$$\sigma^\dagger = \begin{pmatrix} \sigma_{\mathrm{BB}}^\dagger & \sigma_{FB}^\dagger \\ -\sigma_{\mathrm{BF}}^\dagger & \sigma_{FF}^\dagger \end{pmatrix}. \tag{3.15}$$

Thus the adjoint is up to the switch of the sign in the block $-\sigma_{\mathrm{BF}}^\dagger$ the same as for ordinary matrices. When assuming a supermatrix σ where σ_{BB} and σ_{FF} are square matrices we can define the supertrace and the superdeterminant,

$$\mathrm{Str}\sigma = \mathrm{tr}\,\sigma_{\mathrm{BB}} - \mathrm{tr}\,\sigma_{\mathrm{FF}} \quad \text{and} \quad \mathrm{Sdet}\sigma = \frac{\det(\sigma_{\mathrm{BB}} - \sigma_{\mathrm{BF}}\sigma_{\mathrm{FF}}^{-1}\sigma_{\mathrm{FB}})}{\det \sigma_{\mathrm{FF}}}. \tag{3.16}$$

For the definition of the superdeterminant we have also to assume that σ_{FF} is invertible. The definition of the supertrace and the superdeterminant are chosen in such a

way that the circularity property ($\mathrm{Str}\sigma_1\sigma_2 = \mathrm{Str}\sigma_2\sigma_1$), the factorization ($\mathrm{Sdet}\sigma_1\sigma_2 = \mathrm{Sdet}\,\sigma_1\mathrm{Sdet}\,\sigma_2$), and the relation between both functions ($\ln\mathrm{Sdet}\sigma = \mathrm{Str}\ln\sigma$) are natural generalizations from ordinary matrices.

Let us come back to the supersymmetry method in RMT. The main idea of applying supersymmetry is to rewrite the partition function

$$Z_{k|l}(\kappa,\lambda) = \int_{\Sigma_{\mathrm{ord}}} d[H]P(H)\frac{\prod_{j=1}^{l}\det(H-\lambda_j\mathbb{1}_N)}{\prod_{j=1}^{k}\det(H-\kappa_j\mathbb{1}_N)}, \tag{3.17}$$

which is an average over an ordinary $N\times N$ random matrix $H\in\Sigma_{\mathrm{ord}}$, as an integral over a supermatrix $\sigma\in\Sigma_{\mathrm{SUSY}}$

$$Z_{k|l}(\kappa,\lambda) = \int_{\Sigma_{\mathrm{SUSY}}} d[\sigma]Q(\sigma)\mathrm{Sdet}^{-\gamma}(\sigma-M). \tag{3.18}$$

The matrix M is a fixed supermatrix depending on κ and λ and γ is an exponent depending on N and on the symmetry class of H. The dimensions of σ only depend on the numbers of determinants, k and l, we originally average over. The goal of the map from Eq. (3.17) to Eq. (3.18) is to drastically reduce the number of integrations. In full generality the weight P can be arbitrary. Its counterpart Q in superspace crucially depends on P and on the original set of random matrices Σ_{ord}, see [132, P] as well as subsection 3.3 for an explicit dependence for certain classes of random matrix ensembles. Also the set of supermatrices Σ_{SUSY} depends on both quantities, see [76] for a rough classification of the superspaces.

In the case of the correlated real Wishart random matrix model we have $H = WW^T$ and $\Sigma_{\mathrm{ord}} = \mathbb{R}^{p\times n}$, and the probability weight P is given by Eq. (2.7). Then the level density can be calculated via the relation

$$\rho(x) = \frac{1}{\pi p}\lim_{\varepsilon\to 0}\mathrm{Im}\,\frac{\partial}{\partial J}Z_{1|1}(x+\imath\varepsilon, x-J)\Big|_{J=0}. \tag{3.19}$$

In [G, H] we derived the supersymmetric integral for $Z_{1|1}(x+\imath\varepsilon,x-J)$ which is

$$Z_{1|1}(x+\imath\varepsilon,x-J) = \int d[\rho]\int d[\sigma]e^{\imath\mathrm{Str}\rho\sigma}\mathrm{Sdet}^{-n/2}(\mathbb{1}_{2|2}+\imath\sigma)\prod_{j=1}^{p}\mathrm{Sdet}^{-1/2}\left(M-\frac{\Lambda_j}{2}\rho\right) \tag{3.20}$$

with $M = \mathrm{diag}\,(x+\imath\varepsilon, x+\imath\varepsilon, x-J, x-J)$ and Λ_j the eigenvalues of the empirical covariance matrix C. Both supermatrices ρ and σ are of size $(2|2)\times(2|2)$ and satisfy the symmetries

$$\sigma = \sigma^\dagger = \mathrm{diag}\,(\mathbb{1}_2,\tau_2)\sigma^T\mathrm{diag}\,(\mathbb{1}_2,\tau_2) \quad\text{and}\quad \rho = \rho^\dagger = \mathrm{diag}\,(\mathbb{1}_2,\tau_2)\rho^T\mathrm{diag}\,(\mathbb{1}_2,\tau_2) \tag{3.21}$$

with τ_2 the second Pauli matrix. The integral over σ yields Dirac delta functions for ρ_{FF} and a restriction of ρ_{BB} to positive definite matrices. After integrating over the four Grassmann variables of ρ and diagonalizing ρ_{BB} we end up with an integral over three ordinary variables. Those three integrals can be reduced to a finite sum where each summand is a product of three decoupled integrals when taking the derivative and the imaginary part in Eq. (3.19). See [G, H] for the details of the calculation and

the results. As a comparison we also considered the result where W is complex since the results for this case are well-known [6, 133]. We underline that our derivation has been only possible because of the simplicity of the supersymmetric integral. We had not to perform any complicated group integrals as it would be the case in the ordinary matrix space, cf. Eq. (3.13).

This idea of circumventing the group integral (3.13) can be applied to other observables, as well, like the largest and the smallest eigenvalues. The smallest eigenvalues are of particular interest in QCD since they are responsible for the spontaneous breaking of chiral symmetry and are quite suitable for fitting lattice data [71, 86]. Also when studying topological superconductors they become important [134]. In numerics the condition number $c = |\lambda_{\max}/\lambda_{\min}| > 1$ (the ration of the largest and the smallest eigenvalue) is a measure of stability of the solution x of a linear equation $Ax = y$ with respect to the source y, see [135]. When the condition number is large the solution x is very sensitive with respect to small changes in y while it is very hard to change the solution if $c \approx 1$. The distributions of the largest eigenvalues also find applications in time series analysis. They can be used to reduce the dimensionality of the system via projection onto the principal components of the empirical covariance matrix C corresponding to its largest eigenvalues [136]. This approach is one way to extract the system specific correlations. However it does not always cover all important informations. One can find many more applications of the smallest and largest eigenvalues in other fields, too, see [137, 138, 139] for some examples.

The distribution of the smallest and the largest eigenvalue are related to the gap probability that no eigenvalue of $WW^\dagger$ is inside the interval $[a, b]$,

$$E(a, b) = \langle \Theta[(WW^\dagger - b\mathbb{1}_p)(WW^\dagger - a\mathbb{1}_p)] \rangle, \tag{3.22}$$

where we have employed the Heaviside step function for matrix arguments meaning that it is unity if its argument is positive definite and otherwise zero. The brackets $\langle . \rangle$ are the average with respect to the weight (2.7) of W in the case of correlated real Wishart random matrices. Then the distributions of the smallest and the largest eigenvalue are

$$p_{\min}(s) = -\partial_s E(0, s), \qquad p_{\max}(s) = \partial_s E(s, \infty), \tag{3.23}$$

respectively. In the case of an uncorrelated Wishart random ensemble, i.e. $C = \mathbb{1}_p$, and $p/n > 0$ fixed we find the Tracy-Widom distribution [25] in the large n limit,

$$F_2(s) = \exp\left[-\int_s^\infty dx (x - s)^2 q^2(x) dx \right] \text{ with } q''(s) = sq(s) + q^3(s) \text{ and } q(s) \overset{s\to\infty}{\to} \mathrm{Ai}(s) \tag{3.24}$$

for complex matrices and

$$F_1(s) = \exp\left[-\frac{1}{2} \int_s^\infty dx (x - s)^2 q^2(x) dx - \frac{1}{2} \int_s^\infty dx q(x) dx \right] \tag{3.25}$$

for real matrices with the same solution $q(s)$.

When considering the correlated Wishart random matrix ensemble (2.7) we again encounter the same problem of the Itzykson-Zuber integral (3.13). In the complex case this integral can be easily performed and results for the extreme eigenvalues can be derived [140, 141]. For real matrices the whole situation is different because of the

unkown group integral. Thus we look again for a dual integral where the final matrix decouples from the empirical correlation matrix. This was done for the gap probability $E(s,\infty)$ in [J]. Thereby we interpret the Heaviside step function $\Theta(s\mathbb{1}_p - WW^\dagger) = \Theta(s\mathbb{1}_n - W^\dagger W)$ as the result of an Ingham-Siegel integral [142, 143], e.g.

$$\int_{\mathrm{Sym}(n)} d[H] \frac{\exp[\mathrm{tr}\,(\mathbb{1}_n - \imath H)K]}{\det^\gamma(\mathbb{1}_n - \imath H)} \propto \det{}^{\gamma-(n+1)/2} K\Theta(K) \text{ with } \gamma > \frac{n+1}{2} \tag{3.26}$$

for real matrices. Ingham-Siegel integrals regularly appear in RMT, e.g. see [144, 145, 146, P], and can be understood as a multidimensional residue theorem in the $n \times n$ dimensional real symmetric matrix $H = H^T = H^* \in \mathrm{Sym}(n)$. In our case we have to set $\gamma = (n+1)/2$ and $K = s\mathbb{1}_n - W^\dagger W$. After integrating over W we found

$$E(s,\infty) \propto \int_{\mathrm{Sym}(n)} d[H] \frac{\exp[s\mathrm{tr}\,(\mathbb{1}_n - \imath H)]}{\det^{-(n+1)/2}(\mathbb{1}_n - \imath H)} \prod_{j=1}^{p} \det{}^{-1/2}((1+\Lambda_j)\mathbb{1}_n + \imath H), \tag{3.27}$$

see [J], and similarly for the smallest eigenvalue. This integral can be rewritten as a Pfaffian of double integrals when diagonalizing the real symmetric matrix H and integrating over its eigenvalues. Although the resulting integral (3.27) is not very suitable for the large n limit it is at least an explicit analytical result at finite n. Such a result was not accessible before due to the Itzykson-Zuber integral (3.13).

With the help of the result (3.27) we were able to show in [J] that the Tracy-Widom distribution still remains when the empirical eigenvalues Λ_j of the covariance matrix C have a macroscopic distance to the soft edges. This is not very surprising since the universality of the local statistics rely on the separation of scales such that the local statistics does not know anything of the macroscopic one.

We also applied the idea of looking for dual integrals to the uncorrelated real Wishart random matrix ensemble in [I, K]. We solved in this way a long standing problem which even existed for the case $C = \mathbb{1}_p$ where we have no problems with group integrals. Let us briefly explain this problem. The distribution of the smallest eigenvalue and its corresponding gap probability can be written as an average over characteristic polynomials in $WW^\dagger$, i.e.

$$\begin{aligned} E(s,\infty) &\propto e^{-ps} \langle \det{}^{(\nu-1)/2}(WW^\dagger + s\mathbb{1}_p)\rangle_{W\in\mathbb{R}^{p\times(p+1)}}, \\ p_{\min}(s) &\propto s^{(\nu-1)/2} e^{-ps} \langle \det{}^{(\nu-1)/2}(WW^\dagger + s\mathbb{1}_{p-1})\rangle_{W\in\mathbb{R}^{(p-1)\times(p+2)}}. \end{aligned} \tag{3.28}$$

Note that the dimensions of W are different in both expressions and differ from the original dimension $p\times n$ with $\nu = n-p \geq 0$. In the case of an odd index $\nu \in 2\mathbb{N}_0+1$ the exponent of the determinant is an integer and the averages can be readily performed in various ways without any problems [147, 148]. We obtain Pfaffians with the kernel in terms of Laguerre polynomials. So far everything is simple.

The problem arises when the exponent of the determinant is a half-integer, i.e. $\nu \in 2\mathbb{N}_0$. Problems involving averages over square roots of determinants were recently enlisted in [149] and can be found for example in the study of quantum chaotic systems. Such an average for real Wishart random matrices was only available in form of a recursion derived by Edelman [150]. This recursion is highly non-trivial and not very practical. Only for small $\nu = 0, 2$ this recursion could be explicitly solved, see [49, 150, 151].

We closed this gap in [I, K] by applying a mixture of orthogonal polynomial theory [19, 27] and the supersymmetry method [129, 130]. When diagonalizing $WW^\dagger$ the one-point weight in the joint probability density is $w(x) = x^{\nu'}e^{-x}$ with ν' an effective index which is $\nu' = 1$ for the gap probability and $\nu' = 3$ for the distribution of the smallest eigenvalue, cf. Eq. (3.28). Since we average over a half-integer the method of skew-orthogonal polynomials breaks down according to the weight $w(x)$. This can be cured by pushing a square root of the determinant to the weight. Therefore we considered the new weight $w'(x) = x^{\nu'}e^{-x}/\sqrt{x+s}$ and constructed the skew-orthogonal polynomials according to this weight. The polynomials and the kernels were calculated by the supersymmetry method.

The results were surprisingly simple. The gap probability as well as the distribution of the smallest eigenvalue can be expressed in terms of Pfaffians as in the case of odd ν. Yet, the kernel of these Pfaffians are linear combinations of Laguerre polynomials with s-dependent coefficients resulting from the additional square root in the weight. These coefficients are Tricomi's confluent hypergeometric functions [48] for even ν while they are only Laguerre polynomials for odd ν. The hypergeometric functions were also found by Edelman [150] such that it could be expected that they play an important role for even ν. But the complicated recursions made it impossible to deduce the simple structure in terms of a Pfaffian.

Since the Pfaffians are taken of matrices of dimensions proportional to ν this expression is not suitable for the large n limit with p/n fixed. Then the smallest eigenvalue is at a soft edge and the spectrum detaches from the origin. The situation looks completely different in the hard edge scaling limit with $\nu = n - p \geq 0$ fixed. Then the distribution can be written in terms of Bessel functions and can be applied in QCD and the study of topological insulators. The results for this limit can be found in section 5 in [K]. They are completely new. The distributions of the smallest eigenvalues at the hard edge were only explicitly known for ν odd and $\nu = 0, 2$ before our work, see [147, 148].

In principal one can also deduce from our results in [K] the distributions of the second to smallest eigenvalue, of the third to smallest eigenvalue etc. They are given by the average

$$\begin{aligned} p_k(s) \quad \propto \quad & s^{(\nu-1)/2}e^{-(p-k+1)s} \int_{[0,s]^{k-1}} d[x] |\Delta_k(x,-s)| \det{}^{(\nu-1)/2} x \exp[\operatorname{tr} x] \\ & \times \left\langle \det{}^{(\nu-1)/2}(WW^\dagger + s\mathbb{1}_{p-k}) \prod_{j=1}^{k-1} \det(WW^\dagger + (s-x_j)\mathbb{1}_{p-k} \right\rangle_{W \in \mathbb{R}^{(p-k)\times(p-k+3)}} \end{aligned} \tag{3.29}$$

for the distribution of the kth smallest eigenvalue. Thus we have only to average over additional characteristic polynomials. With the help of the structure of the Pfaffian point process we derived it is no problem to extend our result to these averages, too.

3.3 Product Matrices and Meijer G-kernels

Products of matrices are encountered when studying various problems. To name only a few, evolution operators in quantum systems, transfer matrices in condensed matter theory, the Polyakov loop in QCD, or scattering matrices in progressive scattering processes. Also the spectral statistics of products of random matrices are an old problem

which ranges back into the 50s and 60s [152, 153]. Applications of random matrices can be found for example in QCD at finite chemical potential [72], image processing [154], disordered systems (see [155] and references therein), and wireless telecommunication [8]. Exactly the latter application is the reason how we came to consider a particular class of products of random matrices which were fortunately analytically feasible, see [L].

The idea of applying RMT to wireless telecommunications is to describe the information channel between two antenna ensembles called multiple-input-multiple-output array via a random matrix [16], i.e.

$$v_{\text{out}} = A v_{\text{in}} \tag{3.30}$$

with v_{in} the emitting signal, v_{out} the receiving signal and A the random matrix. The reason for such a strong simplification is that in realistic situations the transmitting channel is highly chaotic and mixes the signals of each single emitting antenna as long as the receiver is not in sight of the emitter. Nowadays the situation is slightly more complicated. A mobile phone user does not directly send his signal to the user he called. Between the emitter and the final receiver are many arrays of antenna such that Eq. (3.30) has to be recursively applied with independently chosen random matrices A_j. Then the relation between the input and output signal is

$$v_{\text{out}} = A_M \cdots A_1 v_{\text{in}} = X^{(M)} v_{\text{in}}. \tag{3.31}$$

The matrices A_j can be generically anything even rectangular when the numbers of emitting and receiving antennas are different. Therefore the simplest random matrix model is by choosing A_j as complex Wishart random matrices which was considered in [L] for square matrices and in [16, 8, M] for rectangular ones. The random matrices A_j are chosen complex since a signal consists of a phase and an amplitude. Products of real and quaternion Wishart matrices were considered in [56, 156, N]. The case $M = 2$ appears in the Osborn model [72] for QCD at finite chemical potential though in this model the matrices A_1 and A_2 are statically correlated via the chemical potential.

To analyze the spectral statistics of product matrices we have two choices. Either one considers the generically complex eigenvalues of $X^{(M)}$ assuming that $X^{(M)}$ is a square matrix or one studies the singular values of the product matrix $X^{(M)}$ which are the square roots of the eigenvalues of $X^{(M)} X^{(M)\dagger}$. The eigenvector statistics is always the same since the ensembles are invariant under unitary transformations.

The macroscopic level densities of $X^{(M)}$ as well as of $X^{(M)} X^{(M)\dagger}$ can be calculated with the help of free probability [8, 157, 158, 159] in the limit of large matrix dimensions which is essentially the restriction to planar Feynman diagrams. The level densities exhibit a different singular behavior at the origin compared to a single Wishart matrix ($M = 1$), see [157]. Despite this different behavior the macroscopic level densities are not that enlightening since they can be always transformed by a simple substitution to the results of a single Wishart random matrix, which are the Marčenko-Pastur distribution (2.6) for the singular values and the uniform complex disk centered at the origin for the complex eigenvalues [160, 161]. Thus one has to zoom onto the scale of the local mean level spacing to see essential differences between the level statistics of $X^{(M)}$ for different M.

Before studying the local spectral statistics we have to calculate analytical feasible expressions for the k-point correlation functions,

$$R^{(k,M)}(x) = \left\langle \prod_{j=1}^{k} \delta(x_j - \lambda_j) \right\rangle_{X^{(M)}}, \tag{3.32}$$

with $x = (x_1, \ldots, x_M)$ and λ_j the eigenvalues or singular values of the random matrix $X^{(M)}$. We did this calculation in [L] and in [M] for the singular values of a product of square and rectangular Wishart matrices, respectively. The result is very simple and can be expressed in terms of Meijer G-functions [48],

$$R^{(k,M)}(x) = \det \left[\sum_{n=0}^{N-1} G^{1,0}_{1,M+1} \left(\begin{array}{c} n+1 \\ 0; -\nu_1, \ldots, -\nu_M \end{array} \middle| x_a \right) G^{M,1}_{1,M+1} \left(\begin{array}{c} -n \\ \nu_1, \ldots, \nu_M; 0 \end{array} \middle| x_b \right) \right], \tag{3.33}$$

where N is the number of generic non-zero eigenvalues of $X^{(M)}$ and ν_j is the rectangularity (difference of the two matrix dimensions) of the matrix A_j.

Due to the symmetry of the Meijer G-functions in its indices, in particular $\nu_j \leftrightarrow \nu_i$, we notice a symmetry in reordering the matrices in the product $X^{(M)}$. This symmetry can be also found for the complex eigenvalues [55, 162, O, N] and holds in a more general framework. In [N] we showed that one does not really need that the weights of the random matrices A_j are Gaussian to find this symmetry under reordering. We proved a weak-commutation relation which means the following. Assume two square matrices A_1 and A_2 drawn from the probability weights $P_1(A_1)$ and $P_2(A_2)$ which are invariant under $P_1(A_1) = P_1(V_1 A_1 U_1)$ and $P_2(A_2) = P_2(V_2 A_2 U_2)$ for all $A_1, A_2 \in \mathbb{C}^{N \times N}$ and $V_1, V_2, U_1, U_2 \in \mathrm{U}(N)$. Then the average of an arbitrary matrix-dependent observable F satisfies $\langle F(A_1 A_2) \rangle = \langle F(A_2 A_1) \rangle$. Such an observable has not to be necessarily invariant under unitary transformations, too. Hence this weak-commutation relation can be embedded in a product of square matrices such that it also holds for the average $\langle F(B A_1 A_2 C) \rangle = \langle F(B A_2 A_1 C) \rangle$ with two other matrices B and C which can be random as well as fixed. Additionally our weak-commutation relation applies to products of real and quaternion matrices. One has only to replace the unitary group by the orthogonal and the unitary symplectic group, respectively.

The restriction of the weak-commutation relation to square matrices is only apparent. In [N] we also showed that an arbitrary product of rectangular matrices can be essentially traced back to a product of square matrices. The cost to pay for such a change from rectangular to square matrices, $A_j \to A'_j$, is a change of the probability weights, $P_j(A_j) \to P'_j(A'_j)$. For Gaussian weights, see Eq. (2.5) for real Wishart matrices, the new weight is $P'(A') = \det{}^{\gamma} A' A'^{\dagger} \exp[-\mathrm{tr}\, A' A'^{\dagger}]$ with a certain exponent γ depending on the rectangularity ν of A and whether A is real, complex, or quaternion. The symmetry in the indices ν_j of the k-point correlation function is immediate when considering these new weights in combination with the weak-commutation relation.

In [N] we did not only consider products of random matrices drawn from Wishart ensembles to illustrate the weak-commutation relations but we also considered Jacobi random matrix ensembles. The latter random matrices are truncated orthogonal, unitary or unitary symplectic matrices drawn from the Haar-measure [27, 163, 164]. Applications of such matrices can be found in scattering processes where one is only interested in the transmitting channels such that one considers only sub-matrices of

the S-matrix [155]. The k-point correlation function of the singular values of such matrices for the complex case was calculated in [T] and looks very similar to Eq. (3.33). In [O, N] we studied the complex eigenvalues. The corresponding k-point correlation function is

$$R^{(k,M)}_{\text{complex}}(z) = \det\left[G^{M,0}_{M,M}\left(\begin{array}{c}\mu_1,\ldots,\mu_M\\ \nu_1,\ldots,\nu_M\end{array}\,\middle|\,|z_a|^2\right)\sum_{n=0}^{N-1}\left[\prod_{j=1}^{M}\left(\begin{array}{c}\mu_j+n\\ \nu_j+n\end{array}\right)\right](z_a z_b^*)^n\right], \tag{3.34}$$

where $z = (z_1, \ldots, z_k)$ and μ_j is related to the size of the truncation of the unitary matrix U_j to A_j.

In [O] we also investigated the local statistics in the limit $N \to \infty$ for the case of identically distributed square matrices ($\nu_1 = \ldots = \nu_M = 0$, $\mu_1 = \ldots = \mu_M = \mu$). Thereby we studied two situations, the strong non-unitarity limit where μ is of the order N and the weak non-unitarity limit where μ is fixed and the truncation of the unitary matrices is macroscopically not visible. In the weak non-unitarity the whole spectrum is concentrated in the vicinity of the complex unit circle. Only via an exponential suppressed tail eigenvalues enter the interior of the complex unit disc.

In the strong non-unitarity limit the macroscopic level density of the complex eigenvalues is angular independent and takes a finite support inside the unit disc with a singularity at the origin. We found the Ginibre kernel [165] in the bulk of the complex eigenvalue density and the error-function kernel [166] at the soft edge. Both kernels can be also found for the case $M = 1$ (the complex Ginibre ensemble) and confirm their universality. However at the origin where the singularity is sitting we found another kernel. This kernel is given in terms of Meijer G-functions and is thus called Meijer G-kernel [55, 162, O]. This kernel has analogues for products of real [156, N] and quaternion matrices [56, N] and it was also derived for the singular value statistics, see [58, 57, 59, T]. They can be understood as a generalization of the Bessel kernel, see (2.18) for the two-point function. Indeed the Bessel kernel has also a representation in terms of Meijer G-functions because Bessel functions are a particular kind of Meijer G-functions [48].

Recall that the Bessel kernel can be characterized by the index ν. This index ν is now extended to a whole set of indices, e.g. $\nu_1, \ldots, \nu_{M_1}$ and $\mu_1, \ldots, \mu_{M_2}$ ($M_1, M_2 \leq M$) in the case of truncated unitary matrices where ν_j and μ_i are fixed for all $j = 1, \ldots, M_1$ and $j = 1, \ldots, M_2$. Since Meijer G-functions have all in all four sets of indices, there is a rich classification of hard edge kernels which are all inequivalent. These kernels have only one property in common and this is the fact that they can be always written as double Taylor series in the levels x and their logarithm $\ln x$. In particular this logarithmic behavior of spectral observables is well-known in the infrared regime of QCD. The modified Bessel functions of the second kind, usually denoted as K_ν, have also a logarithmic part and they appear when considering averages of the Green function. These logarithmic contributions are the reason why QCD has to be non-perturbatively dealt at small energies. The physical explanation are the Goldstone bosons which result from the spontaneous breaking of chiral symmetry. This interpretation can be carried over to the general case of the Meijer G-kernels but then two questions arise. What is the corresponding spontaneous breaking of symmetry for such a Meijer G-kernel and what is the non-linear σ-model?

The first question can be easily answered. For this purpose we recall how we obtain

the symmetry breaking pattern of QCD in four dimensions. The Dirac operator has a chiral structure, in particular its characteristic polynomial is

$$\begin{aligned}\det(D\otimes \mathbb{1}_{N_\mathrm{f}}+\mathbb{1}\otimes M) &= \det\begin{pmatrix} \mathbb{1}\otimes M_\mathrm{lr} & W\otimes\mathbb{1}_{N_\mathrm{f}} \\ -W^\dagger\otimes\mathbb{1}_{N_\mathrm{f}} & \mathbb{1}\otimes M_\mathrm{rl}\end{pmatrix} \\ &= \det{}^\nu M_\mathrm{rl}\det(WW^\dagger\otimes\mathbb{1}_{N_\mathrm{f}}+\mathbb{1}\otimes M_\mathrm{lr}M_\mathrm{rl}), \end{aligned} \tag{3.35}$$

such that the quark-dependent part of the Lagrangian can be written as

$$\mathcal{L}_\mathrm{quark} = \mathrm{tr}\,\Psi_\mathrm{r}^\dagger W\Psi_\mathrm{r} - \mathrm{tr}\,\Psi_\mathrm{l}^\dagger W^\dagger\Psi_\mathrm{l} + \mathrm{tr}\,M_\mathrm{lr}\Psi_\mathrm{r}^\dagger\Psi_\mathrm{l} + \mathrm{tr}\,M_\mathrm{rl}\Psi_\mathrm{l}^\dagger\Psi_\mathrm{r}. \tag{3.36}$$

The N_f quark fields are arranged as columns in the right-handed, Ψ_r, and left-handed, Ψ_l, components of the spinors. The quark masses are comprised in the mass matrices M_lr and M_rl which are sometimes also chosen non-diagonal and unequal to generate some observables, e.g. see [167]. When the mass matrices are set to zero, the Lagrangian is invariant under $\Psi_\mathrm{r}\to\Psi_\mathrm{r}U_\mathrm{r}$ and $\Psi_\mathrm{l}\to\Psi_\mathrm{l}U_\mathrm{l}$ with $U_\mathrm{r},U_\mathrm{l}\in\mathrm{U}(N_\mathrm{f})$. Thus the symmetry group is $\mathrm{U}(N_\mathrm{f})\times\mathrm{U}(N_\mathrm{f})$. For topological sectors with a non-trivial index $\nu\neq 0$ the axial group U(1) is anomalously broken because of the integration measure. The order parameter of the spontaneous breaking of chiral symmetry is the chiral condensate which is proportional to the average of $\mathrm{tr}\,(\Psi_\mathrm{r}^\dagger\Psi_\mathrm{l}+\Psi_\mathrm{l}^\dagger\Psi_\mathrm{r})$. Hence the mass is the generating source and the mass term is only invariant under the subgroup $\mathrm{SU}(N_\mathrm{f})\times\mathrm{U}(1)$. This yields the well-known pattern $\mathrm{SU}(N_\mathrm{f})\times\mathrm{SU}(N_\mathrm{f})\times\mathrm{U}(1)\to\mathrm{SU}(N_\mathrm{f})\times\mathrm{U}(1)$.

Let us come back to the product matrix $X^{(M)}$. The analogues of Eqs. (3.35) and (3.36) are

$$\left[\prod_{j=1}^M \det{}^{\gamma_j} Y_j\right]\det(A_M\cdots A_1A_1^\dagger\cdots A_M^\dagger\otimes\mathbb{1}_{N_\mathrm{f}}-\mathbb{1}\otimes Y_1\cdots Y_{2M}) \tag{3.37}$$

$$= \det\begin{pmatrix} \mathbb{1}\otimes Y_{2M} & 0 & & & 0 & A_M^\dagger\otimes\mathbb{1}_{N_\mathrm{f}} \\ A_{M-1}^\dagger\otimes\mathbb{1}_{N_\mathrm{f}} & \mathbb{1}\otimes Y_{2M-1} & 0 & & & 0 \\ 0 & A_{M-2}^\dagger\otimes\mathbb{1}_{N_\mathrm{f}} & \mathbb{1}\otimes Y_{2M-2} & 0 & & \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & & & & A_M\otimes\mathbb{1}_{N_\mathrm{f}} & \mathbb{1}\otimes Y_1 \end{pmatrix},$$

and

$$\mathcal{L} = \sum_{j=1}^M\left(\mathrm{tr}\,\Psi_j^\dagger A_{M+1-j}\Psi_j + \mathrm{tr}\,\Psi_{M+j}^\dagger A_j^\dagger\Psi_{M+j}\right) + \sum_{j=1}^{2M}\mathrm{tr}\,Y_j\Psi_j^\dagger\Psi_{j-1}\ \text{with}\ \Psi_0=\Psi_{2M}, \tag{3.38}$$

respectively. The exponents γ_j depend on the rectangularities ν_j of the matrices A_j and the matrices Y_j play the role of the mass matrices in QCD. From this expression it is obvious that we have the symmetry group $\mathrm{U}^{2M}(N_\mathrm{f})$ for $Y_1\cdots Y_{2M}=0$. Again we might have an anomalous symmetry breaking of the group U(1) due to the measure if it is not a product of square matrices. When assuming a spontaneous symmetry breaking similar to QCD such that the order parameter is given by the Y_j-dependent terms we find the symmetry breaking pattern $\mathrm{SU}^{2M}(N_\mathrm{f})\times\mathrm{U}^{2M-1}(1)\to\mathrm{SU}^M(N_\mathrm{f})\times\mathrm{U}^M(1)$ which was discovered in [S]. In this way we come to the second question what the non-linear σ-model for the Goldstone manifold $\mathrm{SU}^M(N_\mathrm{f})\times\mathrm{U}^{M-1}(1)$ is.

To answer the second question we were fortunate that parallel to the study of product matrices we looked for a way to generalize the supersymmetric projection formula developed in [132] to chiral ensembles, see [P, S]. The projection formula directly relates the superfunction $Q(\sigma)$ in Eq. (3.18) with the original weight $P(H)$ in Eq. (3.17). Therefore it provides a short cut in the calculation. In the case of an $n \times (n+\nu)$ dimensional complex matrix W which is drawn from $P(WW^\dagger) = P(UWW^\dagger U^\dagger)$ for all $U \in \mathrm{U}(n)$, the projection formula yields

$$\begin{aligned} Z(Y) &= \int d[W] P(WW^\dagger) \det(WW^\dagger \otimes \mathbb{1}_{N_\mathrm{f}} + Y) \\ &= \int_{\mathrm{U}(N_\mathrm{f})} d\mu(U) Q(U) \det{}^{-n-\nu} U \det(\mathbb{1}_n \otimes U + Y) \end{aligned} \tag{3.39}$$

for the partition function with

$$Q(U) \propto \int d[W'] d[\chi] P \left[\begin{pmatrix} W'W'^\dagger + \chi\chi^\dagger & \chi\sqrt{U} \\ \sqrt{U}\chi^\dagger & U \end{pmatrix} \right], \tag{3.40}$$

see [P] for the derivation and the general result for averages over products and ratios of characteristic polynomials. The $(n+N_\mathrm{f}) \times (n+\nu+N_\mathrm{f})$ dimensional complex matrix W' only depends on ordinary variables while the matrix entries of the $(n+N_\mathrm{f}) \times N_\mathrm{f}$ dimensional matrix χ are independent Grassmann variables. The source Y is $(nN_\mathrm{f}) \times (nN_\mathrm{f})$ dimensional and can be any matrix. Therefore one can readily apply the projection formula to product matrices, too, via a recursion. This was done in [S] for a product of rectangular complex Wishart random matrices. In the hard edge scaling we found the potential of the non-linear σ-model, the partition function (3.39) for $Y = \mathbb{1}_n \otimes Y'/n^M$ becomes in the limit $n \to \infty$

$$Z(Y) \propto \int \left(\prod_{j=1}^{M} d\mu_j(U_j) \det{}^{\nu_j} U_j \right) \exp\left[\mathrm{tr}\, U_M + \sum_{j=1}^{M-1} \mathrm{tr}\, U_j U_{j+1}^{-1} + \mathrm{tr}\, Y' U_1^{-1} \right]. \tag{3.41}$$

For $M = 1$ we obtain the potential (2.28). Hence, Eq. (3.41) is the non-linear σ-model we were looking for.

In [S] we proposed to prove universality of local spectral statistics via the supersymmetric projection formula and non-linear σ-models as Eq. (3.41). Indeed we showed that the integral (3.40) can be readily performed to obtain Q not only for Gaussian ensembles but also for Cauchy-Lorentz ensembles, see Eq. (3.12), and for Jacobi ensembles [27]. The combination of the projection formula with product matrices extends the supersymmetry approach to a whole class of non-Gaussian random matrices. This result is completely new and a breakthrough in the supersymmetry method which could up to our work [S] only be rigorously done for very few ensembles, e.g. norm-dependent ensembles [19, 168]. Hence also distributions with Levy-tails are now accessible, e.g. it is well-known that the Cauchy-Lorentz ensemble relates to one of the stable distributions, see [123]. We expect that a large class of stable heavy-tailed distributions can be generated with products of random matrices. Coming back to the foreword of section 2, we can say that we are now able to analyze heavy tailed random matrices. In this way we can study also the other possibility mentioned in the beginning of subsection 3.2 where also heavy tails can be the reason for the strong deviations of the financial data shown in Fig. 12 from the Marčenko-Pastur law (2.6).

Furthermore our approach also applies to products of real and quaternion matrices. This is especially advantageous since the derivation of the joint probability density of the singular values for such products needs the knowledge of the Itzykson-Zuber integral similar to Eq. (3.13) which is not known in the real and quaternion case. For the complex case this integral is known and in [T] we derived an analogous integral for Jacobi ensembles. We underline that this kind of integral is not needed for computing the joint probability density of the complex eigenvalues. The derivation for the complex eigenvalues is based on a generalized Schur decomposition where all group integrals factorize, see [55, 56, 156, N, O]. Therefore our results in [S] are the only existing ones for the singular value statistics of products of real and quaternion matrices at finite and infinite matrix dimension N.

The limit $N \to \infty$ is not the only one which can be considered. First results for the singular values of a product matrix consisting of infinitely many matrices, i.e. $M \to \infty$, were already derived in the late 80s [169]. Instead of directly considering the singular values λ_j one considers the normalized logarithm $\kappa_j = M^{-1} \ln \lambda_j$ which are called Lyapunov exponent. The reason for this name comes from the study of dynamical, chaotic systems [67]. As in the physical systems the Lyapunov exponents are considered as a measure for the stability of the system under time evolution where the number M of matrices multiplied is interpreted as a time. Already Newman discovered that when keeping the matrix size fixed in the limit $M \to \infty$ all eigenvalues become deterministic meaning that their distribution is given by Dirac delta functions at specific positions [169]. In [170] it was conjectured that the next order correction of the Lyapunov exponents is Gaussian with a variance of the order $\mathcal{O}(M^{-1/2})$ such that the singular values are given by log-normal distributions. In [Q] we extended these results to a full asymptotic expansions and to the complex eigenvalues for a product of complex Ginibre matrices via the explicit analytical results of the joint probability densities in terms of Meijer G-functions. Due to rotation invariance of the eigenvalue spectrum only the radial parts of the complex eigenvalues become deterministic. Surprisingly, the Dirac delta functions of the radii are at the same positions as for the singular values and also the variances of the log-normal distributions are exactly the same. Hence it seems that product matrices might become normal in the limit $M \to \infty$. However the opposite is the case. The off-diagonal matrix entries may become exponentially large like the diagonal terms but their contribution to the singular values is logarithmically suppressed in M compared to the diagonal entries. We explicitly proved this conjecture for 2×2 matrices distributed with almost arbitrary weight (we only need the invariance under the group $\mathrm{U}(2) \times \mathrm{U}(2)$) in [Q].

Since the individual eigenvalue as well as the individual singular value distributions do not overlap in the large M limit their statistics become independent and the determinantal point process reduces to a permanental point process, see [Q]. This was also found for a product of real and quaternion Ginibre matrices in [171] where our conjecture for these two kinds of product matrices (see [Q]) was proven. The results for these two matrices are qualitatively the same as for an infinite product of complex matrices. The positions of the Dirac delta function are only shifted and the angular dependence of the complex eigenvalues reduces to a Dirac delta function on the real line for real matrices [156, 171] and to $\sin^2 \phi$ for quaternion ones [171].

Finally, let us come back to the Meijer G-kernel. We emphasize that those kernels were found for other random matrix ensembles, as well, see [60, R]. For example we

recently studied a random matrix ensemble which is equivalent to the Bures measure in quantum information theory, see [R]. The Bures metric is a distinguished metric on the manifold of density operators and satisfies certain properties, see [172] for a complete list of these properties and their physical meaning. The property which is important to apply RMT is that the Bures metric is Riemannian. Hence we can explicitly write the corresponding joint probability density of the N eigenvalues $\lambda_j > 0$ of a density operator [173],

$$p_{\text{Bures}}(\lambda) \propto \delta\left(1 - \sum_{j=1}^{N} \lambda_j\right) \left(\prod_{j=1}^{N} \lambda_j^a\right) \prod_{1 \leq a < b \leq N} \frac{(\lambda_a - \lambda_b)^2}{\lambda_a + \lambda_b}. \tag{3.42}$$

The Dirac delta function only reflects the property that the trace of a density operator is normalized to unity. As in the case of other fixed trace ensembles [174, F], it can be replaced by a Fourier-Laplace transformation such that one can calculate the eigenvalue statistics with the help of orthogonal polynomials. We did this computation in [R]. Thereby we used a duality between partition functions of the Bures measure and of the complex Cauchy two-matrix model. The latter matrix model consists of two rectangular matrices W_1 and W_2 which are jointly distributed by [175]

$$P(W_1, W_2) \propto \frac{\exp[-\operatorname{tr} W_1 W_1^\dagger - \operatorname{tr} W_2 W_2^\dagger]}{\det^N (W_1 W_1^\dagger + W_2 W_2^\dagger)}. \tag{3.43}$$

The spectral statistics of the Cauchy two-matrix model was derived in [175, 176, 60] which is a determinantal point process.

How does this help us since the spectral statistics of the Bures measure is a Pfaffian point process? In [175] it was shown that the normalization constant of the Cauchy two-matrix model is essentially the square of the Bures measure. We extended this relation in [R] to any average of products and ratios of characteristic polynomials of a Bures distributed density operator and could also invert this relation. Therefore the full statistics of the Bures measure is given by the statistics of the Cauchy two-matrix model. Interestingly the spectral statistics of the Cauchy two-matrix model is given in terms of Meijer G-function, see [60], as we know it from product matrices. Indeed one can rewrite the statistics of the matrix W_1 as the statistics of a product of a complex Ginibre random matrix times a truncated unitary matrix such that the result of [60] are not very surprising from the point of view of product matrices. Surprisingly a Pfaffian point process as the Bures measure exhibits reminiscent statistics. This is up to now unique for the singular value statistics.

4 Outlook

In the past years there was a rapid development in RMT of new and ground breaking results. On the one hand the spectral statistics could be computed for random matrices whose group invariance is broken including non-Hermitian random matrices. On the other hand a new class of ensembles were discovered whose products are analytically feasible. Both developments open the possibility to improve random matrix models and make them more realistic. We already mentioned some possible directions of future research. We can look at particular lattice discretizations in QCD which also includes the discretization with overlap and domain wall fermions or even study the joined effect of a finite chemical potential and a finite lattice spacing. Also new models for Majorana fermions in disordered systems are possible since the symmetry structure is reminiscent of the one for Wilson fermions. With product matrices one can also try to model the Polyakov loop in QCD. Thereby one needs to draw statistically dependent random matrices which are then multiplied.

Especially the study of heavy tailed random matrices might be very fruitful. Product matrices may serve as "standard candles" in RMT which are analytical feasible as it was the case for the Wishart random matrix ensemble in the last century. From a mathematical point of view these random matrix models are perfect for studying stability and divisibility in the context of free probability. From a statistical point of view one can mix heavy tails and empirical correlation matrices which breaks the invariance under the unitary or orthogonal group. In this way one can study the combined effects which may explain the strong deviations in the tails between real data and the Marčenko-Pastur distribution, cf. Fig. 12. For this purpose the distributions of the individual singular values can be very helpful and have to be calculated. Here the supersymmetry method and in particular the projection formula might be very helpful.

The supersymmetric projection formula has also the potential that one can study the combination of sums and products of random matrices. This includes products of random matrices which are shifted by constant matrices, e.g. $A_j = \mathbb{1} + H_j$. This opens a way to investigate the local spectral statistics of such ensembles in the limit of large matrix dimensions although the analytical results for finite matrix dimensions are still out of reach. Up to now such studies were only possible for the macroscopic level statistics with the help of free probability. In a next step products and sums of statistically dependent matrices should be the goal. Then random matrix ensembles are very near to physical systems.

Summarizing this development, although the complexity of the random matrix models increases the new techniques still allow analytical results. Those results were in the last century only computable for ideal systems. Nowadays the results are much more realistic and they certainly become even better in future. They yield insights of complex systems which have been never found when considering all the details of physical systems. Therefore the reduction to particular spectral observables is as effective as the reduction of classical systems to macroscopic observables like pressure and temperature.

Own Articles attached in the Appendix

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Appendix

A Articles summarized in Subsection 3.1

On the Eigenvalue Density of the non-Hermitian Wilson Dirac Operator

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We find the lattice spacing dependence of the eigenvalue density of the non-Hermitian Wilson Dirac operator in the ϵ-domain. The starting point is the joint probability density of the corresponding random matrix theory. In addition to the density of the complex eigenvalues we also obtain the density of the real eigenvalues separately for positive and negative chiralities as well as an explicit analytical expression for the number of additional real modes.

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Keywords: Wilson Dirac operator, lattice QCD, infrared limit of QCD, random matrix theory

Introduction. In the past two decades, there has been an increasing interest in non-Hermitian random matrix theory (RMT) [1]. To name a few applications, quantum chaos in open systems [2], dissipative systems [3] and QCD at finite chemical potential [4]. Some features of the model we are considering also occur in the condensed matter system analyzed in Ref. [5].

The connection between the infrared limit of QCD and RMT has been well understood in the continuum limit since the early 90's [6]. It is based on the universality of chiral RMT in the microscopic limit (or ϵ domain) [7] with chiral RMT described by the same chiral Lagrangian as QCD. The main advantage of RMT is the availability of powerful methods to derive analytical results, and recently this approach was applied to QCD at finite lattice spacing [8–11]. It was shown that the ϵ limit of the chiral Lagrangian for the Wilson Dirac operator $D_{\rm W}$ [13, 14] can be obtained from an equivalent RMT. Discretization effects of the spectrum of $D_{\rm W}$ have been studied directly by means of chiral Lagrangians [9–11, 15, 16], but using RMT methods will enable us to obtain results that were not accessible previously.

The aim of this paper is to obtain analytical results for the eigenvalue density of $D_{\rm W}$ for the RMT model proposed in Ref. [9]. We consider the quenched case.

RMT. We consider the random matrix theory [9],

$$D_{\rm W} \;=\; \begin{pmatrix} aA & W \\ -W^\dagger & aB \end{pmatrix} \tag{1}$$

distributed by

$$P(D_{\rm W}) \propto \exp\Big[-\frac{n}{2}({\rm tr}A^2+{\rm tr}B^2)-n{\rm tr}WW^\dagger\Big]. \tag{2}$$

The matrices A and B are Hermitian $n\times n$ and $(n+\nu)\times(n+\nu)$ matrices, respectively, and the entries of W are complex and independent. In the microscopic limit, with $n\to\infty$ at fixed rescaled eigenvalues $\widehat{z}=2nz$ and lattice spacing $\widehat{a}^2=na^2/2$, the spectral properties of this RMT become universal and agree with Wilson chiral perturbation theory in the same limit (with n identified as the volume of space-time) apart from the squared trace terms [14, 15]. The finite integer $|\nu|\leq n$ is the index of the Dirac operator and is kept fixed.

The matrix $D_{\rm W}$ is $\gamma_5={\rm diag}(\mathbb{1}_n,-\mathbb{1}_{n+\nu})$-Hermitian, i.e. $D_{\rm W}^\dagger=\gamma_5 D_{\rm W}\gamma_5$. Therefore its eigenvalues are either real or come in complex conjugate pairs. The ν generic zero modes at $a=0$ become the generic real modes of $D_{\rm W}$ at finite lattice spacing. Furthermore, $D_{\rm W}$ may have $2n-2l$ additional real eigenvalues which appear when a pair of complex eigenvalues collides with the real axis.

In Refs. [9–12] the technically simpler case of the Hermitian Wilson Dirac operator $D_5=\gamma_5 D_{\rm W}$ was studied. Although spectra of D_5 have been studied in the lattice literature [17], only the eigenvalues of $D_{\rm W}$ are directly related to chiral symmetry breaking which is our main motivation to study its spectral properties.

The joint probability distribution (jpd). To preserve the γ_5-Hermiticity of $D_{\rm W}$ we can only quasi-diagonalize $D_{\rm W}$ by a non-compact unitary matrix $U\in{\rm U}(n,n+\nu)$,

$$D_{\rm W}=UXU^{-1} \text{ and } X=\begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & y_2 & 0 \\ 0 & -y_2 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix}. \tag{3}$$

In contrast to the diagonalization of a Hermitian matrix such as D_5, the matrix X may only be quasi-diagonal where x_1, x_2, y_2 and x_3 are diagonal matrices of dimension $n-l$, l, l and $n-l+\nu$ with $0\leq l\leq n$ the number of complex conjugate pairs. The complex eigenvalues are given by $(z,z^*)=(x_2+\imath y_2, x_2-\imath y_2)$. The ensemble $D_{\rm W}$ decomposes into $n+1$ disjoint sets of quasi-diagonal matrices (3) with a fixed number of real eigenvalues. The joint probability density of the $2n+\nu$ eigenvalues $Z=(z_{1{\rm r}},\ldots,z_{n{\rm r}},z_{1{\rm l}},\ldots,z_{n+\nu,{\rm l}})\in\mathbb{C}^{2n+\nu}$ can be obtained by integrating over U. This calculation will be discussed in detail elsewhere. We only give the result for $\nu\geq 0$ which is not a restriction because of the symmetry $\nu\to-\nu$. The jpd is given by

$$p(Z)\propto\Delta_{2n+\nu}(Z)\det\begin{bmatrix} \{g_2(z_{a{\rm r}},z_{b{\rm l}})\}_{\substack{1\leq a\leq n\\ 1\leq b\leq n+\nu}} \\ \{z_{b{\rm l}}^{a-1}g_1(z_{b{\rm l}})\}_{\substack{1\leq a\leq \nu\\ 1\leq b\leq n+\nu}} \end{bmatrix},$$

$$g_1(z)=\sqrt{\frac{n}{2\pi a^2}}\exp\Big[-\frac{n}{2a^2}x^2\Big]\,\delta(y), \tag{4}$$

$$
\begin{aligned}
g_2(z_1,z_2) &= \sqrt{\frac{n^3}{4\pi a^2(1+a^2)}}\,\frac{z_1^*-z_2^*}{|z_1-z_2|} \\
&\times \left[\exp\left[-\frac{n(x_1+x_2)^2}{4a^2}-\frac{n(y_1-y_2)^2}{4}\right]\delta^{(2)}(z_1-z_2^*)\right. \\
&+ \left.\frac{1}{2}\exp\left[-\frac{n}{4a^2}(x_1+x_2)^2+\frac{n}{4}(x_1-x_2)^2\right]\right. \\
&\times \left.\mathrm{erfc}\left[\frac{\sqrt{n(1+a^2)}}{2a}|x_1-x_2|\right]\delta(y_1)\delta(y_2)\right], \\
&\equiv g_{2\,\mathrm{c}}(z_1)\delta^{(2)}(z_1-z_2^*)+g_{2\,\mathrm{r}}(x_1,x_2)\delta(y_1)\delta(y_2),
\end{aligned}
$$

FIG. 1: The projection of ρ_{c} onto the imaginary axis for $\nu = 1$ and $\widehat{a} = 1/\sqrt{8}$. The Monte Carlo simulation (histogram, bin size=0.4) contains 200000 matrices with $n = 50$. This simulation nicely confirms our analytical result (blue curve) and shows the deviations from the $\widehat{a} = 0$ result (red curve).

where erfc is the complementary error function and $\delta^{(2)}(x+\imath y) = \delta(x)\delta(y)$. Due to γ_5 the permutation group $\mathfrak{S}(2n+\nu)$ is broken to $\mathfrak{S}(n)\times\mathfrak{S}(n+\nu)$ which reflects itself in the product of the Vandermonde determinant $\Delta_{2n+\nu}(Z)$ and the other determinant in Eq. (4). The expansion of the delta functions yields the jpd for each of the $n+1$ subsets with a fixed number of complex eigenvalue pairs. The two-point distribution g_2 splits into one term for the real eigenvalues $g_{2\mathrm{r}}$ and one for the complex conjugated pairs $g_{2\mathrm{c}}$ as it is also known for the real Ginibre ensemble and its chiral counterpart [18].

The eigenvalue densities for the real and complex eigenvalues can be obtained by integrating over all eigenvalues except one. The spectral density can be decomposed into the density of real modes, ρ_{r} for positive chirality ($\langle\psi|\gamma_5|\psi\rangle > 0$), ρ_{l} for negative chirality ($\langle\psi|\gamma_5|\psi\rangle < 0$), and the density of complex pairs, ρ_{c},

$$
\int p(Z)\prod_{z_j\neq z_{1\mathrm{r}}} d[z_j] = \rho_{\mathrm{r}}(x_{1\mathrm{r}})\delta(y_{1\mathrm{r}})+\frac{\rho_{\mathrm{c}}(z_{1\mathrm{r}})}{2}, \quad (5)
$$

$$
\int p(Z)\prod_{z_j\neq z_{1\mathrm{l}}} d[z_j] = \rho_{\mathrm{l}}(x_{1\mathrm{l}})\delta(y_{1\mathrm{l}})+\frac{\rho_{\mathrm{c}}(z_{1\mathrm{l}})}{2}. \quad (6)
$$

Note that the chirality reflects the conventions of the RMT. By expanding the first row of the determinant in Eq. (4) and re-expressing the additional factors from $\Delta_{2n+\nu}(Z)$ as $N_{\mathrm{f}} = 2$ partition functions we obtain

$$
\rho_{\mathrm{c}}(z) = g_{2\mathrm{c}}(z)(z-z^*)Z^{\nu}_{N_{\mathrm{f}}=2}(z,z^*;a), \quad (7)
$$

$$
\rho_{\mathrm{r}}(x) = \int_{\mathbb{R}} g_{2\mathrm{r}}(x,x')(x-x')Z^{\nu}_{N_{\mathrm{f}}=2}(x,x';a)dx'. \quad (8)
$$

A similar factorized structure was found in Ref. [19].

Expanding the first column of the determinant and integrating over all eigenvalues except z_{1l}, we find the same expression for ρ_{c} and the density ρ_{l} of the real modes originating from g_2 (using Eq. (6)). However, there is an additional contribution to ρ_{r} due to the last ν rows which gives the distribution of chirality over the real eigenvalues

$$
\rho_\chi = \rho_{\mathrm{l}} - \rho_{\mathrm{r}}. \quad (9)
$$

Additional rows of some of the determinants have to be expanded to express them into known partition functions. We have checked for $\nu = 1$ and $\nu = 2$ that the result agrees with previously derived expressions [9, 11].

In the microscopic limit, the partition functions in Eqs. (7) and (8) can be expressed in terms of integrals over U(2). They can be simplified using the eigenvalues of the U(2)-matrices as integration variables resulting in

$$
\rho_{\mathrm{c}}\left(\frac{z}{2n}\right) = \frac{e^{-x^2/8\widehat{a}^2}|y|}{(2\pi)^{5/2}2\widehat{a}}\int_{[0,2\pi]^2} e^{x(\cos\varphi_1+\cos\varphi_2)-4\widehat{a}^2(\cos^2\varphi_1+\cos^2\varphi_2)}\mathrm{sinc}[y(\cos\varphi_1-\cos\varphi_2)]\cos\nu(\varphi_1+\varphi_2)D\varphi_k, \quad (10)
$$

$$
\rho_{\mathrm{r}}\left(\frac{x}{2n}\right) = \frac{1}{16\pi^2}\int_{[0,2\pi]^2}\frac{\exp[\Delta_1^2-\Delta_2^2]\mathrm{erf}\left[\Delta_1,\sqrt{2}\Delta_1\right]-\exp[\Delta_2^2-\Delta_1^2]\mathrm{erf}\left[\Delta_2,\sqrt{2}\Delta_2\right]}{\cos\varphi_1-\cos\varphi_2}\cos\nu(\varphi_1+\varphi_2)D\varphi_k, \quad (11)
$$

$$
\rho_\chi\left(\frac{x}{2n}\right) = \frac{(-1)^\nu}{16\pi\widehat{a}^2}\int_{\mathbb{R}^2}\frac{e^{-((s_1-x)^2+(s_2+\imath x)^2)/16\widehat{a}^2}}{s_1-\imath s_2}s_1^\nu\left[s_1K_{\nu+1}(s_1)I_\nu(\imath s_2)+\imath s_2K_\nu(s_1)I_{\nu+1}(\imath s_2)\right]\frac{\delta^{(\nu-1)}(s_1)}{(\nu-1)!}ds_1ds_2 \quad (12)
$$

The functions sinc, erf, I_l, K_l and $\delta^{(l)}$ are the *sinus cardinalis*, the generalized incomplete error function

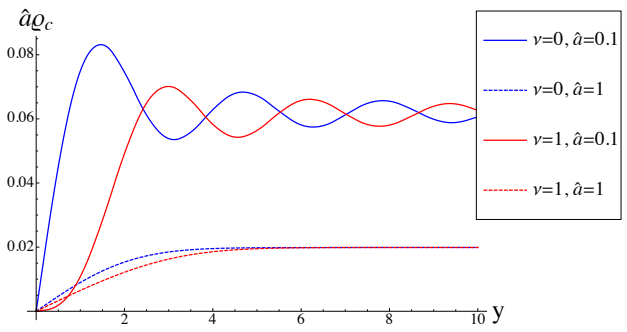

FIG. 2: Along the imaginary axis the difference of ρ_c for different ν is much clearer at small $\widehat{a}$ (solid curves) than at large $\widehat{a}$ (dashed curves) where they are almost the same.

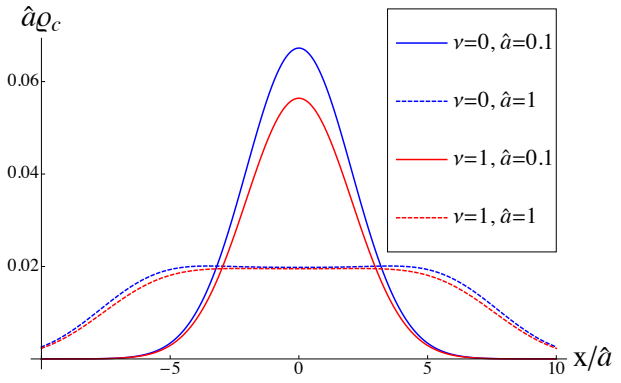

FIG. 3: The distribution ρ_c along a parallel axis to the x-axis (here at $y = 40\widehat{a}^2$) is Gaussian shaped for small $\widehat{a}$ (solid curves) and develops a plateau for large lattice spacing (dashed curves).

$(\mathrm{erf}(b,c) = \mathrm{erf}(c) - \mathrm{erf}(b))$, modified Bessel function of the first and second kind and the l-th derivative of the Dirac delta function, respectively. The integration measure is induced by the invariant U(2) measure, $D\varphi_k = \sin^2((\varphi_1-\varphi_2)/2)d\varphi_1 d\varphi_2$ and $\Delta_j = 2\widehat{a}\left(\cos\varphi_j - x/8\widehat{a}^2\right)$. Because of the δ-function only the algebraic singular part of the K_ν contributes to ρ_χ (which was already obtained in Refs. [9, 11]). The distribution ρ_χ vanishes for $\nu = 0$ and can be obtained from the generating function for the eigenvalue density of $\gamma_5(D_W+m)$. [9] Comparisons of the analytical results with simulations of the random matrix model (1) are shown in Figs. 1 and 4. The normalizations are chosen such that the integral over ρ_χ is equal to ν. The other constants are already fixed by this choice.

For small increasing $\widehat{a}$ the complex eigenvalues move parallel to the real axis according to a Gaussian distribution with a width of $2\widehat{a}$ (See Fig. 3). Therefore the density of the projection of these eigenvalues on the imaginary axis is very close to the $\widehat{a} = 0$ result. For large $\widehat{a}$ the distribution of the real parts of the complex eigenvalues develops a box-like shape from $-8\widehat{a}^2$ to $8\widehat{a}^2$ which can be derived from a saddle point approximation of Eq. (10) (See Fig. 3) and the oscillations disappear (See Fig. 2). Along the imaginary axis $\rho_c(\widehat{a} \gg 1)$ becomes $\widehat{a}^{-2}\mathrm{erf}(y/4\widehat{a})$.

Near the real axis ρ_c behaves as $y^{\nu+1}$ for small $\widehat{a}$ but is linear in y for $\widehat{a}$ large enough (See Fig. 2). In the continuum limit it peaks around the imaginary axis and eventually gets the form of the continuum microscopic eigenvalue density.

Real modes. For a Wilson Dirac operator with index ν there are at least ν real modes. The additional real modes result when complex conjugate eigenvalue pairs enter the real axis. The average number of these modes follows from the integral

$$N_{\mathrm{add}} = 2\int_{\mathbb{R}} \rho_{\mathrm{r}}\left(\frac{x}{2n}\right) dx \qquad (13)$$

$$= \int_{[0,2\pi]^2} \frac{1 - e^{-2\widehat{a}^2(\cos\varphi_1 - \cos\varphi_2)^2}}{8\pi^2 \sin^2((\varphi_1+\varphi_2)/2)} \cos\nu(\varphi_1+\varphi_2) d\varphi_1 d\varphi_2.$$

In the limits for small and large lattice spacing we find

$$N_{\mathrm{add}} \propto \begin{cases} \widehat{a}^{2(\nu+1)}, & \widehat{a} \ll 1, \\ \widehat{a}, & \widehat{a} \gg 1. \end{cases} \qquad (14)$$

This is shown in Fig. 4. For large lattice spacing the contribution to N_{add} becomes independent of the index ν whereas for sufficiently small lattice spacing only $\nu = 0$ contributes significantly.

For small lattice spacing, the distribution ρ_{r} has a Gaussian shape with a width of $2\widehat{a}$, and for $\widehat{a} \gg 1$, it develops a plateau with sharp edges at $\pm 8\widehat{a}^2$, cf. Fig. 6. The height of ρ_{r} at the origin scales like $\widehat{a}^{2\nu+1}$ for small lattice spacing and like $\widehat{a}^{-1}$ for large $\widehat{a}$.

The distribution of chirality over the real eigenvalues ρ_χ is shown in Fig. 5. For small $\widehat{a}$ we observe the spec-

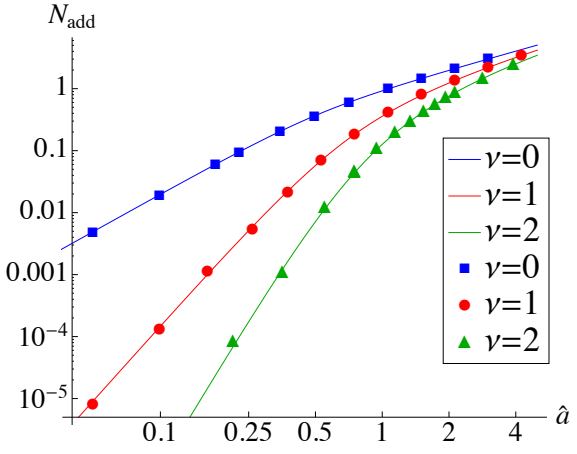

FIG. 4: Log-log-plot of the additional real eigenvalues versus $\widehat{a}$ for various ν. The number of matrices and its size vary in this plot for the Monte Carlo simulations (symbols). The statistical error of the numerics varies between 0.1% and 10% around the analytic result (solid curves).

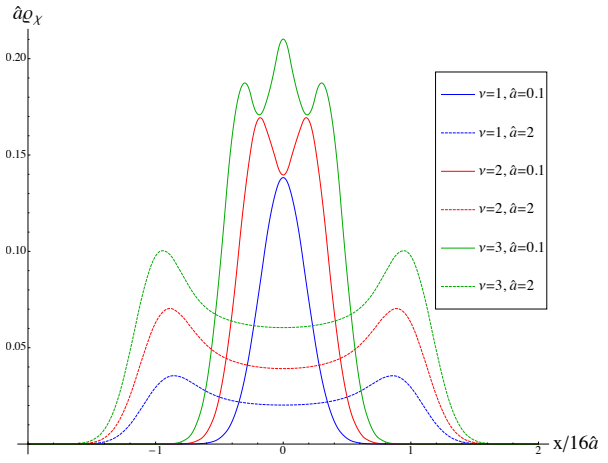

FIG. 5: For small lattice spacing (solid curves) the distribution ρ_χ is given by the GUE (See the legend for the values of the index and the lattice spacing). For $\widehat{a} \gg 1$ (dashed curves) the shape becomes ν-independent with two peaks at $\pm 8\widehat{a}^2$ that behave as $1/\sqrt{(8\widehat{a}^2)^2 - x^2}$ for $\widehat{a} \gg 1$.

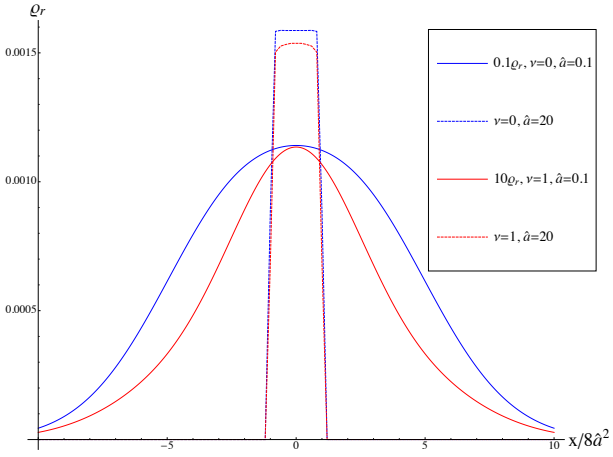

FIG. 6: The eigenvalue distribution $\rho_{\rm r}$ has a Gaussian shape for small $\widehat{a}$ (solid lines) but becomes box like with increasing lattice spacing (dashed curves). Notice that $\rho_{\rm r}$ for $((\nu, \widehat{a}) = (0, 0.1))$ and $((\nu, \widehat{a}) = (1, 0.1))$ is one order larger and one order smaller than shown in the diagram.

tral density of the ν-dimensional Gaussian unitary ensemble. For large lattice spacing it deforms into a curve with two peaks at $\pm 8\widehat{a}^2$ that up to an overall normalization is independent of ν and evolves into inverse square root singularities for $\widehat{a} \to \infty$.

Conclusions. Discretization effects become strong for $\widehat{a} \approx 0.5$. The oscillations of the spectral density in the continuum limit are no longer visible while the density of the complex eigenvalues develops a plateau with a width of $16\widehat{a}^2$. In terms of physical parameters, $\widehat{a} = \widetilde{a}\sqrt{W_8 V}$, with W_8 a low energy constant [10] and V the volume of space time, we have the condition that $\widetilde{a} \ll 1/\sqrt{W_8 V}$ to be close to the continuum limit.

In the regime of small lattice spacing, $\widehat{a} \approx 0.1$, the width of the distribution of the complex eigenvalues is given by $\sigma = 2\widetilde{a}\sqrt{W_8/V}/\Sigma$ whereas the spacing of the projection of these eigenvalues onto the imaginary axis is equal to $\Delta\lambda = \pi/\Sigma V$. We thus have that $\sigma/\Delta\lambda = 2\widehat{a}/\pi$, which allows us to extract a numerical value for W_8 from lattice simulations.

An important result is that the number of additional real modes is strongly suppressed for large ν. This implies that for large volumes when most configurations have an index $|\nu| > 0$, additional real modes are not much of a problem for lattice QCD simulations with Wilson fermions provided that $W_8 a^2 V \ll 1$.

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Surprising Pfaffian factorizations in Random Matrix Theory with Dyson index $\beta = 2$

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Abstract. In the past decades, determinants and Pfaffians were found for eigenvalue correlations of various random matrix ensembles. These structures simplify the average over a large number of ratios of characteristic polynomials to integrations over one and two characteristic polynomials only. Up to now it was thought that determinants occur for ensembles with Dyson index $\beta = 2$ whereas Pfaffians only for ensembles with $\beta = 1, 4$. We derive a non-trivial Pfaffian determinant for $\beta = 2$ random matrix ensembles which is similar to the one for $\beta = 1, 4$. Thus, it unveils a hidden universality of this structure. We also give a general relation between the orthogonal polynomials related to the determinantal structure and the skew-orthogonal polynomials corresponding to the Pfaffian. As a particular example we consider the chiral unitary ensembles in great detail.

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1. Introduction

Random matrix ensembles serve as simple models in a wide range of applications [1, 2, 3, 4, 5] which can be found in number theory [6, 7], disordered systems [1], quantum chaos [8], empirical data analysis [9, 10, 11], information theory [12], and quantum chromodynamics (QCD) [13]. The complexity of most systems prevents derivations of correlation functions whereas analytic results are accessible for the corresponding random matrix model. The reason for the applicability of random matrix theory lies in the universality of spectral statistics on certain scales like the local mean level spacing [14, 15, 16] or on the global scale [17, 18, 19, 20]. If the Lagrangian of the physical system drastically simplifies such that it is effectively described by global symmetries there might be a random matrix model fulfilling the same symmetries.

Already in the 60's and 70's [21, 22, 23, 24, 25], the k-point correlation functions of the Gaussian and circular ensembles for the three symmetries of orthogonal ($\beta = 1$; GOE/COE), unitary ($\beta = 2$; GUE/CUE) and unitary-symplectic ($\beta = 4$; GSE/CSE)

invariance were derived. They can be expressed as a single determinant for the unitary case and a single Pfaffian for $\beta \in \{1, 4\}$ where the integrals are pulled inside of these structures. Their matrix elements only depend on two eigenvalues which is a drastic simplification of the integrand. Since then many other random matrix ensembles were studied, e.g. the Ginibre ensembles [26, 27, 28, 29] and the the other two rotation groups $\mathrm{O}(N)$ and $\mathrm{USp}\,(2N)$ [30]. The k-point correlation functions as well as the averages over ratios of characteristic polynomials for many of these ensembles are determinants and Pfaffians with relatively simple entries only depending on one or two eigenvalues [31, 32, 33]. For a long time it was thought that determinants appear for ensembles with $\beta = 2$ and Pfaffians for the other two cases. In Refs. [34, 35] the general conditions where derived to find these structures. Thus all these particular random matrix ensembles were unified in one procedure to derive these structures.

Very recently a random matrix model for the Wilson Dirac operator was introduced [36] in lattice QCD. It generalizes the chiral GUE which was studied in a Hermitian version [36, 37, 38, 39] and a non-Hermitian one [40]. The eigenvalue correlations exhibit Pfaffians for the Hermitian [39] as well as for the non-Hermitian case [41] reflecting the structure found in Ref. [35]. This structure has to be also valid in the continuum limit which is the chiral GUE. Hence the question arises if the Pfaffian determinants obtained for the k-point correlation functions and thus for the averages over ratios of characteristic polynomials are much more general than conjectured in the broad literature.

Also in other intermediate random matrix ensembles Pfaffians were found. For example a similar situation arises in the transition from GUE to GOE or GSE [42, 34]. If the ensemble is purely a GUE then then the eigenvalue correlations can be cast into determinants whereas the smallest interaction with a GOE or a GSE yields a Pfaffian. It would be of theoretical, technical and numerical interest if all ensembles corresponding to $\beta = 2$ exhibit this phenomenon when coupling it to another random matrix ensemble. Such a property simplifies the spectral statistics of intermediate ensembles onto the behavior of the entries of the Pfaffian which are averages of one or two characteristic polynomials only.

Recently, Forrester and Sinclair introduced Pfaffians at $\beta = 2$. In Ref. [43] Sinclair extends the Pfaffian found for the partition function with $\beta = 1, 4$ to Hyperpfaffians with $\beta = L^2, L^2 + 1$ ($L \in \mathbb{N}$) which also comprises the $\beta = 2$ case. With help of these results the authors of Ref. [44] studied a log-gas on a ring with two interacting species. One component of this gas is described by a $\beta = 4$ log gas and the other one by a $\beta = 1, 2$ log gas. The Pfaffian determinants found in Refs. [43, 44] are similar to but not the same as the one derived in Sec. 4.

We derive Pfaffian determinants for averages over ratios of characteristic polynomials weighted by a joint probability density function factorizing in weights of the single eigenvalues apart from a squared Vandermonde determinant. This squared Vandermonde determinant can be cast into one determinant similar to the $\beta = 4$ case. Thus it fulfills the same condition as presented in Ref. [35] which implies a Pfaffian. This unifies all ten symmetry classes in the Cartan classification [45, 46] and exhibits a

hidden universal algebraic property in all of these ensembles.

An introduction of the main idea and of the important functions for the technique used here is given in Sec. 2. In Sec. 3, we recall some basics known about the determinantal structure obtained for averages over ratios of characteristic polynomials with respect to chiral unitary random matrix ensembles. In contrast to this structure we derive Pfaffians for the same correlation functions in Sec. 4. Thereby we discuss the Wilson-Dirac random matrix ensemble as a neat application and a good motivation of the derived Pfaffian determinant at the end of this section. The skew-orthogonal polynomials corresponding to the Pfaffian determinants are indeed closely related to the orthogonal polynomials which are found in the determinantal structures. This relation is shown in Sec. 5. In Sec. 6, we discuss the generalization of these results for chiral unitary ensembles to other random matrix ensembles like GUE and CUE.

2. Preliminaries

Structures found in supersymmetry are the key ingredient for the technique used in the ensuing sections. These structures allow to derive determinants as well as Pfaffians of averaged ratios of characteristic polynomials and, thus, k-point correlation functions for a large class of random matrix ensembles in a direct way. The main idea is to recognize that these structures are a pure algebraic property of the random matrix ensemble and not an analytic one. By an algebraic rearrangement of the integrand one gets the determinants and Pfaffians without explicitly calculating any integrals. This idea was first proposed in Refs. [34, 35].

The requirements to obtain determinants was traced back to a factorization of the probability density of the random matrix ensemble into densities for the single eigenvalues times two Vandermonde determinants (see Ref. [34]), i.e. the measure for the single eigenvalues has to be

$$d\mu(z) = \prod_{j=1}^{N} g_1(z_j) d[z_j] |\Delta_N(z)|^2 \tag{2.1}$$

with the Vandermonde determinant

$$\Delta_N(z) = \prod_{1 \le a < b \le N} (z_a - z_b) = (-1)^{N(N-1)/2} \det \left[z_a^{b-1} \right]_{1 \le a,b \le N}. \tag{2.2}$$

The variables z can be complex which correspond to ensembles related to biorthogonal polynomials [47]. For Pfaffians this requirement changes to a weight for pairs of eigenvalues and a single Vandermonde determinant [35], i.e.

$$d\mu(z) = \prod_{j=1}^{N} g_2(z_{2j-1}, z_{2j}) d[z_{2j-1}] d[z_{2j}] \Delta_{2N}(z). \tag{2.3}$$

If one of these two conditions are fulfilled then the technique presented in Refs. [34, 35] circumvents the integration theorem by Dyson and Metha [24, 25, 4, 48]. Moreover the approach of Refs. [34, 35] makes an integration theorem unnecessary at the end since it

is automatically fulfilled for random matrix ensembles traced back to measures of the form (2.1) or (2.3). This can be readily seen by the combination of the determinantal and Pfaffian factorization for averages over ratios of characteristic polynomials [34, 35], the representation of the orthogonal and skew-orthogonal polynomials as averages of the corresponding ensemble [49, 50, 47, 4, 48, 51] and the expressions of the kernels of the determinants and Pfaffians in orthogonal and skew-orthogonal polynomials [4, 48]. In Sections 3 and 4 we derive the k-point correlation function without using the integration theorem by Dyson and Metha.

Although, we do not explicitly need supersymmetry, in particular a superspace, some functions are quite useful to write the algebraic expressions of the calculations in a very compact, constructive and intuitive way. These functions have their origin in the theory of supermatrices. For the interested reader, good introductions in supersymmetry are given in Ref. [52] and in the appendix of Ref. [53]. Here we only recall some of these useful algebraic functions and notions.

A diagonal $(p/q) \times (p/q)$ supermatrix x consists of two blocks, $x = \mathrm{diag}\,(x_1, x_2)$. The $p \times p$ matrix x_1 and the $q \times q$ matrix x_2 are indeed diagonal, too. The supertrace "Str" and the superdeterminant "Sdet" of s is then defined by

$$\begin{aligned} \mathrm{Str}\, x &= \mathrm{tr}\, x_1 - \mathrm{tr}\, x_2 = \sum_{j=1}^{p} x_{j1} - \sum_{i=1}^{q} x_{i2}, \\ \mathrm{Sdet}\, x &= \frac{\det x_1}{\det x_2} = \frac{\prod_{j=1}^{p} x_{j1}}{\prod_{i=1}^{q} x_{i2}}. \end{aligned} \tag{2.4}$$

The crucial function of the method used here is

$$\begin{aligned} \mathrm{B}_{p/q}(x) &= \frac{\Delta_p(x_1)\Delta_q(x_2)}{\prod_{a,b}(x_{a1} - x_{b2})} \\ &= (-1)^{q(q-1)/2+(q+1)p} \det \begin{bmatrix} \left\{ \dfrac{1}{x_{a1} - x_{b2}} \right\}_{\substack{1 \le a \le p \\ 1 \le b \le q}} \\ \left\{ x_{b2}^{a-1} \right\}_{\substack{1 \le a \le q-p \\ 1 \le b \le q}} \end{bmatrix} \end{aligned} \tag{2.5}$$

for $p \le q$. It is the square root of a Berezinian,

$$\mathrm{B}^{\,2}_{p/q}(x) = \mathrm{Ber}^{\,(2)}_{p/q}(x), \tag{2.6}$$

which is the Jacobian in superspace when diagonalizing a Hermitian $(p/q) \times (p/q)$ supermatrix. The notation on the right hand side of Eq. (2.6) refers to the one used in Refs. [34, 35].

Everything we need for the method of Refs. [34, 35] are the functions "Sdet" and "B" embedded in an ordinary space like $\mathbb{R}^{p+q}$ or $\mathbb{C}^{p+q}$. Hence those readers who are not accustomed to supersymmetry may consider these functions as ordinary, rational functions.

3. Review of chiral unitary random matrices

We consider the anti-Hermitian random matrix

$$D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} \tag{3.1}$$

which is distributed by the density

$$P(D)d[D] = \exp[-\alpha \operatorname{tr} V(WW^\dagger)] \prod_{a,b} d\,\operatorname{Re} W_{ab}\; d\,\operatorname{Im} W_{ab} \tag{3.2}$$

with a non-zero normalization constant. In particular it serves as a model for the Dirac operator in QCD [13]. The constant α is proportional to n. The matrix W is a $n\times(n+\nu)$ rectangular matrix. Each of the $n(n+\nu)$ entries of W is a complex number which might be statistically coupled by the arbitrary density P. The parameter ν with $0 \le \nu \le n$ is the topological charge or also known as index of the Dirac operator such that D has ν generic zero eigenmodes. The potential V is invariant under the group $\mathrm{U}(n)$, i.e.

$$V(UWW^\dagger U^\dagger) = UV(WW^\dagger)U^\dagger, \tag{3.3}$$

and is chosen such that all moments of the ensemble over $\mathbb{C}^{n\times(n+\nu)}$ exist. In the simplest case P is Gaussian. Nevertheless the arguments given here are also true for an arbitrary potential. We only need the property

$$P\left(\left[\begin{array}{c|cc} 0 & \Lambda & 0 \\ \hline -\Lambda & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]\right) = \exp[-\alpha \operatorname{tr} V(\Lambda^2)] = \prod_{j=1}^{n} \exp[-\alpha V(\lambda_j^2)] \tag{3.4}$$

for the matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with the singular values $0 \le \lambda_1 \le \ldots \le \lambda_n$ of W, i.e. there are $U \in \mathrm{U}(n)$ and $V \in \mathrm{U}(n+\nu)$ with

$$D = \operatorname{diag}(U, V) \left[\begin{array}{c|cc} 0 & \Lambda & 0 \\ \hline -\Lambda & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \operatorname{diag}(U^\dagger, V^\dagger). \tag{3.5}$$

In this basis the measure (3.2) can be written as

$$\begin{aligned} P(D)d[D] = \frac{\operatorname{Vol}_n \operatorname{Vol}_{n+\nu}}{\operatorname{Vol}_1^n \operatorname{Vol}_\nu} \Delta_n^2(\Lambda^2) \prod_{j=1}^{n} \exp[-\alpha V(\lambda_j^2)] \lambda_j^{2\nu+1} d\lambda_j \\ \times\; d\mu_{\mathrm{U}(n)/\mathrm{U}^n(1)}(U) d\mu_{\mathrm{U}(n+\nu)/\mathrm{U}(\nu)}(V). \end{aligned} \tag{3.6}$$

The abbreviation of the constant

$$\operatorname{Vol}_l = \prod_{j=1}^{l} \frac{2\pi^j}{(j-1)!} \tag{3.7}$$

refers to the volume of the unitary group $\mathrm{U}(l)$. Thus, the prefactor in Eq. (3.6) is the volume of the coset $[\mathrm{U}(n) \times \mathrm{U}(n+\nu)]/[\mathrm{U}^n(1) \times \mathrm{U}(\nu)]$. The measure $d\mu_{\mathfrak{G}}$ is the normalized Haar measure of the coset $\mathfrak{G}$.

An important quantity to analyze the eigenvalue statistics of this ensemble is the average over ratios of characteristic polynomials with respect to D, i.e.

$$Z^{(n,\nu)}_{k_1/k_2}(\kappa) = \int\limits_{\mathbb{C}^{n\times(n+\nu)}} \frac{\prod_{j=1}^{k_2} \det(D - \imath\kappa_{j2}\mathbb{1}_{2n+\nu})}{\prod_{j=1}^{k_1} \det(D - \imath\kappa_{j1}\mathbb{1}_{2n+\nu})} P(D)d[D] \tag{3.8}$$

with the diagonal, non-degenerate $(k_1/k_2)\times(k_1/k_2)$ supermatrix $\kappa = \mathrm{diag}\,(\kappa_1,\kappa_2) = \mathrm{diag}\,(\kappa_{11},\ldots,\kappa_{k_1 1},\ \kappa_{12},\ldots,\kappa_{k_2 2})$ and the $2n+\nu$ dimensional unit matrix $\mathbb{1}_{2n+\nu}$. This average is also known as the partition function with k_1 bosonic and k_2 fermionic flavors in QCD [54, 55, 15]. The variables κ_{j1} are complex numbers with a non-vanishing imaginary part such that the integral is well defined. The partition function (3.8) is simply related to the matrix Green function and, thus, to the k-point correlation function by derivatives with respect to κ.

The joint probability density (3.6) is of the class studied in Ref. [34] and can, therefore, be written as a determinant. This was derived in many articles before [56, 54, 55]. The crucial idea presented in Ref. [34] is the combination of the ratio of characteristic polynomials (3.8) with the two Vandermonde determinants (3.6) to square roots of Berezinians (2.5), i.e.

$$\Delta_n^2(\Lambda^2)\frac{\prod_{j=1}^{k_2} \det(\Lambda^2 - \kappa_{j2}^2\mathbb{1}_n)}{\prod_{j=1}^{k_1} \det(\Lambda^2 - \kappa_{j1}^2\mathbb{1}_n)} = \frac{\mathrm{B}_{\,l_{11}/l_{21}+n}(\widetilde{\kappa}_1^2,\Lambda^2)\mathrm{B}_{\,l_{12}/l_{22}+n}(\widetilde{\kappa}_2^2,\Lambda^2)}{\mathrm{B}_{\,l_{11}/l_{21}}(\widetilde{\kappa}_1^2)\mathrm{B}_{\,l_{12}/l_{22}}(\widetilde{\kappa}_1^2)} \tag{3.9}$$

for any choice of natural numbers $l_{11}+l_{12}=k_1$ and $l_{21}+l_{22}=k_2$.

In Eq. (3.9), we split the supermatrix κ into the two sets $\widetilde{\kappa}_1 = \mathrm{diag}\,(\widetilde{\kappa}_{11},\widetilde{\kappa}_{21}) = \mathrm{diag}\,(\kappa_{11},\ldots,\kappa_{l_{11}1},\kappa_{12},\ldots,\kappa_{l_{21}2})$ and $\widetilde{\kappa}_2 = \mathrm{diag}\,(\widetilde{\kappa}_{12},\widetilde{\kappa}_{22}) = \mathrm{diag}\,(\kappa_{l_{11}+1,1},\ldots,\kappa_{k_1 1},\kappa_{l_{21}+1,2},\ldots,\kappa_{k_2 2})$. The choice how we split this set is arbitrary and, thus, we get equivalent but not trivially related results. This was already recognized by the authors of Ref. [57] for products of characteristic polynomials. Let $d_1 = n + l_{21} - l_{11}$ and $d_2 = n + l_{22} - l_{12}$. The interesting case is $d_1, d_2 \geq 0$ because we want to discuss the limit $n\to\infty$ and k_1, k_2 fixed, at the end of this section. The other cases are discussed in Ref. [34].

Without loss of generality we assume $d_1 \leq d_2$. We rearrange the integrand (3.8) with the help of Eq. (3.9) which yields

$$\begin{aligned} Z^{(n,\nu)}_{k_1/k_2}(\kappa) &\propto \mathrm{Sdet}^{\,-\nu}\kappa \int \frac{\mathrm{B}_{\,l_{11}/l_{21}+n}(\widetilde{\kappa}_1^2,\Lambda^2)\mathrm{B}_{\,l_{12}/l_{22}+n}(\widetilde{\kappa}_2^2,\Lambda^2)}{\mathrm{B}_{\,l_{11}/l_{21}}(\widetilde{\kappa}_1^2)\mathrm{B}_{\,l_{12}/l_{22}}(\widetilde{\kappa}_1^2)} \prod_{j=1}^{n} \exp[-\alpha V(\lambda_j^2)]\lambda_j^{2\nu+1}d\lambda_j \\ &\propto \mathrm{Sdet}^{\,-\nu}\kappa \int \frac{\prod_{j=1}^{n} \exp[-\alpha V(\lambda_j^2)]\lambda_j^{2\nu+1}d\lambda_j}{\mathrm{B}_{\,l_{11}/l_{21}}(\widetilde{\kappa}_1^2)\mathrm{B}_{\,l_{12}/l_{22}}(\widetilde{\kappa}_1^2)} \end{aligned}$$

$$\times \det \begin{bmatrix} \left\{ \dfrac{1}{\kappa_{a1}^2 - \kappa_{b2}^2} \right\}_{\substack{1\le a\le l_{11}\\ 1\le b\le l_{21}}} & \left\{ \dfrac{1}{\kappa_{a1}^2 - \lambda_{b2}^2} \right\}_{\substack{1\le a\le l_{11}\\ 1\le b\le n}} \\ \left\{ \kappa_{b2}^{2(a-1)} \right\}_{\substack{1\le a\le d_1\\ 1\le b\le l_{21}}} & \left\{ \lambda_{b2}^{2(a-1)} \right\}_{\substack{1\le a\le d_1\\ 1\le b\le n}} \end{bmatrix}$$

$$\times \det \begin{bmatrix} \left\{ \dfrac{1}{\kappa_{a1}^2 - \kappa_{b2}^2} \right\}_{\substack{l_{11}+1\le a\le k_1\\ l_{21}+1\le b\le k_2}} & \left\{ \dfrac{1}{\kappa_{a1}^2 - \lambda_{b2}^2} \right\}_{\substack{l_{11}+1\le a\le k_1\\ 1\le b\le n}} \\ \left\{ \kappa_{b2}^{2(a-1)} \right\}_{\substack{1\le a\le d_2\\ l_{21}+1\le b\le k_2}} & \left\{ \lambda_{b2}^{2(a-1)} \right\}_{\substack{1\le a\le d_2\\ 1\le b\le n}} \end{bmatrix}. \tag{3.10}$$

Applying the generalized Andréief integration theorem [58, 34] we obtain

$$Z_{k_1/k_2}^{(n,\nu)}(\kappa) \propto \frac{\mathrm{Sdet}^{-\nu}\kappa}{\mathrm{B}_{l_{11}/l_{21}}(\widetilde{\kappa}_1^2)\mathrm{B}_{l_{12}/l_{22}}(\widetilde{\kappa}_1^2)} \tag{3.11}$$

$$\times \det \begin{bmatrix} 0 & \left\{ \dfrac{1}{\kappa_{b1}^2 - \kappa_{a2}^2} \right\}_{\substack{l_{21}+1\le a\le k_2\\ l_{11}+1\le b\le k_1}} & \left\{ \kappa_{a2}^{2(b-1)} \right\}_{\substack{l_{21}+1\le a\le k_2\\ 1\le b\le d_2}} \\ \left\{ \dfrac{1}{\kappa_{a1}^2 - \kappa_{b2}^2} \right\}_{\substack{1\le a\le l_{11}\\ 1\le b\le l_{21}}} & \{F(\kappa_{a1}, \kappa_{b1})\}_{\substack{1\le a\le l_{11}\\ l_{11}+1\le b\le k_1}} & \{F_b(\kappa_{a1})\}_{\substack{1\le a\le l_{11}\\ 1\le b\le d_2}} \\ \left\{ \kappa_{b2}^{2(a-1)} \right\}_{\substack{1\le a\le d_1\\ 1\le b\le l_{21}}} & \{F_a(\kappa_{b1})\}_{\substack{1\le a\le d_1\\ l_{11}+1\le b\le k_1}} & \{M_{ab}\}_{\substack{1\le a\le d_1\\ 1\le b\le d_2}} \end{bmatrix}$$

Notice that Andréief's integration theorem as well as its generalization is only an algebraic rearrangement of the integrals without explicitly calculating any integral. The functions F and F_a are one dimensional integrals and their explicit expressions are not so important as we will see in the discussion after Eq. (3.15). For the interested reader we refer to Ref. [34] where the explicit integrals are given for general random matrix ensembles corresponding to determinants ($\beta = 2$). The constant $d_1 \times d_2$ matrix $M = [M_{ab}]$ is given by

$$M_{ab} = \int_{\mathbb{R}} \lambda^{2(a+b-2)} \exp[-\alpha V(\lambda^2)] \lambda^{2\nu+1} d\lambda \tag{3.12}$$

and thus generates the moments of the measure.

In the next step we use the identity

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det[A - BD^{-1}C] \tag{3.13}$$

for arbitrary matrices A, B and C and an invertible matrix D. For the matrix D we choose the $d_1 \times d_1$ matrix

$$D = [M_{ab}]_{1\le a,b\le d_1} \tag{3.14}$$

which is only a part of the full rectangular matrix M appearing in Eq. (3.11). The determinant of D is proportional to the normalization constant of the ensemble (4.2) and M is therefore invertible. Employing Eq. (3.13) we find

$$Z_{k_1/k_2}^{(n,\nu)}(\kappa) = \frac{1}{\mathrm{B}_{l_{11}/l_{21}}(\widetilde{\kappa}_1^2)\mathrm{B}_{l_{12}/l_{22}}(\widetilde{\kappa}_2^2)} \tag{3.15}$$

$$\times \det \begin{bmatrix} \left\{ G_1^{(d_1)}(\kappa_{a2}, \kappa_{b2}) \right\}_{\substack{1\le a\le l_{21} \\ l_{21}+1\le b\le k_2}} & \left\{ G_2^{(d_1)}(\kappa_{b1}, \kappa_{a2}) \right\}_{\substack{1\le a\le l_{21} \\ 1\le b\le l_{11}}} \\ \left\{ G_2^{(d_1)}(\kappa_{a1}, \kappa_{b2}) \right\}_{\substack{l_{11}+1\le a\le k_1 \\ l_{21}+1\le b\le k_2}} & \left\{ G_3^{(d_1)}(\kappa_{a1}, \kappa_{b1}) \right\}_{\substack{l_{11}+1\le a\le k_1 \\ 1\le b\le l_{11}}} \\ \left\{ H_1^{(a)}(\kappa_{b2}) \right\}_{\substack{d_1+1\le a\le d_2 \\ l_{21}+1\le b\le k_2}} & \left\{ H_2^{(a)}(\kappa_{b1}) \right\}_{\substack{d_1+1\le a\le d_2 \\ 1\le b\le l_{11}}} \end{bmatrix}.$$

In the last step we identify the functions $G_1^{(d_1)}$, $G_2^{(d_1)}$, $G_3^{(d_1)}$, $H_1^{(a)}$ and $H_2^{(a)}$ by considering the particular choices $(l_{11}, l_{12}, l_{21}, l_{22}) \in \{(0,0,1,1), (1,0,1,0), (1,1,0,0), (0,0,0,1), (1,0,0,0)\}$. In all of these cases the determinant reduces to one of the entries. Then we obtain

$$\frac{Z_{k_1/k_2}^{(n,\nu)}(\kappa)}{Z_{0/0}^{(n,\nu)}} = \frac{(-1)^{k_1(k_1-1)/2+(l_{21}+1)(k_1+1)+(l_{11}+1)(k_2+1)}}{\mathrm{B}_{\,l_{11}/l_{21}}(\widetilde{\kappa}_1^2)\mathrm{B}_{\,l_{12}/l_{22}}(\widetilde{\kappa}_2^2)} \frac{\prod_{j=0}^{d_1-1} h_j^{(\nu)}}{\prod_{j=0}^{n-1} h_j^{(\nu)}} \tag{3.16}$$

$$\times \det \begin{bmatrix} \left\{ -\dfrac{Z_{0/2}^{(d_1-1,\nu)}(\kappa_{a2}, \kappa_{b2})}{h_{d_1-1}^{(\nu)} Z_{0/0}^{(d_1-1,\nu)}} \right\}_{\substack{1\le a\le l_{21} \\ l_{21}+1\le b\le k_2}} & \left\{ \dfrac{1}{Z_{0/0}^{(d_1,\nu)}} \dfrac{Z_{1/1}^{(d_1,\nu)}(\kappa_{b1}, \kappa_{a2})}{(\kappa_{b1}^2 - \kappa_{a2}^2)} \right\}_{\substack{1\le a\le l_{21} \\ 1\le b\le l_{11}}} \\ \left\{ \dfrac{1}{Z_{0/0}^{(d_1,\nu)}} \dfrac{Z_{1/1}^{(d_1,\nu)}(\kappa_{a1}, \kappa_{b2})}{(\kappa_{a1}^2 - \kappa_{b2}^2)} \right\}_{\substack{l_{11}+1\le a\le k_1 \\ l_{21}+1\le b\le k_2}} & \left\{ \dfrac{h_{d_1}^{(\nu)}}{Z_{0/0}^{(d_1+1,\nu)}} Z_{2/0}^{(d_1+1,\nu)}(\kappa_{a1}, \kappa_{b1}) \right\}_{\substack{l_{11}+1\le a\le k_1 \\ 1\le b\le l_{11}}} \\ \left\{ \dfrac{Z_{0/1}^{(a-1,\nu)}(\kappa_{b2})}{Z_{0/0}^{(a-1,\nu)}} \right\}_{\substack{d_1+1\le a\le d_2 \\ l_{21}+1\le b\le k_2}} & \left\{ \dfrac{h_{a-1}^{(\nu)}}{Z_{0/0}^{(a,\nu)}} Z_{1/0}^{(a,\nu)}(\kappa_{b1}) \right\}_{\substack{d_1+1\le a\le d_2 \\ 1\le b\le l_{11}}} \end{bmatrix}$$

for the partition function (3.8) which is a particular result of the general one derived in Ref. [34].

The determinant (3.16) interpolates between one-point and two-point kernels as the entries of the determinant. We emphasize again the choice of the numbers $0 \le l_{11} \le k_1$ and $0 \le l_{21} \le k_1$ and the splitting of κ are arbitrary. The particular choice $l_{11} = k_1$ and $l_{21} = 0$ yields the $k_1 + k_2$ dimensional determinant with one-point kernels considered in Refs. [54, 15]. This choice is suitable for the microscopic limit in chiral random matrix theory. For bulk and soft edge correlations [15] the representation in two point correlations are the better choice to make contact with other random matrix ensembles [17, 18, 19, 16]. This case relates to the choice $l_{11} = k_1$ and $l_{21} = k_2$ for $k_2 \le k_1$ and $l_{11} = 0$ and $l_{21} = 0$ for $k_2 \ge k_1$.

The k-point correlation function at the k variables $x = \mathrm{diag}\,(x_1, \ldots, x_k)$ is given by

$$\begin{aligned} R_k^{(n,\nu)}(x) &\propto \int_{\mathbb{R}_+^{n-k}} \Delta_n^2(\mathrm{diag}\,(x^2, \Lambda^2)) \exp[-\alpha\,\mathrm{tr}\,V(x^2) - \alpha\,\mathrm{tr}\,V(\Lambda^2)] \det{}^{2\nu+1} x \prod_{j=1}^{n-k} \lambda_j^{2\nu+1} d\lambda_j \\ &\propto \Delta_k^2(x^2) \det x \exp[-\alpha\,\mathrm{tr}\,V(x^2)] Z_{0/2k}^{(n-k,\nu)}(\mathrm{diag}\,(x, -x)). \end{aligned} \tag{3.17}$$

Now we employ the formula (3.16) for $(l_{11}, l_{12}, l_{21}, l_{22}) = (0, 0, k, k)$ and find the result

$$R_k^{(n,\nu)}(x) \propto \det \left[\sqrt{x_a x_b} \exp[-\alpha(V(x_a^2) + V(x_b^2))/2] Z_{0/2}^{(n-1,\nu)}(x_a, -x_b) \right]_{1\le a,b\le k}. \tag{3.18}$$

Since $Z_{0/2}^{(n-1,\nu)}(x_a, -x_b) = (-1)^\nu Z_{0/2}^{(n-1,\nu)}(x_a, x_b)$ this agrees with the general formula for $\beta = 2$ ensembles [4]. Please notice that we derived this formula without using the integration theorem by Dyson and Mehta [24, 25, 4, 48].

The constant $h_j^{(\nu)}$ in Eq. (3.16) is the normalization constant of the orthogonal polynomial

$$p_j^{(\nu)}(x^2) = \frac{(-1)^j}{(-\imath x)^\nu} \frac{Z_{0/1}^{(j,\nu)}(x)}{Z_{0/0}^{(j,\nu)}}. \tag{3.19}$$

These polynomials solve the orthogonality relation

$$\int_0^\infty p_j^{(\nu)}(x^2) p_i^{(\nu)}(x^2) x^{2\nu+1} \exp[-\alpha V(x^2)] dx = h_j^{(\nu)} \delta_{ji}. \tag{3.20}$$

The authors of Ref. [59] have shown that these polynomials fulfill a recursion relation with respect to the topological charge ν by

$$\frac{p_j^{(\nu+1)}(x)}{p_j^{(\nu+1)}(0)} = \frac{1}{x} \frac{p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x) - p_{j+1}^{(\nu)}(0) p_j^{(\nu)}(x)}{p_j^{(\nu)}(0) p_{j+1}^{(\nu)\prime}(0) - p_{j+1}^{(\nu)}(0) p_j^{(\nu)\prime}(0)} \tag{3.21}$$

which is quite useful by taking the limit $n \to \infty$. This relation follows when setting $m = 0$ in Eq. (12) of Ref. [59]. One can readily prove identity (3.21) by showing the orthogonality relation (3.20) for the right hand side with respect to the $\nu + 1$ measure, i.e.

$$\begin{aligned}
&\int_0^\infty \frac{p_j^{(\nu)}(0) p_{j+1}^{(\nu)}(x^2) - p_{j+1}^{(\nu)}(0) p_j^{(\nu)}(x^2)}{x^2} \frac{p_l^{(\nu)}(0) p_{l+1}^{(\nu)}(x^2) - p_{l+1}^{(\nu)}(0) p_l^{(\nu)}(x^2)}{x^2} x^{2\nu+3} e^{-\alpha V(x^2)} dx \\
&\propto \int_0^\infty \sum_{a=0}^{j} \frac{p_a^{(\nu)}(0) p_a^{(\nu)}(x^2)}{h_a^{(\nu)}} (p_l^{(\nu)}(0) p_{l+1}^{(\nu)}(x^2) - p_{l+1}^{(\nu)}(0) p_l^{(\nu)}(x^2)) x^{2\nu+1} e^{-\alpha V(x^2)} dx \\
&\propto \delta_{jl},
\end{aligned} \tag{3.22}$$

where we used the Christoffel-Darboux formula. The monic normalization of $p_j^{(\nu)}(x) = x^j + \ldots$ for all j and ν explains the choice of the constants.

The Cauchy transform of $p_j^{(\nu)}$ is related to the partition function with one bosonic flavor by

$$\begin{aligned}
\widehat{p}_j^{(\nu)}(x^2) &= \int_0^\infty \frac{p_j^{(\nu)}(\lambda^2)}{\lambda^2 - x^2} \lambda^{2\nu+1} \exp[-\alpha V(\lambda^2)] d\lambda \\
&= (-1)^j (-\imath x)^\nu \frac{h_j^{(\nu)}}{Z_{0/0}^{(j+1,\nu)}} Z_{1/0}^{(j+1,\nu)}(x).
\end{aligned} \tag{3.23}$$

In the result (3.16) we recognize that the choices $(l_{11}, l_{12}, l_{21}, l_{22}) = (0,0,1,1), (1,0,1,0), (1,1,0,0)$ yield the same partition functions as the choices $(l_{11}, l_{12}, l_{21}, l_{22}) = (0,0,0,2), (1,0,0,1), (2,0,0,0)$, respectively. Therefore the two-flavor partition

functions in Eq. (3.16) can also be expressed in the orthogonal polynomials (3.19) and their Cauchy transforms (3.23), i.e.

$$\frac{Z_{0/2}^{(d_1-1,\nu)}(\kappa_{a2},\kappa_{b2})}{Z_{0/0}^{(d_1-1,\nu)}} = -\frac{(-\kappa_{a2}\kappa_{b2})^\nu}{\kappa_{a2}^2-\kappa_{b2}^2}\det\begin{bmatrix} p_{d_1-1}^{(\nu)}(\kappa_{a2}^2) & p_{d_1-1}^{(\nu)}(\kappa_{b2}^2) \\ p_{d_1}^{(\nu)}(\kappa_{a2}^2) & p_{d_1}^{(\nu)}(\kappa_{b2}^2)\end{bmatrix}, \tag{3.24}$$

$$\frac{Z_{2/0}^{(d_1+1,\nu)}(\kappa_{a1},\kappa_{b1})}{Z_{0/0}^{(d_1+1,\nu)}} = \frac{1}{h_{d_1}^{(\nu)}h_{d_1-1}^{(\nu)}}\frac{1}{(-\kappa_{a1}\kappa_{b1})^\nu(\kappa_{a1}^2-\kappa_{b1}^2)}\det\begin{bmatrix} \widehat{p}_{d_1-1}^{(\nu)}(\kappa_{a1}^2) & \widehat{p}_{d_1-1}^{(\nu)}(\kappa_{b1}^2) \\ \widehat{p}_{d_1}^{(\nu)}(\kappa_{a1}^2) & \widehat{p}_{d_1}^{(\nu)}(\kappa_{b1}^2)\end{bmatrix}, \tag{3.25}$$

$$\frac{Z_{1/1}^{(d_1,\nu)}(\kappa_{a1},\kappa_{b2})}{Z_{0/0}^{(d_1,\nu)}} = \frac{1}{h_{d_1-1}^{(\nu)}}\left(\frac{\kappa_{b2}}{\kappa_{a1}}\right)^\nu\det\begin{bmatrix} \widehat{p}_{d_1-1}^{(\nu)}(\kappa_{a1}^2) & p_{d_1-1}^{(\nu)}(\kappa_{b2}^2) \\ \widehat{p}_{d_1}^{(\nu)}(\kappa_{a1}^2) & p_{d_1}^{(\nu)}(\kappa_{b2}^2)\end{bmatrix}. \tag{3.26}$$

These three relations are already well known [4, 48]. They can also be derived with help of the Christoffel-Darboux formula.

The structure (3.16) is a general property of ensembles with a joint probability density including a squared Vandermonde determinant as considered in Sec. 4.2 of Ref. [34] whereas the relations (3.24)-(3.26) have to be slightly modified for other ensembles.

In the microscopic limit the authors of Refs. [14, 15] have shown that for a generic potential V the orthogonal polynomials and their Cauchy transforms become

$$p_n^{(\nu)}\left(\frac{x^2}{(cn)^2}\right) \overset{n\gg1}{\propto} \frac{J_\nu(x)}{x^\nu}, \tag{3.27}$$

$$\widehat{p}_n^{(\nu)}\left(\frac{x^2}{(cn)^2}\right) \overset{n\gg1}{\propto} x^\nu K_\nu(x), \tag{3.28}$$

where c is a constant depending on the potential V. The functions J_ν and K_ν are the Bessel function of the first kind and the modified one of the second kind, respectively. Hence in the microscopic limit the partition function (3.8) is

$$Z_{k_1/k_2}^{(n,\nu)}\left(\frac{\kappa}{cn}\right) \overset{n\gg1}{\propto} \frac{1}{\mathrm{B}_{l_{11}/l_{21}}(\widetilde{\kappa}_1^2)\mathrm{B}_{l_{12}/l_{22}}(\widetilde{\kappa}_2^2)} \times \det\begin{bmatrix} \left\{I_\nu^{(1)}(\kappa_{a2},\kappa_{b2})\right\}_{\substack{1\le a\le l_{21}\\ l_{21}+1\le b\le k_2}} & \left\{I_\nu^{(2)}(\kappa_{b1},\kappa_{a2})\right\}_{\substack{1\le a\le l_{21}\\ 1\le b\le l_{11}}} \\ \left\{I_\nu^{(2)}(\kappa_{a1},\kappa_{b2})\right\}_{\substack{l_{11}+1\le a\le k_1\\ l_{21}+1\le b\le k_2}} & \left\{I_\nu^{(3)}(\kappa_{a1},\kappa_{b1})\right\}_{\substack{l_{11}+1\le a\le k_1\\ 1\le b\le l_{11}}} \\ \left\{\kappa_{b2}^a J_{\nu+a}(\kappa_{b2})\right\}_{\substack{0\le a\le d_2-d_1-1\\ l_{21}+1\le b\le k_2}} & \left\{\kappa_{b1}^a K_{\nu+a}(\kappa_{b1})\right\}_{\substack{0\le a\le d_2-d_1-1\\ 1\le b\le l_{11}}}\end{bmatrix}, \tag{3.29}$$

where

$$I_\nu^{(1)}(\kappa_{a2},\kappa_{b2}) = \begin{cases} \dfrac{\kappa_{a2}J_{\nu-1}(\kappa_{a2})J_\nu(\kappa_{b2}) - \kappa_{b2}J_\nu(\kappa_{a2})J_{\nu-1}(\kappa_{b2})}{\kappa_{a2}^2-\kappa_{b2}^2}, & a\neq b, \\ \dfrac{J_{\nu+1}(\kappa_{a2})J_{\nu-1}(\kappa_{a2}) - J_\nu^2(\kappa_{a2})}{2}, & a=b, \end{cases} \tag{3.30}$$

$$I_\nu^{(2)}(\kappa_{a1},\kappa_{b2}) = \frac{\kappa_{a1}K_{\nu-1}(\kappa_{a1})J_\nu(\kappa_{b2}) - \kappa_{b2}K_\nu(\kappa_{a1})J_{\nu-1}(\kappa_{b2})}{\kappa_{a1}^2-\kappa_{b2}^2}, \tag{3.31}$$

$$I_\nu^{(3)}(\kappa_{a1},\kappa_{b1}) = \begin{cases} \dfrac{\kappa_{a1}K_{\nu-1}(\kappa_{a1})K_\nu(\kappa_{b1}) - \kappa_{b1}K_\nu(\kappa_{a1})K_{\nu-1}(\kappa_{b1})}{\kappa_{a1}^2 - \kappa_{b1}^2}, & a \neq b, \\ \dfrac{K_{\nu+1}(\kappa_{a1})K_{\nu-1}(\kappa_{a1}) - K_\nu^2(\kappa_{a1})}{2}, & a = b. \end{cases} \tag{3.32}$$

This is the well known result found in the literature [56, 54, 55].

4. Derivation of the Pfaffian determinant

In subsection 4.1 we derive a Pfaffian determinant for the same class of chiral random matrix ensembles discussed in Sec. 3. A neat application of this Pfaffian is presented in subsection 4.2. This example is the random matrix model for the Wilson-Dirac operator in lattice QCD [36, 37, 38, 39, 40].

4.1. Pfaffian determinants in chiral random matrix theory

We show that the representations in determinants (3.16) are not the only existing ones for chiral unitary ensembles. A non-trivial Pfaffian can be derived for the partition function by noticing that the square of the Vandermonde in the measure (3.6) can be rewritten as one Vandermonde determinant of the variables $\pm\lambda_j$, i.e.

$$\Delta_n^2(\Lambda^2) = (-1)^{n(n-1)/2}\frac{\Delta_{2n}(\Lambda,-\Lambda)}{2^n \det\Lambda}. \tag{4.1}$$

The determinant of Λ will be put into the weight later on, cf. Eqs. (4.2) and (4.3) below. Considering the Wilson random matrix theory [36, 37, 38, 39, 40] such a splitting arises in a natural way for finite lattice spacing. Then an eigenvalue pair $\pm\imath\lambda_j$ becomes either a complex conjugated pair or two independent real eigenvalues corresponding to a pair of eigenvectors with positive and negative chirality. Hence, the Pfaffian resulting from the single Vandermonde determinant (4.1) is the one which is generalized to non-zero lattice spacing and not the determinant [39, 41].

This allows us to define an anti-symmetric two-point measure on $\mathbb{R}^2$

$$g(x_1,x_2) = \frac{|x_1x_2|^\nu}{4}\exp\left[-\alpha\frac{V(x_1^2)+V(x_2^2)}{2}\right]\delta(x_1+x_2)[\Theta(x_1)-\Theta(x_2)], \tag{4.2}$$

where Θ is the Heaviside distribution. Then we consider the measure

$$D[\lambda] = \frac{\mathrm{Vol}_n\mathrm{Vol}_{n+\nu}}{\mathrm{Vol}_1^n\mathrm{Vol}_\nu}\Delta_{2n}(\lambda)\prod_{j=1}^n g(\lambda_{2j-1},\lambda_{2j})d\lambda_{2j}d\lambda_{2j-1} \tag{4.3}$$

over $2n$ independent eigenvalues instead of the measure (3.6). This measure fulfills the general condition for finding a Pfaffian, cf. Ref. [35] and see also Eq. (2.3).

The partition function (3.8) can be expressed in terms of this new measure,

$$Z_{k_1/k_2}^{(n,\nu)}(\kappa) = \frac{(-1)^{n(k_1+k_2)}}{n!}\mathrm{Sdet}^{-\nu}(-\imath\kappa)\int_{\mathbb{R}^{2n}}\prod_{a=1}^{2n}\frac{\prod_{j=1}^{k_2}(\kappa_{j2}-\lambda_a)}{\prod_{j=1}^{k_1}(\kappa_{j1}-\lambda_a)}D[\lambda]. \tag{4.4}$$

In the first step we extend the Vandermonde determinant (4.1) with the characteristic polynomials,

$$Z^{(n,\nu)}_{k_1/k_2}(\kappa) = (-1)^{n(k_1+k_2)} \frac{\mathrm{Vol}_n \mathrm{Vol}_{n+\nu}}{n! \mathrm{Vol}_1^n \mathrm{Vol}_\nu} \mathrm{Sdet}^{-\nu}(-\imath\kappa) \tag{4.5}$$
$$\times \int\limits_{\mathbb{R}^{2n}} \frac{\mathrm{B}_{k_1/k_2+2n}(\kappa,\lambda)}{\mathrm{B}_{k_1/k_2}(\kappa)} \prod_{j=1}^{n} g(\lambda_{2j},\lambda_{2j-1}) d\lambda_{2j} d\lambda_{2j-1}.$$

This representation is apart from the z_{2N+1}-integral of the form as in Eq. (3.3) in Ref. [35]. Notice that in this extension we do not have the same freedom as in the determinantal case (3.9) since there is only one Vandermonde determinant in the integrand (4.3). Let $d = 2n + k_2 - k_1 \geq 0$. Then we employ the representation of the function "B" as a determinant, see Eq. (2.5),

$$Z^{(n,\nu)}_{k_1/k_2}(\kappa) \propto \frac{\mathrm{Sdet}^{-\nu}\kappa}{\mathrm{B}_{k_1/k_2}(\kappa)} \int\limits_{\mathbb{R}^{2n}} \prod_{j=1}^{n} g(\lambda_{2j},\lambda_{2j-1}) d\lambda_{2j} d\lambda_{2j-1} \tag{4.6}$$
$$\times \det \begin{bmatrix} \left\{ \dfrac{1}{\kappa_{a1} - \kappa_{b2}} \right\}_{\substack{1\leq a\leq k_1 \\ 1\leq b\leq k_2}} & \left\{ \dfrac{1}{\kappa_{a1} - \lambda_{b2}} \right\}_{\substack{1\leq a\leq k_1 \\ 1\leq b\leq 2n}} \\ \left\{ \kappa_{b2}^{a-1} \right\}_{\substack{1\leq a\leq d \\ 1\leq b\leq k_2}} & \left\{ \lambda_{b2}^{a-1} \right\}_{\substack{1\leq a\leq d \\ 1\leq b\leq 2n}} \end{bmatrix}.$$

The generalized de Bruijn integration theorem [60, 34] can be applied now which yields

$$Z^{(n,\nu)}_{k_1/k_2}(\kappa) \propto \frac{\mathrm{Sdet}^{-\nu}\kappa}{\mathrm{B}_{k_1/k_2}(\kappa)} \tag{4.7}$$
$$\times \mathrm{Pf} \begin{bmatrix} 0 & \left\{ \dfrac{1}{\kappa_{b1} - \kappa_{a2}} \right\}_{\substack{1\leq a\leq k_2 \\ 1\leq b\leq k_1}} & \left\{ \kappa_{a2}^{b-1} \right\}_{\substack{1\leq a\leq k_2 \\ 1\leq b\leq d}} \\ \left\{ -\dfrac{1}{\kappa_{a1} - \kappa_{b2}} \right\}_{\substack{1\leq a\leq k_1 \\ 1\leq b\leq k_2}} & \left\{ \widetilde{F}(\kappa_{a1},\kappa_{b1}) \right\}_{1\leq a,b\leq k_1} & \left\{ \widetilde{F}_b(\kappa_{a1}) \right\}_{\substack{1\leq a\leq k_1 \\ 1\leq b\leq d}} \\ \left\{ -\kappa_{b2}^{a-1} \right\}_{\substack{1\leq a\leq d \\ 1\leq b\leq k_2}} & \left\{ -\widetilde{F}_a(\kappa_{b1}) \right\}_{\substack{1\leq a\leq d \\ 1\leq b\leq k_1}} & \left\{ \widetilde{M}_{ab} \right\}_{1\leq a,b\leq d} \end{bmatrix}.$$

As Andréief's integration theorem the generalized de Bruijn integration theorem is only an algebraic rearrangement of the integrals without calculating any of them. The functions $\widetilde{F}$ and $\widetilde{F}_a$ are two-fold integrals and are again not much of importance, see the discussion after Eq. (4.10). Explicit expressions of them are given in Ref. [35] for general random matrix ensembles corresponding to Pfaffians comprising the measure (4.3), too.

The $d \times d$ anti-symmetric matrix $\widetilde{M} = [\widetilde{M}_{ab}]$ consists of the moments

$$\widetilde{M}_{ab} = \int_{\mathbb{R}^2} (\lambda_1^{a-1}\lambda_2^{b-1} - \lambda_1^{b-1}\lambda_2^{a-1}) g(\lambda_1,\lambda_2) d\lambda_1 d\lambda_2. \tag{4.8}$$

Analogously to Eq. (3.13), we employ the identity

$$\mathrm{Pf} \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} = \mathrm{Pf}\, C\, \mathrm{Pf}\, [A + BC^{-1}B^T] \tag{4.9}$$

with an arbitrary matrix B, an arbitrary antisymmetric matrix A and an arbitrary even dimensional, antisymmetric matrix C which has to be invertible. Let $k_1 + k_2$ be even. Then d is also even and the Pfaffian of the matrix $\widetilde{M}$ is proportional to the normalization constant of the ensemble (3.2). Hence the choice $C = \widetilde{M}$ is well-defined. This yields

$$Z^{(n,\nu)}_{k_1/k_2}(\kappa) \propto \frac{1}{\mathrm{B}_{\,k_1/k_2}(\kappa)} \times \mathrm{Pf} \begin{bmatrix} \left\{ \widetilde{G}^{(d)}_1(\kappa_{a2}, \kappa_{b2}) \right\}_{1\le a,b\le k_2} & \left\{ \widetilde{G}^{(d)}_2(\kappa_{b1}, \kappa_{a2}) \right\}_{\substack{1\le a\le k_2 \\ 1\le b\le k_1}} \\ \left\{ -\widetilde{G}^{(d)}_2(\kappa_{a1}, \kappa_{b2}) \right\}_{\substack{1\le a\le k_1 \\ 1\le b\le k_2}} & \left\{ \widetilde{G}^{(d)}_3(\kappa_{a1}, \kappa_{b1}) \right\}_{1\le a,b\le k_1} \end{bmatrix}. \tag{4.10}$$

The functions $\widetilde{G}^{(d)}_1$, $\widetilde{G}^{(d)}_2$ and $\widetilde{G}^{(d)}_3$ can be obtained by considering the cases $(k_1/k_2) = (0/2), (1/1), (2/0)$, respectively. In each of these cases the Pfaffian (4.10) reduces to a single term. This leads to a particular case of the general result derived in Ref. [35]. We find our main result of this article

$$\frac{Z^{(n,\nu)}_{k_1/k_2}(\kappa)}{Z^{(n,\nu)}_{0/0}} = \frac{(-1)^{k_2(k_2+1)/2}}{\mathrm{B}_{\,k_1/k_2}(\kappa)} \frac{\prod_{j=0}^{d-1} h^{(\nu)}_j}{\prod_{j=0}^{n-1} h^{(\nu)}_j} \times \mathrm{Pf} \left[\begin{array}{c|c} \dfrac{\kappa_{b2}-\kappa_{a2}}{h^{(\nu)}_{d/2-1} Z^{(d/2-1,\nu)}_{0/0}} Z^{(d/2-1,\nu)}_{0/2}(\kappa_{a2}, \kappa_{b2}) & \dfrac{1}{Z^{(d/2,\nu)}_{0/0}} \dfrac{Z^{(d/2,\nu)}_{1/1}(\kappa_{b1}, \kappa_{a2})}{(\kappa_{a2}-\kappa_{b1})} \\ \hline \dfrac{1}{Z^{(d/2,\nu)}_{0/0}} \dfrac{Z^{(d/2,\nu)}_{1/1}(\kappa_{a1}, \kappa_{b2})}{(\kappa_{a1}-\kappa_{b2})} & \dfrac{h^{(\nu)}_{d/2}(\kappa_{a1}-\kappa_{b1})}{Z^{(d/2+1,\nu)}_{0/0}} Z^{(d/2+1,\nu)}_{2/0}(\kappa_{a1}, \kappa_{b1}) \end{array} \right] \tag{4.11}$$

for even $k_1 + k_2$. The indices a and b run from 1 to k_2 in the first columns and the first rows and from 1 to k_1 in the last ones. The result for odd $k_2 + k_1$ can be readily obtained by introducing an additional fermionic flavor and sending it to infinity. This shifts the parameter d to $d + 1$ and adds a row and a column to the matrix in the Pfaffian (4.11) with the partition functions $Z^{((d-1)/2,\nu)}_{0/1}(\kappa_{b2})$ and $Z^{((d+1)/2,\nu)}_{1/0}(\kappa_{b1})$ which are apart from a factor κ^ν an orthogonal polynomial and its Cauchy-transform, cf. Eqs. (3.19) and (3.23). Notice that the matrix in the Pfaffian (4.11) is indeed antisymmetric because $Z^{(d/2-1,\nu)}_{0/2}$ and $Z^{(d/2+1,\nu)}_{2/0}$ are symmetric under a permutation of the entries.

Indeed, Eq. (4.11) cannot be traced back to the identity

$$\mathrm{Pf} \begin{bmatrix} 0 & X \\ -X^T & 0 \end{bmatrix} = (-1)^{p(p-1)/2} \det X \tag{4.12}$$

with an arbitrary $p \times p$ matrix X. We refer to the relation (4.12) as a trivial Pfaffian extension of a determinant. The Pfaffian (4.11) seems to be the result of recursion relations of the orthogonal polynomials (3.19). It is difficult to see how these recursions have to be performed to map the Pfaffian (4.11) to the determinant (3.16). However the construction of this structure seems to be the same for a broad class of ensembles. This is

confirmed by the fact that the result (4.11) can be extended to all factorizing ensembles with a squared Vandermonde determinant in the joint probability density (2.1). This will be shown in Sec. 6.

Again one can consider the k-point correlation function (3.17) and what it looks like with the Pfaffian determinant. Using the result (4.11) we find for the k-point correlation function

$$\begin{aligned} R_k^{(n,\nu)}(x) &\propto \exp[-\alpha \operatorname{tr} V(x)] \\ &\quad \times \operatorname{Pf} \left[\begin{array}{c|c} (x_a - x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & (x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, -x_b) \\ \hline -(x_a + x_b) Z_{0/2}^{(n-1,\nu)}(-x_a, x_b) & -(x_a - x_b) Z_{0/2}^{(n-1,\nu)}(-x_a, -x_b) \end{array} \right]_{1 \le a,b \le k} \\ &\propto \exp[-\alpha \operatorname{tr} V(x)] \\ &\quad \times \operatorname{Pf} \left[\begin{array}{c|c} (x_a - x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & (x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \\ \hline -(x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & -(x_a - x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \end{array} \right]_{1 \le a,b \le k} . \end{aligned} \tag{4.13}$$

Again we have not employed the integration theorem by Dyson and Mehta [24, 25, 4, 48]. To see that Eq. (4.13) indeed agrees with the determinant (3.18) one can consider the square of the Pfaffian,

$$\begin{aligned} &\operatorname{Pf}^2 \left[\begin{array}{c|c} (x_a - x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & (x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \\ \hline -(x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & -(x_a - x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \end{array} \right]_{1 \le a,b \le k} \\ &= \det \left[\begin{array}{c|c} (x_a - x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & (x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \\ \hline -(x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & -(x_a - x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \end{array} \right]_{1 \le a,b \le k} \\ &= 2^k \det \left[\begin{array}{c|c} -x_b Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & x_b Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \\ \hline -(x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & -(x_a - x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \end{array} \right]_{1 \le a,b \le k} \\ &= 2^{2k} \det \left[\begin{array}{c|c} -x_b Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & 0 \\ \hline -(x_a + x_b) Z_{0/2}^{(n-1,\nu)}(x_a, x_b) & -x_a Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \end{array} \right]_{1 \le a,b \le k} \\ &= 2^{2k} \det{}^2 x \det{}^2 \left[Z_{0/2}^{(n-1,\nu)}(x_a, x_b) \right]_{1 \le a,b \le k} . \end{aligned} \tag{4.14}$$

The square root of Eq. (4.14) yields Eq. (3.18).

In the large n limit, we employ Eqs. (3.24-3.28) and (3.30-3.32) and obtain

$$\begin{aligned} Z_{k_1/k_2}^{(n,\nu)}\left(\frac{\kappa}{cn}\right) &\overset{n \gg 1}{\propto} \frac{1}{\mathrm{B}_{k_1/k_2}(\kappa)} \\ &\quad \times \operatorname{Pf} \left[\begin{array}{c|c} (\kappa_{a2} - \kappa_{b2}) I_\nu^{(1)}(\kappa_{a2}, \kappa_{b2}) & (\kappa_{b1} + \kappa_{a2}) I_\nu^{(2)}(\kappa_{b1}, \kappa_{a2}) \\ \hline -(\kappa_{a1} + \kappa_{b2}) I_\nu^{(2)}(\kappa_{a1}, \kappa_{b2}) & (\kappa_{a1} - \kappa_{b1}) I_\nu^{(3)}(\kappa_{a1}, \kappa_{b1}) \end{array} \right] \end{aligned} \tag{4.15}$$

for even $k_1 + k_2$ and

$$Z_{k_1/k_2}^{(n,\nu)}\left(\frac{\kappa}{cn}\right) \overset{n \gg 1}{\propto} \frac{1}{\mathrm{B}_{k_1/k_2}(\kappa)} \tag{4.16}$$

$$\times \quad \mathrm{Pf} \left[\begin{array}{c|c|c} 0 & J_\nu(\kappa_{b2}) & K_\nu(\kappa_{b1}) \\ \hline -J_\nu(\kappa_{a2}) & (\kappa_{a2}-\kappa_{b2})I_\nu^{(1)}(\kappa_{a2},\kappa_{b2}) & (\kappa_{b1}+\kappa_{a2})I_\nu^{(2)}(\kappa_{b1},\kappa_{a2}) \\ \hline -K_\nu(\kappa_{a1}) & -(\kappa_{a1}+\kappa_{b2})I_\nu^{(2)}(\kappa_{a1},\kappa_{b2}) & (\kappa_{a1}-\kappa_{b1})I_\nu^{(3)}(\kappa_{a1},\kappa_{b1}) \end{array}\right]$$

for odd $k_1 + k_2$. These Pfaffians carry over to the Wilson Dirac random matrix model [39, 41]. For small numbers of bosonic and fermionic flavors these results were checked by the recursion relations of the Bessel functions [61].

Please notice the difference in the prefactor of Eqs. (3.29), (4.15) and (4.16). The entries of the Berezinian are the squares of the variables κ for the determinantal structure (3.29) whereas it is only κ for the Pfaffian. This yields a technical advantage when calculating eigenvalue correlations of the random matrix models for the Wilson Dirac operator.

4.2. An application: Wilson-Dirac random matrix

The Wilson-Dirac operator is a modified Dirac operator on a lattice. In the infrared limit this operator can be modeled by the Wilson-Dirac random matrix [36, 37, 38, 40] which is a $(2n+\nu)\times(2n+\nu)$ Hermitian matrix

$$D_{\mathrm{W}} = \left[\begin{array}{cc} aA & W \\ -W^\dagger & aB \end{array}\right] \tag{4.17}$$

distributed by the Gaussian

$$P(D_{\mathrm{W}}) = \exp\left[-\frac{n}{2}(\mathrm{tr}\, A^2 + \mathrm{tr}\, B^2) - n\,\mathrm{tr}\, WW^\dagger\right]. \tag{4.18}$$

The variable a plays the role of the lattice spacing. The chiral symmetry is explicitly broken by the Hermitian matrices A and B, i.e.

$$\gamma_5\, D_{\mathrm{W}}|_{m=0}\,\gamma_5 \neq -\, D_{\mathrm{W}}|_{m=0} \quad \text{with} \quad \gamma_5 = \mathrm{diag}\,(\mathbb{1}_n, -\mathbb{1}_{n+\nu}), \tag{4.19}$$

which have the dimensions $n\times n$ and $(n+\nu)\times(n+\nu)$, respectively. Hence, A and B model the Wilson-term.

We consider the partition function with N_f fermionic flavors,

$$Z_{N_{\mathrm{f}}}^{(n,\nu)}(m,a) = \int \prod_{j=1}^{N_{\mathrm{f}}} \det(D_{\mathrm{W}} + m_j \mathbb{1}_{2n+\nu}) P(D_{\mathrm{W}}) d[D_{\mathrm{W}}]. \tag{4.20}$$

The external variables $m = \mathrm{diag}\,(m_1,\ldots,m_{N_{\mathrm{f}}})$ play the role of the quark masses. Indeed one can also consider bosonic flavors. However, we restrict ourself to fermionic flavors to keep the example as simple as possible

In the microscopic limit ($n\to\infty$), $\widehat{m} = 2nm$, $\widehat{a} = \sqrt{n}a/2$ and ν are kept fixed. This yields the integral

$$Z_{N_{\mathrm{f}}}^{(n,\nu)}(\widehat{m},\widehat{a}) \stackrel{n\gg 1}{=} \int\limits_{\mathrm{U}\,(N_f)} \exp\left[\frac{1}{2}\,\mathrm{tr}\,\widehat{m}(U+U^{-1}) - \widehat{a}^2\,\mathrm{tr}\left(U^2+U^{-2}\right)\right] \det{}^\nu U d\mu(U). \tag{4.21}$$

For a derivation of this result we refer to Refs. [36, 37]. Exactly the integral (4.21) makes contact with lattice QCD [62, 63, 64, 65].

At zero lattice spacing ($\widehat{a} = 0$) this partition function can be identified with the one considered in Sec. 3,

$$Z^{(n,\nu)}_{N_{\rm f}}(m, a = 0) \propto Z^{(n,\nu)}_{0/N_{\rm f}}(\imath m). \tag{4.22}$$

Considering again the microscopic limit (4.21), we trace the integral back to the $a = 0$ result by introducing a $N_{\rm f} \times N_{\rm f}$ Hermitian random matrix σ similar to the calculation in Ref. [38, 66],

$$\begin{aligned} Z^{(n,\nu)}_{N_{\rm f}}(\widehat{m}, \widehat{a}) \overset{n\gg 1}{\propto} & \int \exp\left[-\frac{1}{4\widehat{a}^2}\,\mathrm{tr}(\sigma - \imath\widehat{m})^2 - 2(\widehat{a}N_{\rm f})^2\right] \\ & \times \int\limits_{\mathrm{U}\,(N_f)} \exp\left[-\imath\,\mathrm{tr}\,\sigma(U + U^{-1})\right] \det{}^{\nu} U d\mu(U) d[\sigma] \\ & \propto \int \exp\left[-\frac{1}{4\widehat{a}^2}\,\mathrm{tr}(\sigma + \imath\widehat{m})^2 - 2(\widehat{a}N_{\rm f})^2\right] Z^{(n,\nu)}_{0/N_{\rm f}}(\sigma) d[\sigma]. \end{aligned} \tag{4.23}$$

Notice that σ is an ordinary matrix and not a supermatrix because we consider fermionic flavors, only. The constant $\exp[-2(\widehat{a}N_{\rm f})^2]$ can be shifted into the normalization constant and can, thus, be omitted in the ensuing calculations.

A diagonalization of $\sigma = VsV^{\dagger}$ with $V \in \mathrm{U}\,(N_{\rm f})$ yields a Harish-Chandra-Itzykson-Zuber-integral [67, 68] in the Gaussian term. The partition function $Z^{(n,\nu)}_{0/N_{\rm f}}$ is invariant under $\mathrm{U}\,(N_{\rm f})$. We find

$$Z^{(n,\nu)}_{N_{\rm f}}(\widehat{m}, \widehat{a}) \propto \int \frac{\det\left[\exp\left[-(s_j - \imath\widehat{m}_i)^2/4\widehat{a}^2\right]\right]_{1\le j,i\le N_{\rm f}}}{\Delta_{N_{\rm f}}(\widehat{m})} Z^{(n,\nu)}_{0/N_{\rm f}}(s)\Delta_{N_{\rm f}}(s) d[s]. \tag{4.24}$$

Employing the result as a determinant of the microscopic limit of $\widehat{a} = 0$ partition function, cf. Eq. (3.29), we end up with a complicated expression,

$$\begin{aligned} Z^{(n,\nu)}_{N_{\rm f}}(\widehat{m}, \widehat{a}) \overset{n\gg 1}{\propto} & \int \frac{\det\left[\exp\left[-(s_j - \imath\widehat{m}_i)^2/4\widehat{a}^2\right]\right]_{1\le j,i\le N_{\rm f}}}{\Delta_{N_{\rm f}}(\widehat{m})} \\ & \times \det\left[s_i^{j-1} J_{\nu-1+j}(s)\right]_{1\le j,i\le N_{\rm f}} \frac{\Delta_{N_{\rm f}}(s)}{\Delta_{N_{\rm f}}(s^2)} d[s]. \end{aligned} \tag{4.25}$$

There is no obvious way to further simplify the integral (4.25) due to the factor $\Delta_{N_{\rm f}}(s)/\Delta_{N_{\rm f}}(s^2)$. This was not much of a problem for the authors of Refs. [38, 66] because they only considered a small numbers of flavors. However the problem is highly non-trivial for an arbitrary number of flavors.

This problem can be solved by using the Pfaffian expressions (4.15) and (4.16) instead of the determinant. Let $N_{\rm f}$ be even to keep the expressions as simple as possible. Then we have for the microscopic limit (4.21)

$$\begin{aligned} Z^{(n,\nu)}_{N_{\rm f}}(\widehat{m}, \widehat{a}) \overset{n\gg 1}{\propto} & \int \frac{\det\left[\exp\left[-(s_j - \imath\widehat{m}_i)^2/4\widehat{a}^2\right]\right]_{1\le j,i\le N_{\rm f}}}{\Delta_{N_{\rm f}}(\widehat{m})} \\ & \times \mathrm{Pf}\left[\frac{s_j J_{\nu-1}(s_j) J_{\nu}(s_i) - s_i J_{\nu}(s_j) J_{\nu-1}(s_i)}{s_j + s_i}\right]_{1\le j,i\le N_{\rm f}} d[s]. \end{aligned} \tag{4.26}$$

After expanding the determinant no term hinders us to pull the integrals into the Pfaffian. We obtain the compact result

$$Z_{N_\mathrm{f}}^{(n,\nu)}(\widehat{m},\widehat{a}) \propto \frac{1}{\Delta_{N_\mathrm{f}}(\widehat{m})}\mathrm{Pf}\,\left[(\widehat{m}_j-\widehat{m}_i)Z_2^{(n,\nu)}(\widehat{m}_j,\widehat{m}_i,\widehat{a})\right]_{1\le j,i\le N_\mathrm{f}}$$

with

$$\begin{aligned} Z_2^{(n,\nu)}(\widehat{m}_1,\widehat{m}_2,\widehat{a}) \propto \frac{1}{\widehat{m}_1-\widehat{m}_2}\int_{\mathbb{R}^2}\exp\left[-\frac{(s_1-\imath\widehat{m}_1)^2+(s_2-\imath\widehat{m}_2)^2}{4\widehat{a}^2}\right] \\ \times\,\frac{s_1J_{\nu-1}(s_1)J_\nu(s_2)-s_2J_\nu(s_1)J_{\nu-1}(s_2)}{s_1+s_2}ds_1ds_2. \end{aligned} \tag{4.27}$$

This is a drastic simplification of the problem compared to Eq. (4.25).

5. Skew-orthogonal polynomials

What are the skew-orthogonal polynomials which correspond to the Pfaffian (4.11)? In order to solve this problem we consider the two-point measure (4.2). The skew orthogonal polynomials q_j are defined by

$$\begin{aligned} &\int_{\mathbb{R}^2}\det\begin{bmatrix} q_{2j-1}(x_1) & q_{2j-1}(x_2)\\ q_{2i-1}(x_1) & q_{2i-1}(x_2)\end{bmatrix}g(x_1,x_2)dx_1dx_2\\ &=\int_{\mathbb{R}^2}\det\begin{bmatrix} q_{2j}(x_1) & q_{2j}(x_2)\\ q_{2i}(x_1) & q_{2i}(x_2)\end{bmatrix}g(x_1,x_2)dx_1dx_2=0, \end{aligned} \tag{5.1}$$

and

$$\int_{\mathbb{R}^2}\det\begin{bmatrix} q_{2j+1}(x_1) & q_{2j+1}(x_2)\\ q_{2i}(x_1) & q_{2i}(x_2)\end{bmatrix}g(x_1,x_2)dx_1dx_2=\widehat{h}_i^{(\nu)}\delta_{ij}. \tag{5.2}$$

Moreover one has to assume that q_l is a polynomial of order l.

The integral over the measure (4.2) for two arbitrary and conveniently integrable functions f_1 and f_2 can be simplified to

$$\begin{aligned} &\int_{\mathbb{R}^2}\det\begin{bmatrix} f_1(x_1) & f_1(x_2)\\ f_2(x_1) & f_2(x_2)\end{bmatrix}g(x_1,x_2)dx_1dx_2\\ &=\frac{1}{2}\int_0^\infty\det\begin{bmatrix} f_1(x) & f_1(-x)\\ f_2(x) & f_2(-x)\end{bmatrix}x^{2\nu}\exp\left[-nV(x^2)\right]dx. \end{aligned} \tag{5.3}$$

Due to this identity the skew-orthogonal polynomials $q_l^{(\nu)}$ are related by the orthogonal polynomials p_l in the following way

$$q_{2l}^{(\nu)}(x)=p_l^{(\nu)}(x^2) \tag{5.4}$$

for the even polynomials and

$$q_{2l+1}^{(\nu)}(x)=xp_l^{(\nu)}(x^2)+\text{const.}\,p_l^{(\nu)}(x^2) \tag{5.5}$$

for the odd polynomials. Notice that these skew-orthogonal polynomials for $V(x) = x$ (the Laguerre ensemble) are similar to but not completely the same as the one for $\beta = 1$ and $\beta = 4$ shown in Ref. [4, 48] for the Laguerre ensemble. The reason is the two point weight which is

$$g_{\mathrm{chGOE}}(x_1, x_2) = (x_1 x_2)^\nu \exp\left[-\alpha(x_1^2 + x_2^2)\right] \frac{x_1 - x_2}{|x_1 - x_2|}, \tag{5.6}$$

$$g_{\mathrm{chGSE}}(x_1, x_2) = (x_1 x_2)^{2\nu+3/2} \exp\left[-\alpha(x_1^2 + x_2^2)\right] \delta'(x_1 - x_2), \tag{5.7}$$

in comparison see Eq. (4.2) for $\beta = 2$. The labels "chGOE" and "chGSE" refer to the chiral Gaussian orthogonal ensemble ($\beta = 1$) and to the chiral Gaussian symplectic ensemble ($\beta = 4$), respectively. The sign function $(x_1 - x_2)/|x_1 - x_2|$ generate the modulus of the Vandermonde determinant for $\beta = 1$. The distribution δ' is the first derivative of the Dirac delta function and cancels with these terms of the Vandermonde determinant which are zero at the support of the Dirac delta functions. This generates Cramers degeneracy in the quaternion case ($\beta = 4$).

The solution of Eqs. (5.1) and (5.2) is not unique which is reflected by the arbitrary constant in the odd polynomials (5.5). One can readily confirm that this choice of the polynomials solves the conditions (5.1) and (5.2) by recognizing the symmetry $q_j(-x) = (-1)^j q_j(x)$ and the orthogonality relation (3.20) for p_j. The normalization constant is

$$\widehat{h}_i^{(\nu)} = h_i^{(\nu)}. \tag{5.8}$$

This relation between orthogonal and skew-orthogonal polynomials seems so trivial because of the particular and simple structure of the two-point weight (4.2).

6. A few more ensembles with Dyson index $\beta = 2$ and Pfaffians

The algebraic rearrangement for chiral unitary ensembles described in Sec. 4 can be extended to other random matrix ensembles which have a squared Vandermonde determinant in the joint probability density function. By the same trick as in Eq. (4.1) we write

$$\Delta_N^2(z) = (-1)^{N(N-1)/2} \frac{\Delta_{2N}(\sqrt{z}, -\sqrt{z})}{2^N \sqrt{\det z}}, \tag{6.1}$$

where the variables $z = \mathrm{diag}\,(z_1, \ldots, z_N)$ might be complex. The square root is the positive one but this is without loss of generality since the right hand side of Eq. (6.1) comprises both roots. Again the determinant of z will be put to the measure $d\mu$ for a single eigenvalue.

We consider an average over ratios of characteristic polynomials for random ensembles like GUE and CUE, i.e.

$$\widetilde{Z}^{(N)}_{k_1/k_2}(\kappa) = \int_{\mathbb{C}^N} \Delta_N^2(z) \prod_{i=1}^{N} \frac{\prod_{j=1}^{k_2} (z_i - \kappa_{j2})}{\prod_{j=1}^{k_1} (z_i - \kappa_{j1})} d\mu(z_i), \tag{6.2}$$

where $d\mu$ is a measure on $\mathbb{C}$ and κ is chosen such that the integrals exist. Notice that there is no modulus of the Vandermonde determinant which is a necessary property of the following discussion. A modulus of the Vandermonde is an obstacle to map Eq. (6.2) to the general joint probability density corresponding to the Pfaffian, see Ref. [35], which we have not managed yet. A modulus corresponds to the biorthogonal polynomials [47] whereas the choice without the modulus corresponds to the orthogonal polynomials, only. Apart from the modulus of the Vandermonde it is exactly the correlation function discussed in Sec. 4.2 of Ref. [34].

With the help of the derivation in Sec. 4 the integral (6.2) can be written as

$$\widetilde{Z}^{(N)}_{k_1/k_2}(\kappa) \propto \frac{1}{\mathrm{B}_{\,k_1/k_2}(\sqrt{\kappa})} \times \mathrm{Pf}\left[\begin{array}{c|c} \dfrac{\widetilde{d}(\sqrt{\kappa_{b2}}-\sqrt{\kappa_{a2}})}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\widetilde{Z}^{(\widetilde{d}-1)}_{0/2}(\kappa_{a2},\kappa_{b2}) & \dfrac{1}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\dfrac{\widetilde{Z}^{(\widetilde{d})}_{1/1}(\kappa_{b1},\kappa_{a2})}{(\sqrt{\kappa_{a2}}-\sqrt{\kappa_{b1}})} \\ \hline \dfrac{1}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\dfrac{\widetilde{Z}^{(\widetilde{d})}_{1/1}(\kappa_{a1},\kappa_{b2})}{(\sqrt{\kappa_{a1}}-\sqrt{\kappa_{b2}})} & \dfrac{\sqrt{\kappa_{a1}}-\sqrt{\kappa_{b1}}}{(\widetilde{d}+1)\widetilde{Z}^{(\widetilde{d})}_{0/0}}\widetilde{Z}^{(\widetilde{d}+1)}_{2/0}(\kappa_{a1},\kappa_{b1}) \end{array}\right] \tag{6.3}$$

for $k_1 + k_2$ even and

$$\widetilde{Z}^{(N)}_{k_1/k_2}(\kappa) \propto \frac{1}{\mathrm{B}_{\,k_1/k_2}(\sqrt{\kappa})} \times \mathrm{Pf}\left[\begin{array}{c|c|c} 0 & -\dfrac{\widetilde{d}}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\widetilde{Z}^{(\widetilde{d}-1)}_{0/1}(\kappa_{b2}) & \dfrac{1}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\widetilde{Z}^{(\widetilde{d})}_{1/0}(\kappa_{b1}) \\ \hline \dfrac{\widetilde{d}}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\widetilde{Z}^{(\widetilde{d}-1)}_{0/1}(\kappa_{a2}) & \dfrac{\widetilde{d}(\sqrt{\kappa_{b2}}-\sqrt{\kappa_{a2}})}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\widetilde{Z}^{(\widetilde{d}-1)}_{0/2}(\kappa_{a2},\kappa_{b2}) & \dfrac{1}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\dfrac{\widetilde{Z}^{(\widetilde{d})}_{1/1}(\kappa_{b1},\kappa_{a2})}{(\sqrt{\kappa_{a2}}-\sqrt{\kappa_{b1}})} \\ \hline -\dfrac{1}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\widetilde{Z}^{(\widetilde{d})}_{1/0}(\kappa_{a1}) & \dfrac{1}{\widetilde{Z}^{(\widetilde{d})}_{0/0}}\dfrac{\widetilde{Z}^{(\widetilde{d})}_{1/1}(\kappa_{a1},\kappa_{b2})}{(\sqrt{\kappa_{a1}}-\sqrt{\kappa_{b2}})} & \dfrac{\sqrt{\kappa_{a1}}-\sqrt{\kappa_{b1}}}{(\widetilde{d}+1)\widetilde{Z}^{(\widetilde{d})}_{0/0}}\widetilde{Z}^{(\widetilde{d}+1)}_{2/0}(\kappa_{a1},\kappa_{b1}) \end{array}\right] \tag{6.4}$$

for $k_1 + k_2$ odd. The variable $\widetilde{d}$ is

$$\widetilde{d} = \begin{cases} N + (k_2 - k_1)/2, & k_2 + k_1 \in 2\mathbb{N}, \\ N + (k_2 - k_1 + 1)/2, & k_2 + k_1 + 1 \in 2\mathbb{N}. \end{cases} \tag{6.5}$$

The indices a and b run from 1 to k_1 for κ_1 and from 1 to k_2 for κ_2. Apart from the square roots of the variables κ these structures are exactly the same as those of random matrix ensembles with Dyson index $\beta \in \{1, 4\}$. Hence, it seems to be that the Pfaffian determinants (6.3) and (6.4) for the average over characteristic polynomials are more general than the determinant derived in Ref. [34].

Random matrix ensembles whose generating functions can be cast into the form (6.2) have this non-trivial expression as a Pfaffian. The Hermitian Gaussian unitary ensemble as well as its generalization with other potentials fulfill *a priori* this

requirement since the joint probability density has a squared Vandermonde determinant without the modulus. More generally our derivation applies to each ensemble with a real spectrum, a squared Vandermonde determinant and a factorizing probability distribution, cf. Eq. (3.4). Also the CUE (unitary group) can be cast into the form (6.2). More ensembles can be found in the tables 1 and 2 of Ref. [34]. The Ginibre ensemble as well as its chiral counterpart are not in this class. Their joint probability density incorporates a modulus of the Vandermonde determinant and is, thus, in the class for the bi-orthogonal polynomials. Therefore it is possible that their eigenvalue correlation functions cannot be expressed in Pfaffians like Eq. (6.3) and (6.4).

The skew-orthogonal polynomials corresponding to the Pfaffians (6.3) and (6.4) have the same relation to the orthogonal polynomials as chiral unitary ensembles, see Eqs. (5.4-5.8). By construction this relation is so simple.

7. Remarks and conclusions

We derived a non-trivial Pfaffian determinant for the average over ratios of characteristic polynomials of a large class of random matrix ensembles with Dyson index $\beta = 2$. This structure is similar to the one for $\beta \in \{1, 4\}$, cf. Ref. [35]. Hence, it is universal and unifies most of the symmetry classes known in the literature, particularly the Cartan classification [45, 46]. It is unclear how far beyond this classification [69] this structure is applicable. It is only known that there are some of them which share the identity (4.11). For example the real and quaternion Ginibre ensembles as well as there chiral counterpart fulfill an identity similar to Eq. (4.11).

For many random matrix ensembles like the GUE it seems an academical question if one can derive a Pfaffian or not since there are no applications, yet. However, for the chiral GUE it is important to know this due to the new results obtained for the Wilson Dirac random matrix ensemble discussed in Refs. [36, 37, 38, 39, 40, 41, 66]. Pfaffians were found there for finite lattice spacing. On the level of the joint probability density the authors of Ref. [39] checked that the ensemble is the chiral GUE as well as the GUE at certain values of the lattice spacing. However for the eigenvalue correlation functions the continuum limit has not yielded the known determinant (3.29). With this work we clarified this puzzle.

For intermediate ensembles in general our result might be helpful to understand the structure appearing by switching the interaction between the two ensembles on. It is numerically advantageous to think about spectral correlations of intermediate ensembles as kernels of Pfaffians since the integrand drastically simplifies. In combination with the supersymmetry method [70, 71, 72, 73] also the number of integrals reduces a lot.

Moreover, we derived the relation between the orthogonal polynomials and the skew-orthogonal polynomials corresponding to the determinants and the Pfaffians, respectively. This relation, see Eqs. (5.4-5.8), is not only quite simple but also universal since it applies to all random matrix ensembles discussed in this work. The relation between the orthogonal and skew-orthogonal polynomials for $\beta = 2$ slightly differs to

those found in Ref. [48] for the cases $\beta = 1$ and $\beta = 4$. The difference in the two-point weight is the reason for this. Based on the representations (3.16), (6.3) and (6.4) shared by all random matrix ensembles with $\beta = 2$ as well as checks of these representations [61], we conjecture that the recursion relation of the orthogonal polynomials connects the determinant and the Pfaffian and this has to be done in a general way.

The Pfaffian found for the average over characteristic polynomials carries over to the k-point correlation functions. This structure is valid in the large matrix limit, too. It should not depend on which scaling limit is chosen since the Pfaffian is independent of the matrix size. Hence, the correlation functions appearing as kernels of this Pfaffian have non-trivial recursion relations mapping the determinant to the Pfaffian.

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The Realization of the Sharpe-Singleton Scenario

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Abstract

The microscopic spectral density of the Wilson Dirac operator for two flavor lattice QCD is analyzed. The computation includes the leading order a^2 corrections of the chiral Lagrangian in the microscopic limit. The result is used to demonstrate how the Sharpe-Singleton first order scenario is realized in terms of the eigenvalues of the Wilson Dirac operator. We show that the Sharpe-Singleton scenario only takes place in the theory with dynamical fermions whereas the Aoki phase can be realized in the quenched as well as the unquenched theory. Moreover, we give constraints imposed by γ_5-Hermiticity on the additional low energy constants of Wilson chiral perturbation theory.

I. INTRODUCTION

In the deep chiral limit, with almost massless quarks, lattice QCD with Wilson fermions has a highly nontrivial phase structure. As in continuum QCD, it is the deep chiral limit which reveals the spontaneous breaking of chiral symmetry on the lattice. In addition, the interplay between the continuum and the chiral limit in lattice QCD with Wilson fermions leads to new phase structures known as the Aoki phase [1] and the Sharpe-Singleton scenario [2]. These phases have no direct analogues in the continuum theory, and dominate if the chiral limit is performed prior to the continuum limit. While this at first may seem like a highly undesirable artifact of Wilson fermions it can in fact be turned to our advantage: The Aoki phase is reached through a second order phase transition and at the boundary of this transition the pions are massless. This opens the possibility to study nonperturbative QCD at extremely small pion masses even at a nonzero lattice spacing. On the contrary the Sharpe-Singleton scenario is a first order phase transition in which the pions are massive even in the chiral limit at nonzero lattice spacing.

These phase structures of lattice QCD with Wilson fermions can be described within the framework of Wilson chiral perturbation theory [2–8]. This low energy effective theory of lattice QCD with Wilson fermions describes discretization effects by means of additional terms in the chiral Lagrangian (see [9, 10] for reviews). Each of these new terms come with a new low energy constant. The sign and magnitude of these constants reflect whether lattice QCD with Wilson fermions will enter the Aoki phase or the Sharpe-Singleton scenario. Considerable progress, both analytically [11–16] and numerically [17–23], has been made recently in the determination of these constants. However, a complete picture has not yet emerged. For example, the observation that quenched lattice simulations consistently observe the Aoki phase [24–27], while in unquenched simulations both the Aoki and the Sharpe-Singleton scenario [20, 23, 28–37] has been observed, remains a puzzle.

The spontaneous breaking of chiral symmetry is tightly connected to the smallest eigenvalues of the Dirac operator [38, 39]. Moreover, the Aoki phase manifests itself in the smallest eigenvalues of the Wilson Dirac operator [11, 40]. Here we show that the behavior of the smallest eigenvalues of the Wilson Dirac operator is also directly related to the Sharpe-Singleton scenario. In particular, we explain that in the Sharpe-Singleton scenario the Wilson Dirac eigenvalues undergo a collective macroscopic jump as the quark mass

changes sign. Moreover, we show that this collective jump only occurs in the presence of dynamical fermions. The quenched theory has no analogue of this and hence the Sharpe-Singleton scenario is not possible in the quenched theory. This conclusion is verified by a direct computation of the microscopic quenched and unquenched chiral condensate.

In order to establish these results we explicitly derive the unquenched microscopic spectral density of the Wilson Dirac operator. This calculation makes use of both Wilson random matrix theory as well as Wilson chiral perturbation theory. By means of an underlying Pfaffian structure we uncover a compact factorized form of the exact unquenched microscopic eigenvalue density. This form makes it possible to understand the full dependence of the eigenvalue density on the low energy constants. We analyze this dependence in the mean field limit which can also be directly derived from Wilson chiral perturbation theory.

The mean field limit of the microscopic spectral density corresponds to the leading order result of Wilson chiral perturbation theory in the p-regime. This will allow us to close the circle by explaining the original p-regime results of Sharpe and Singleton in terms of the behavior of the Wilson Dirac eigenvalues. In particular, we will explain how the nonzero minimal value of the pion mass in the Sharpe-Singleton scenario is connected to the collective jump of the Wilson Dirac eigenvalues.

The approach to the Wilson Dirac spectrum followed in this paper has been applied previously in Refs. [11, 12, 15–17, 41–46] and results from these studies will be used.

The study of the smallest eigenvalues of the Wilson Dirac eigenvalues not only explains the way in which the Aoki phase and the Sharpe-Singleton scenarios are realized, it also gives direct information on the sign and magnitude of the low energy constants of Wilson chiral perturbation theory. We will show that the spectral properties of the Wilson Dirac operator determine the sign of all three additional low energy constants of the leading order chiral Lagrangian of Wilson chiral perturbation theory in the microscopic limit.

The results for the unquenched spectral density of the Wilson Dirac operator presented here also offer a direct way to measure the low energy constants of Wilson chiral perturbation theory by matching the predictions against results from lattice QCD. The first quenched studies of this nature appeared recently [21, 22].

This paper is organized as follows. After a brief presentation of the properties of the Wilson Dirac operator in Section II we recall the basics of Wilson chiral perturbation theory in section III. In section IV we determine constraints on the additional low energy parameters

of Wilson chiral perturbation theory in terms of the spectral properties of the Wilson Dirac operator. The unquenched microscopic spectrum of the Wilson Dirac operator is analyzed in section V. Finally, the realization of the Sharpe-Singleton scenario is the topic of section VI. Section VII contains our summary and conclusions. Wilson random matrix theory, the factorization properties of the spectral density and the details of the mean field calculation are discussed in Appendix A, Appendix B and Appendix C, respectively.

II. THE WILSON DIRAC OPERATOR

Here we recall a few basic properties of the Wilson Dirac operator. The Wilson term in the lattice discretized covariant derivative

$$D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{ar}{2}\nabla_\mu\nabla^*_\mu \tag{1}$$

breaks the anti-Hermiticity as well as the axial symmetry of the continuum Dirac operator. However, D_W is γ_5-Hermitian

$$\gamma_5 D_W \gamma_5 = D_W^\dagger \tag{2}$$

and the product with γ_5, $D_5(m) \equiv \gamma_5(D_W + m)$ is therefore Hermitian.

The eigenvalues, z_k, of D_W consists of complex conjugated pairs as well as exactly real eigenvalues [47]. Only the real eigenmodes have nonzero chirality and determine the index, ν, of the Wilson Dirac operator

$$\nu = \sum_k \mathrm{sign}(\langle k|\gamma_5|k\rangle). \tag{3}$$

Here $|k\rangle$ denotes the k'th eigenstate of D_W. The eigenvalues, λ^5, of $D_5(m)$ are unpaired when $a \neq 0$.

In section IV below we will use these properties to constrain the parameters of Wilson chiral perturbation theory.

III. WILSON CHIRAL PERTURBATION THEORY

In the microscopic limit at nonzero lattice spacing where (m is the quark mass, ζ the axial quark mass, z an eigenvalue of D_W, and a is the lattice spacing)

$$mV, \quad \zeta V, \quad zV \quad \text{and} \quad a^2V \tag{4}$$

are kept fixed as $V \to \infty$, the microscopic partition function of [48] extends to [11]

$$Z^{\nu}_{N_f}(m,\zeta;a) = \int_{U(N_f)} dU \, \det{}^{\nu} U \; e^{S[U]}, \tag{5}$$

where the action $S[U]$ for degenerate quark masses is given by [2–4]

$$\begin{aligned} S \; = \; & \frac{m}{2}\Sigma V \mathrm{Tr}(U+U^{\dagger}) + \frac{\zeta}{2}\Sigma V \mathrm{Tr}(U-U^{\dagger}) \\ & -a^2 V W_6 [\mathrm{Tr}\left(U+U^{\dagger}\right)]^2 - a^2 V W_7 [\mathrm{Tr}\left(U-U^{\dagger}\right)]^2 - a^2 V W_8 \mathrm{Tr}(U^2 + {U^{\dagger}}^2). \end{aligned} \tag{6}$$

In addition to the chiral condensate, Σ, the action also contains the low energy constants W_6, W_7 and W_8 as parameters [55].

In order to lighten the notation we introduce the rescaled, dimensionless variables

$$\hat{a}_i^2 = a^2 V W_i, \qquad \hat{m} = mV\Sigma, \qquad \hat{z} = zV\Sigma \quad \text{and} \qquad \hat{\zeta} = \zeta V \Sigma. \tag{7}$$

The generating functional for the eigenvalue density of D_W in the complex plane is the graded extension of Eq. (5). Because of the non-Hermiticity of D_W, the graded extension

$$Z^{\nu}_{N_f+2|2}(\hat{z},\hat{z}^*,\hat{z}',\hat{z}'^*,\hat{m};\hat{a}_i) \tag{8}$$

requires an extra pair of conjugate quarks with masses $\hat{z}$ and $\hat{z}^*$, as well as a conjugate pair of bosonic quarks, with masses $\hat{z}'$ and $\hat{z}'^*$ [49]. The graded mass term becomes

$$\mathrm{Trg}\big(\mathcal{M}U + \mathcal{M}U^{-1}\big) \quad \text{with} \quad \mathcal{M} = \mathrm{diag}(\hat{m}_1,\ldots,\hat{m}_{N_f},\hat{z},\hat{z}^*,\hat{z}',\hat{z}'^*), \tag{9}$$

where Trg denotes the graded trace $\mathrm{Trg}A = \mathrm{Tr}(A_f) - \mathrm{Tr}(A_b)$, with A_f the fermion-fermion block of A and A_b its boson-boson block. The eigenvalue density of D_W in the complex plane is

$$\rho^{\nu}_{c,N_f}(\hat{z},\hat{z}^*,\hat{m};\hat{a}_i) \; = \; \partial_{\hat{z}^*} \lim_{\hat{z}' \to \hat{z}} \partial_{\hat{z}} \log Z^{\nu}_{N_f+2|2}(\hat{z},\hat{z}^*,\hat{z}',\hat{z}'^*,\hat{m};\hat{a}_i). \tag{10}$$

The sign and magnitude of W_6, W_7 and W_8 determine the phase structure at small mass [2]: for $W_8 + 2W_6 > 0$ the Aoki phase dominates if $|m|\Sigma < 8(W_8+2W_6)a^2$ while for $W_8 + 2W_6 < 0$ the Sharpe-Singleton scenario takes place. It is therefore of considerable interest to understand if it is possible to determine the signs of the additional low energy constants. In the next section we show how these signs follow from the γ_5-Hermiticity of the Wilson Dirac operator.

IV. CONSTRAINTS ON W_6, W_7 AND W_8 DUE TO γ_5-HERMITICITY

In Refs. [12, 13, 16] it was shown that properties of the partition function and the correlation functions due to γ_5-Hermiticity lead to bounds on W_6, W_7 and W_8. The bounds that where found are [12, 13] $W_8 > 0$ (independent of the value of W_6 and W_7 [13]) and [12, 16] $W_8 - W_6 - W_7 > 0$. In addition it was argued in [16] that $W_8 + 2W_6 > 0$ provided that disconnected diagrams are suppressed. Note that lattice studies [18] have found that disconnected diagrams can have a significant contribution.

Here we show that the signs of W_6 and W_7 can be determined from γ_5-Hermiticity if we consider the spectral properties of the Wilson Dirac operator. There are two implicit assumptions that have been well established in the study of Dirac spectra. First, that for a given value of the low-energy constants the chiral Lagrangian can be extended to partially quenched QCD with the same low-energy constants. Second, there is a one-to-one relation between spectral properties in the microscopic domain and the partially quenched chiral Lagrangian.

Let us first recall why γ_5-Hermiticity implies that $W_8 > 0$ when $W_6 = W_7 = 0$ [12]. As shown by explicit calculations in [11, 12, 42, 43] the microscopic graded generating functional corresponding to

$$\mathcal{L}(U) \;=\; \frac{1}{2}m\Sigma\mathrm{Tr}(U+U^\dagger) + \frac{1}{2}\zeta\Sigma\mathrm{Tr}(U-U^\dagger) - a^2W_8\mathrm{Tr}(U^2+{U^\dagger}^2) \tag{11}$$

with $W_8 > 0$ gives predictions for the spectrum of the γ_5-Hermitian D_W and the Hermitian D_5. This was further confirmed by its equivalence to a γ_5-Hermitian Wilson Random Matrix Theory.

On the contrary if $W_8 < 0$, it was explicitly shown in [12] that the graded generating functional corresponding to Eq. (11) is the generating functional for the spectral fluctuations in a lattice theory with iWilson fermions defined as

$$D_{iW} = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - i\frac{ar}{2}\nabla_\mu\nabla^*_\mu, \tag{12}$$

which is anti-Hermitian rather than γ_5-Hermitian. This conclusion was again confirmed by the equivalence to an anti-Hermitian iWilson Random Matrix Theory. Note that D_W and D_{iW} only differ by a factor of i in the Wilson term, and that D_{iW} is *not* γ_5-Hermitian.

Therefore we understand the effective theory, Eq. (11), for both signs of W_8 and that the Hermiticity properties of the Wilson Dirac operator determine this sign. For Wilson

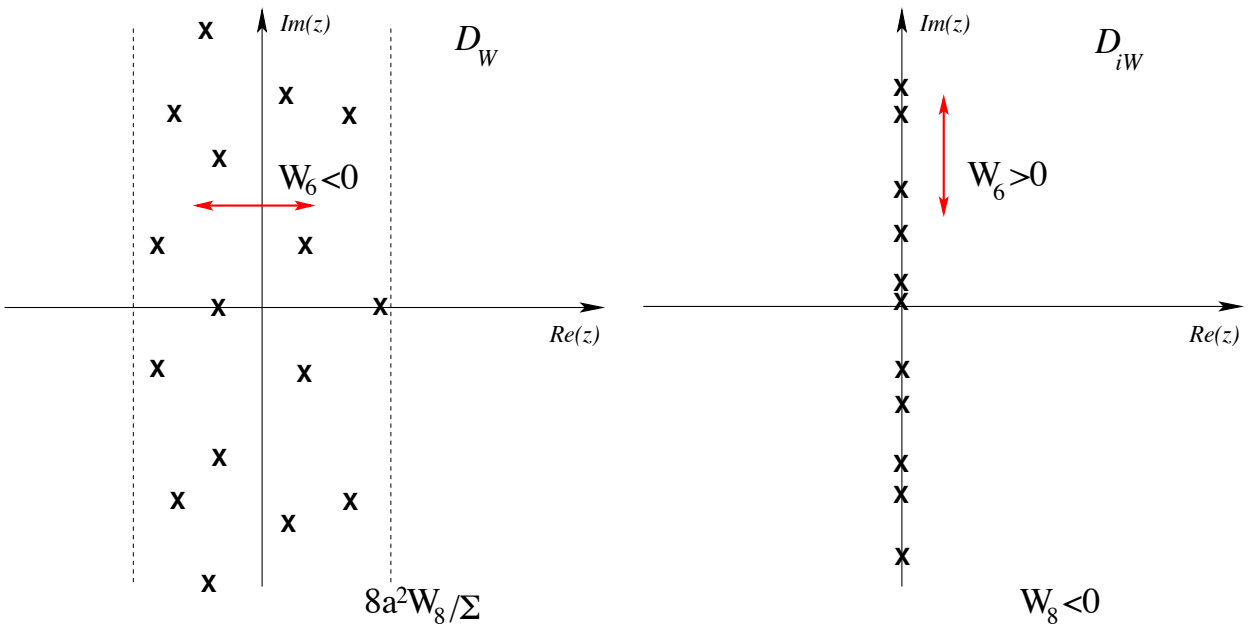

FIG. 1: Illustration of the fluctuations of the Dirac eigenvalues. **Left:** A negative value of W_6 corresponds to a γ_5-Hermitian Wilson Dirac operator, i.e. with eigenvalues that are either real or come in complex conjugate pairs. **Right:** The Dirac operator corresponding to $W_6 > 0$ is in the Hermiticity class of D_{iW} with purely imaginary eigenvalues.

fermions we have $W_8 > 0$, whereas for iWilson fermions the constraint is $W_8 < 0$. This is fully consistent with the results from QCD inequalities [12, 13].

Let us now extend the argument to also include W_6 and W_7. We will show that Wilson chiral perturbation theory with $W_6 < 0$, $W_7 < 0$ and $W_8 > 0$ gives predictions for the spectrum of a γ_5-Hermitian D_W. On the contrary Wilson chiral perturbation theory with $W_6 > 0$, $W_7 > 0$ and $W_8 < 0$ gives predictions for the spectrum of D_{iW}.

The fact that all three signs are reversed when changing between Wilson and iWilson fermions is not accidental. Since the Wilson term and the iWilson term break chiral symmetry in exactly the same way, the respective low energy effective theories, must have the same symmetry breaking terms in the chiral Lagrangian. Moreover, since the explicit symmetry breaking terms at order a^2 have their origin in the Wilson term, the two effective fermionic Lagrangians are related by a combined change of sign of W_6, W_7 and W_8 [56].

In order to see which sign of W_6 and W_7 corresponds to Wilson fermions let us rewrite

the trace squared terms in Wilson chiral perturbation theory as

$$Z^{\nu}_{N_f}(\hat{m},\hat{\zeta};\hat{a}_6,\hat{a}_7,\hat{a}_8) \;=\; \frac{1}{16\pi|\hat{a}_6\hat{a}_7|}\int_{-\infty}^{\infty} dy_6 dy_7 \;\exp\left[-\frac{y_6^2}{16|\hat{a}_6^2|}-\frac{y_7^2}{16|\hat{a}_7^2|}\right] \times Z^{\nu}_{N_f}(\hat{m}-y_6,\hat{\zeta}-y_7;\hat{a}_6=0,\hat{a}_7=0,\hat{a}_8), \tag{13}$$

valid for $W_6<0$ and $W_7<0$ and

$$Z^{\nu}_{N_f}(\hat{m},\hat{\zeta};\hat{a}_6,\hat{a}_7,\hat{a}_8) \;=\; \frac{1}{16\pi|\hat{a}_6\hat{a}_7|}\int_{-\infty}^{\infty} dy_6 dy_7 \;\exp\left[-\frac{y_6^2}{16|\hat{a}_6^2|}-\frac{y_7^2}{16|\hat{a}_7^2|}\right] \times Z^{\nu}_{N_f}(\hat{m}-iy_6,\hat{\zeta}-iy_7;\hat{a}_6=0,\hat{a}_7=0,\hat{a}_8), \tag{14}$$

valid for $W_6>0$ and $W_7>0$.

Let us first consider the case $W_7=0$. A negative value of W_6 corresponds to a Dirac operator that is compatible with the γ_5-Hermiticity of the Wilson Dirac operator. The additional fluctuations can be interpreted as collective fluctuations of the eigenvalues, z_k, of D_W parallel to the real z-axis. To see this, extend Eq. (13) to the graded generating functional, Eq. (8), and include y_6 in the graded mass matrix

$$\mathcal{M}-y_6=\mathrm{diag}(\hat{m}_1-y_6,\ldots,\hat{m}_{N_f}-y_6,\hat{z}-y_6,\hat{z}^*-y_6,\hat{z}'-y_6,\hat{z}'^*-y_6) \tag{15}$$

(see Eq. (21) below for further details). Such fluctuations are allowed for Wilson fermions since the eigenvalues of D_W come in pairs (z,z^*) or are strictly real. This is illustrated in the left hand panel of figure 1.

For a positive value of W_6 the corresponding Dirac operator is in a different Hermiticity class than the Wilson Dirac operator and will have different spectral properties. Therefore, we necessarily have $W_6<0$ for the Wilson Dirac operator. For the iWilson-lattice theory on the other hand, we have that $D^{\dagger}_{iW}=-D_{iW}$ and consequently purely imaginary eigenvalues. Moreover, since the eigenvalues are *not* paired with equal and opposite sign (for $a\neq 0$) the spectrum of iD_W can fluctuate along the imaginary axis, see the right hand panel of figure 1 for an illustration. The Dirac operator corresponding to $W_6>0$ is hence in the Hermiticity class of D_{iW}. In perfect agreement with the above conclusion for Wilson fermions and the fact that the two effective theories should have opposite signs for all three W_i's.

The story for W_7 is analogous: A negative value of W_7 corresponds to real fluctuations of the axial quark mass, which are compatible with the Hermiticity properties of the Wilson Dirac operator. These fluctuations can be interpreted as collective fluctuations of the eigenvalues, λ^5, of $D_5\equiv\gamma_5(D_W+m)$ parallel to the real λ^5-axis. Such fluctuations are allowed for Wilson fermions since D_5 is Hermitian and the symmetry $(\lambda^5,-\lambda^5)$ is violated when $a\neq 0$.

For iWilson fermions the product $\gamma_5 D_{iW}$ has complex eigenvalues which come in pairs with opposite real part (or are strictly imaginary), hence their fluctuations can only take part in the imaginary direction. This is consistent with $W_7 > 0$ in the chiral Lagrangian for iWilson fermions and in perfect agreement with the fact that this sign should be opposite to that of the chiral Lagrangian for Wilson fermions.

Finally, when W_6 and W_7 have opposite signs the Hermiticity properties of the shifted Dirac operator always differ from the one realized at $W_6 = W_7 = 0$. The corresponding Dirac operator therefore is neither γ_5-Hermitian nor anti-Hermitian. The same is true if all W_i have the same sign.

In conclusion, we explained that the signs of the low energy constants of Wilson chiral perturbation theory follow from the γ_5-Hermiticity of the Wilson Dirac operator. We have, $W_6 < 0$, $W_7 < 0$ and $W_8 > 0$. Note that both the Aoki phase with $W_8 + 2W_6 > 0$ and the Sharpe-Singleton scenario with $W_8 + 2W_6 < 0$ are allowed by γ_5-Hermiticity.

In the reminder of this paper we will work with $W_6 < 0$, $W_7 < 0$ and $W_8 > 0$. Moreover, since the low energy constant W_7 does not affect the competition between the Aoki phase and the Sharpe-Singleton scenario we will set $W_7 = 0$.

In section VI below we show how a collective effect on the eigenvalues of D_W induced by $W_6 < 0$ leads to a shift between the Aoki and the Sharpe-Singleton scenario. To establish this result we will first derive the unquenched microscopic eigenvalue density of D_W.

V. THE UNQUENCHED SPECTRUM OF D_W

In this section we calculate the microscopic spectral density of the Wilson Dirac operator, D_W, in the presence of two dynamical flavors. We first carry through the calculation with $W_6 = W_7 = 0$ and subsequently introduce the effects of W_6. In order to derive the microscopic spectral density of D_W it is convenient to use Wilson chiral random matrix theory introduced in [11], which is reviewed in Appendix A for completeness.

We start from the joint eigenvalue probability distribution of the random matrix partition function Eq. (42). To obtain the eigenvalue density in the complex plane we integrate over all but a complex pair of eigenvalues. Using the properties of the Vandermonde determinant

we obtain ($\hat{z} = \hat{x} + i\hat{y}$)

$$\rho^{\nu}_{c,N_f=2}(\hat{z},\hat{z}^*,\hat{m};\hat{a}_8) \;=\; e^{-\hat{x}^2/(8\hat{a}_8^2)}\frac{|\hat{y}|e^{-4\hat{a}_8^2}}{16(2\pi)^{5/2}2\hat{a}_8}(\hat{z}-\hat{m})^2(\hat{z}^*-\hat{m})^2\frac{Z^{\nu}_4(\hat{z},\hat{z}^*,\hat{m},\hat{m};\hat{a}_8)}{Z^{\nu}_2(\hat{m},\hat{m};\hat{a}_8)}. \tag{16}$$

This amazingly compact form can be simplified further. In [46] it was shown that the four flavor partition function Z^{ν}_4 can be expressed in terms of two flavor partition functions. A proof in terms of chiral Lagrangians is given in Appendix B. This leads to the final form for the microscopic spectral density of D_W with two dynamical flavors

$$\begin{aligned}\rho^{\nu}_{c,N_f=2}(\hat{z},\hat{z}^*,\hat{m};\hat{a}_8) \;=\; & e^{-\hat{x}^2/(8\hat{a}_8^2)}\frac{|\hat{y}|e^{-4\hat{a}_8^2}}{16(2\pi)^{5/2}2\hat{a}_8}Z^{\nu}_2(\hat{z},\hat{z}^*;\hat{a}_8) \qquad (17)\\ & \times\Big(1-\frac{1}{2i\hat{y}}\frac{\partial_{\hat{m}}[\hat{Z}^{\nu}_2(\hat{z},\hat{m};\hat{a}_8)]\hat{Z}^{\nu}_2(\hat{z}^*,\hat{m};\hat{a}_8)-\hat{Z}^{\nu}_2(\hat{z},\hat{m};\hat{a}_8)\partial_{\hat{m}}[\hat{Z}^{\nu}_2(\hat{z}^*,\hat{m};\hat{a}_8)]}{Z^{\nu}_2(\hat{m},\hat{m};\hat{a}_8)Z^{\nu}_2(\hat{z},\hat{z}^*;\hat{a}_8)}\Big),\end{aligned}$$

where the two flavor partition function is given by [43]

$$\begin{aligned}Z^{\nu}_{N_f=2}(\hat{m}_1,\hat{m}_2;\hat{a}_8) \;=\; & \frac{e^{4\hat{a}_8^2}}{\pi 8\hat{a}_8^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}ds_1ds_2\frac{(is_1-is_2)}{\hat{m}_1-\hat{m}_2}(is_1)^{\nu}(is_2)^{\nu}\tilde{Z}^{\nu}_2(is_1,is_2;\hat{a}_8=0)\\ & \times\exp\left[-\frac{1}{16\hat{a}_8^2}[(s_1+i\hat{m}_1)^2+(s_2+i\hat{m}_2)^2]\right], \qquad (18)\end{aligned}$$

with

$$\tilde{Z}^{\nu}_2(x_1,x_2;\hat{a}_8=0)=\frac{2}{x_1^{\nu}x_2^{\nu}(x_2^2-x_1^2)}\det\begin{vmatrix} I_{\nu}(x_1) & x_1I_{\nu+1}(x_1)\\ I_{\nu}(x_2) & x_2I_{\nu+1}(x_2)\end{vmatrix}, \tag{19}$$

and we have introduced the notation $\hat{Z}^{\nu}_2(\hat{m}_1,\hat{m}_2;\hat{a}_8)\equiv(\hat{m}_1-\hat{m}_2)Z^{\nu}_2(\hat{m}_1,\hat{m}_2;\hat{a}_8)$.

The expression in the first line of Eq. (17) is the quenched eigenvalue density of D_W [44]. The correction factor in the second line is responsible for the eigenvalue repulsion from the quark mass. A plot of the eigenvalue density of the Wilson Dirac operator in the complex plane for two dynamical flavors is given in figure 2.

Note the strong similarity with the result for the eigenvalue density of the continuum Dirac operator at nonzero chemical potential in phase quenched QCD [50]. In that case the eigenvalue density follows from the integrable Toda lattice hierarchy [51]. The analytical form of the eigenvalue density of the Wilson Dirac operator, Eq. (16), strongly suggests that a similar integrable structure is present in the microscopic limit of the Wilson lattice QCD partition function.

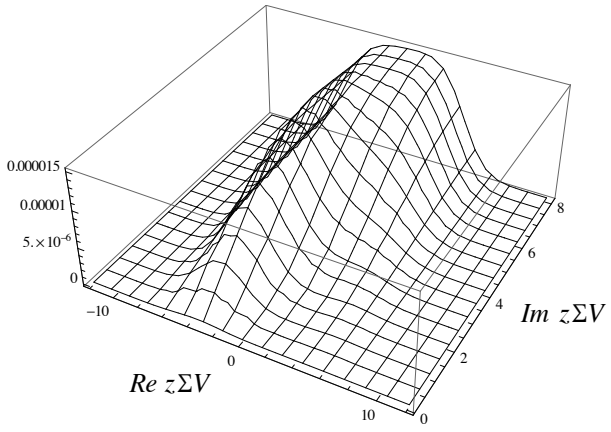

FIG. 2: The microscopic spectral density of the Wilson Dirac operator for $N_f = 2$ flavors of equal mass $\hat{m} = 2$ and $\hat{a}_8 = 0.8$ ($\hat{a}_6 = \hat{a}_7 = 0$) in the sector $\nu = 0$. The eigenvalues form a strip centered on the imaginary axis. Note the repulsion of the eigenvalues from the quark mass.

A. Including the effect of W_6

As pointed out in [12] the graded generating function for the eigenvalue density can be extended to include the effect of W_6 and W_7 by a Gaussian integral as in Eq. (13). Since this works for the graded generating functional it also works for the spectral density itself [12]. In the unquenched case, however, one must be careful with the normalization factor $1/Z^{\nu}_{N_f}(\hat{m}; \hat{a}_8)$.

Let us start with the case where $W_6 = W_7 = 0$. Then the density of D_W in the complex plane is obtained from the graded generating function as follows

$$\begin{aligned} \rho^{\nu}_{c,N_f}(\hat{z}, \hat{z}^*, \hat{m}; \hat{a}_8) &= \partial_{\hat{z}^*} \Sigma^{\nu}_{N_f+2|2}(\hat{z}, \hat{z}^*, \hat{m}; \hat{a}_8) \\ &= \partial_{\hat{z}^*} \lim_{\hat{z}' \to \hat{z}} \partial_{\hat{z}} \log Z^{\nu}_{N_f+2|2}(\hat{z}, \hat{z}^*, \hat{z}', \hat{z}'^*, \hat{m}; \hat{a}_8) \ , \end{aligned} \tag{20}$$

where the graded generating functional, $Z_{N_f+2|2}$, was introduced in Eq. (8).

To extend this to $W_6 < 0$ we first note that the Gaussian trick, Eq. (13), also works for

the graded generating functional. Using this we find

$$\begin{aligned}
\rho^{\nu}_{c,N_f}(\hat{z},\hat{z}^*,\hat{m};\hat{a}_6,\hat{a}_8) &= \partial_{\hat{z}^*}\lim_{\hat{z}'\to\hat{z}}\partial_{\hat{z}}\log Z^{\nu}_{N_f+2|2}(\hat{z},\hat{z}^*,\hat{z}',\hat{z}'^*,\hat{m};\hat{a}_6,\hat{a}_8) \qquad (21)\\
&= \partial_{\hat{z}^*}\lim_{\hat{z}'\to\hat{z}}\partial_{\hat{z}}\log\int[dy]Z^{\nu}_{N_f+2|2}(\hat{z}-y,\hat{z}^*-y,\hat{z}'-y,\hat{z}'^*-y,\hat{m}-y;\hat{a}_8)\\
&= \frac{1}{Z^{\nu}_{N_f}(\hat{m};\hat{a}_6,\hat{a}_8)}\int[dy]\ Z^{\nu}_{N_f}(\hat{m}-y;\hat{a}_8)\ \partial_{\hat{z}^*}\Sigma^{\nu}_{N_f+2|2}(\hat{z}-y,\hat{z}^*-y,\hat{m}-y;\hat{a}_8)\\
&= \frac{1}{Z^{\nu}_{N_f}(\hat{m};\hat{a}_6,\hat{a}_8)}\int[dy]\ Z^{\nu}_{N_f}(\hat{m}-y;\hat{a}_8)\rho^{\nu}_{c,N_f}(\hat{z}-y,\hat{z}^*-y,\hat{m}-y;\hat{a}_8),
\end{aligned}$$

where we will recall the notation: $[dy] = dy/(4\sqrt{\pi}|\hat{a}_6|)\exp(-y^2/(16|\hat{a}_6^2|))$.

In order to understand the effect of W_6 on the unquenched spectral density of D_W we will analyze the mean field limit of Eq. (21). As is shown in the next section the factor of $Z^{\nu}_{N_f}$ in the integrand, is essential for the realization of the Sharpe-Singleton scenario.

VI. THE SHARPE-SINGLETON SCENARIO IN THE SPECTRUM OF D_W

Here we show that the Sharpe-Singleton scenario can be understood in terms of a collective effect of the eigenvalues of D_W induced by $W_6 < 0$ when the quark mass changes sign. The Sharpe-Singleton scenario is therefore not realized in the quenched theory even if $W_8 + 2W_6 < 0$.

Before we give the proof let us first consider an electrostatic analogy which can help set the stage. The quenched chiral condensate

$$\int d^2z\ \frac{\rho_{N_f=0}(z,z^*;a)}{z-m} \qquad (22)$$

can be thought of as the electric field (in two dimensions) created by positive charges located at the positions of the eigenvalues z of D_W and measured at the position m (which can be thought of as a test charge). At the point where the quark mass hits the strip of eigenvalues of D_W centered on the imaginary axis, the mass dependence of the chiral condensate (electric field) shows a kink. As the quark mass is lowered further (the test charge passes through the strip of eigenvalues) the condensate (electric field) drops linearly to zero at $m = 0$. The drop is linear because the eigenvalue density is uniform.

For the unquenched chiral condensate we reach an identical conclusion provided that the quark mass (test charge) only has a local effect on the eigenvalues, i.e. it only affects

eigenvalues close to the quark mass. This is the case for the Aoki phase when the quark mass is inside the strip of eigenvalues of D_W.

On the contrary, in order to realize the first order Sharpe-Singleton scenario the quark mass must have a collective effect on the eigenvalues of D_W such that the strip of eigenvalues is entirely to the left of the quark mass for small positive values of m and then at $m = 0$ the strip collectively jumps to the opposite side of the origin such that for small negative values of the quark mass the strip of eigenvalues is to the right of m. The collective jump of the eigenvalues at $m = 0$ flips the sign of the chiral condensate (electric field) in agreement with the Sharpe-Singleton scenario.

In order to show that the Sharpe-Singleton scenario is indeed realized in terms of the eigenvalues of D_W in the manner described above let us analyze the effect of $W_6 < 0$ on the eigenvalues of D_W.

A. The mean field eigenvalue density of D_W

In the mean field limit the density of eigenvalues of D_W at $\hat{a}_6 = 0$ is simply given by a uniform strip of half width $8\hat{a}_8^2/\Sigma$ centered on the imaginary axis (the deriviation of this result is analogous to the one for nonzero chemical potential, see [49, 52])

$$\rho^{\rm MF}_{c,N_f=2}(\hat{x}, \hat{m}; \hat{a}_8) = \theta(8\hat{a}_8^2 - |\hat{x}|). \tag{23}$$

This result is identical to the quenched mean field spectral density since the correction factor in the second line of Eq. (17) only has an effect on the microscopic scale (the direct repulsion of the eigenvalues from the quark mass has a microscopic range).

To include the effect of $\hat{a}_6$ we use the Gaussian trick discussed in Eq. (21). The simplest way to proceed is to take the mean field limit before the y_6-integration, we find

$$\rho^{\rm MF}_{c,N_f=2}(\hat{x}, \hat{m}; \hat{a}_6, \hat{a}_8) = \frac{1}{Z_2^{\rm MF}(\hat{m}; \hat{a}_6, \hat{a}_8)} \int dy_6 \; e^{-y_6^2/16|\hat{a}_6^2|} Z_2^{\rm MF}(\hat{m} - y_6; \hat{a}_8)\theta(8\hat{a}_8^2 - |\hat{x} - y_6|). \tag{24}$$

Note the essential way in which the two flavor partition function enters both in numerator and the denominator. This is what separates the mean field calculation with dynamical fermions from the quenched analogue.

The mean field result for the two flavor partition function with $\hat{a}_6 = 0$ is given by

$$Z_2^{\rm MF}(\hat{m}; \hat{a}_8) = e^{2\hat{m}-4\hat{a}_8^2} + e^{-2\hat{m}-4\hat{a}_8^2} + \theta(8\hat{a}_8^2 - |\hat{m}|)e^{\hat{m}^2/8\hat{a}_8^2+4\hat{a}_8^2}. \tag{25}$$

The $\hat{a}_6$ dependence can again be restored by means of introducing an additional Gaussian integral. In the mean field limit this results in

$$Z_2^{\rm MF}(\hat{m};\hat{a}_6,\hat{a}_8) \;=\; e^{2\hat{m}+16|\hat{a}_6^2|-4\hat{a}_8^2}+e^{-2\hat{m}+16|\hat{a}_6^2|-4\hat{a}_8^2} \\ +\theta(8(\hat{a}_8^2+2\hat{a}_6^2)-|\hat{m}|)e^{\hat{m}^2/8(\hat{a}_8^2-2|\hat{a}_6^2|)+4\hat{a}_8^2}. \tag{26}$$

Note that when $2\hat{a}_6^2+\hat{a}_8^2<0$ the term in the second line of this equation is absent. The final result for the mean field two flavor eigenvalue density of D_W is

$$\rho^{\rm MF}_{c,N_f=2}(\hat{x},\hat{m};\hat{a}_6,\hat{a}_8) \;=\; \frac{1}{Z_2^{\rm MF}(\hat{m};\hat{a}_6,\hat{a}_8)} \\ \times\Big\{e^{2\hat{m}+16|\hat{a}_6^2|-4\hat{a}_8^2}\theta(8\hat{a}_8^2-|\hat{x}+16|\hat{a}_6|^2|) \\ +e^{-2\hat{m}+16|\hat{a}_6^2|-4\hat{a}_8^2}\theta(8\hat{a}_8^2-|\hat{x}-16|\hat{a}_6|^2|) \\ +\theta(8(\hat{a}_8^2+2\hat{a}_6^2)-|\hat{m}|)\theta\left(8\hat{a}_8^2-\left|\hat{x}+\frac{2|\hat{a}_6|^2\hat{m}}{(\hat{a}_8^2-2|\hat{a}_6|^2)}\right|\right)e^{\hat{m}^2/8(\hat{a}_8^2-2|\hat{a}_6^2|)+4\hat{a}_8^2}\Big\}. \tag{27}$$

A derivation of this result which includes the fluctuations around the saddle points is given in Appendix C.

In order to access the Sharpe-Singleton scenario let us consider the case where $\hat{m}$ is small compared to $16|\hat{a}_6^2|-8\hat{a}_8^2$ which is taken large and positive.

The terms in the second line of Eq. (27) give rise to a strip of eigenvalues of half width $8\hat{a}_8^2/\Sigma$ centered at -$16|\hat{a}_6^2|/\Sigma$ while the term in the third line gives rise to a strip of eigenvalues of half width $8\hat{a}_8^2/\Sigma$ centered at $16|\hat{a}_6^2|/\Sigma$. The relative height of the two strips is $\exp(4\hat{m})$. Therefore even though the magnitude of $\hat{m}$ is relatively small it has a dramatic effect: As the sign of $\hat{m}$ changes from positive to negative values the entire strip of eigenvalues jumps from its position around -$16|\hat{a}_6^2|/\Sigma$ to the new position around $16|\hat{a}_6^2|/\Sigma$. For a plot see figure 3. Because of the exponential suppression of one of the strips, the jump of the support of the spectrum occurs on a scale of $\hat{m}\sim O(1)$ or $m\sim 1/V\Sigma$ and leads to the first order discontinuity of the chiral condensate at $m=0$ as predicted by the Sharpe-Singleton scenario.

In the continuum limit the chiral condensate also jumps from Σ to $-\Sigma$ on a scale of $\hat{m}\sim O(1)$ or $m\sim 1/V\Sigma$, but in this case the difference in the potential between the two minima is of $O(\hat{m})$ as opposed to $O(\hat{a}_6^2)$ for the Sharpe-Singleton scenario.

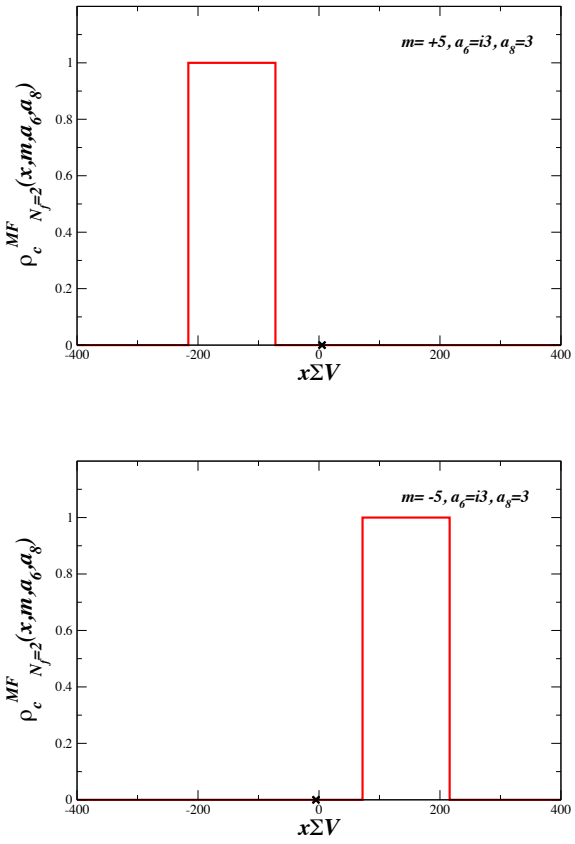

FIG. 3: The Wilson Dirac spectrum for the Sharpe-Singleton scenario: Shown is the mean field spectral density of the Wilson Dirac operator for $N_f = 2$ with $\hat{a}_6 = 3i$ and $\hat{a}_8 = 3$ ($\hat{a}_7 = 0$) as a function of $\hat{x} = \mathrm{Re}[\hat{z}]$ (the mean field density is independent of $\hat{y} = \mathrm{Im}[\hat{z}]$). The choice of $\hat{a}_6$ and $\hat{a}_8$ corresponds to a negative value of $W_8 + 2W_6$ and hence the Sharpe-Singleton scenario. The two flavors have equal mass $\hat{m} = 5$ (**top**) and $\hat{m} = -5$ (**bottom**). Even though the quark mass, marked by **x** on the x-axis, only changes by a small amount compared to the size of the gap the entire strip of eigenvalues jumps to the opposite side of the origin. This leads to the first order jump of the chiral condensate at $m = 0$.

The terms in the mean field two flavor partition function, see Eq. (25), are directly responsible for the jump of the eigenvalue density at $\hat{m} = 0$ in the theory with dynamical quarks. In the corresponding quenched computation we simply have

$$\rho^{\mathrm{MF}}_{c,N_f=0}(\hat{x};\hat{a}_6,\hat{a}_8) = \int dy_6 \; e^{-y_6^2/16|\hat{a}_6^2|}\theta(8\hat{a}_8^2 - |\hat{x} - y_6|), \tag{28}$$

which leads to a single strip of eigenvalues centered at the imaginary axis independent of

the value of W_6

$$\rho^{\rm MF}_{c,N_f=0}(\hat{x};\hat{a}_6,\hat{a}_8)=\theta(8\hat{a}_8^2-|\hat{x}|). \tag{29}$$

B. The connection to the mean field results of Sharpe and Singleton

From the results of the previous subsection we see that the gap from the quark mass to the edge of the strip of eigenvalues of D_W is given by

$$|m|-8(W_8+2W_6)a^2/\Sigma. \tag{30}$$

In [2] it was found that the pion masses for $|m|\Sigma>8(W_8+2W_6)a^2$ are given by

$$\frac{m_\pi^2F_\pi^2}{2}=|m|\Sigma-8(W_8+2W_6)a^2. \tag{31}$$

Hence the gap from the quark mass to the edge of the strip of eigenvalues of D_W can be thought of as the effective quark mass that enters the standard form of the GOR-relation. In particular, note that for $W_8+2W_6<0$ the mass never reaches the strip of eigenvalues. Correspondingly, the minimal value of the pion mass is given by

$$\frac{m_\pi^2F_\pi^2}{2}=-8(W_8+2W_6)a^2, \tag{32}$$

again in perfect agreement with the leading order p-regime computation of [2].

C. Direct computation of the quenched and unquenched condensate

From the essential part played by the dynamical fermion determinant in the realization of the Sharpe-Singleton scenario in terms of the eigenvalues of the Wilson Dirac operator we conclude that the Sharpe-Singleton first order scenario only takes place in the theory with dynamical quarks. Here we explicitly compute the quenched and unquenched microscopic chiral condensate and directly verify that the first order jump of the chiral condensate at $m=0$ only takes place in the theory with dynamical quarks.

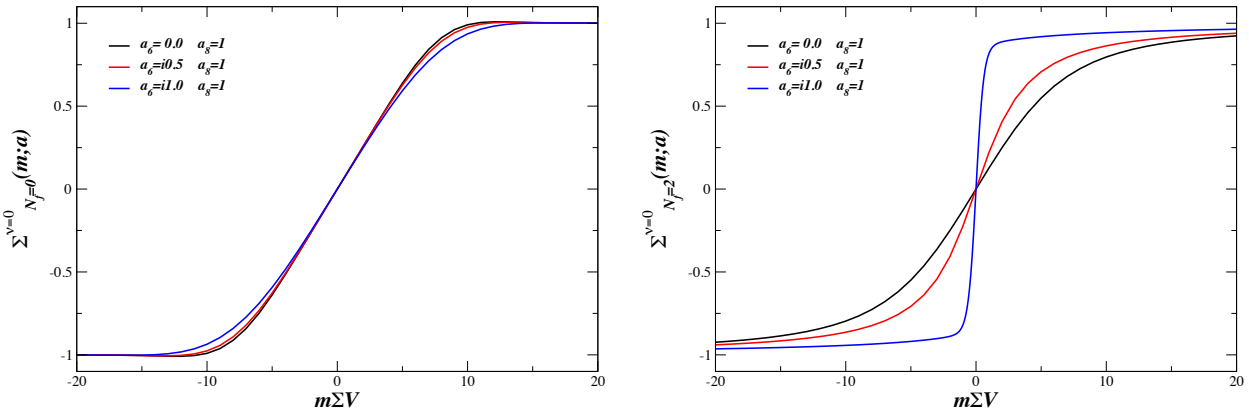

FIG. 4: The Sharpe-Singleton first order phase transition is due to dynamical quarks and is not present in the quenched case even if $W_8 + 2W_6 < 0$. Shown is the microscopic chiral condensate as a function of the quark mass for $\hat{a}_8 = 1$ and $\hat{a}_6 = 0, 0.5i$ and i corresponding to $W_8 + 2W_6 > 0$, $W_8 + 2W_6 = 0$ and $W_8 + 2W_6 < 0$, respectively. **Left** $N_f = 0$: In the quenched case there is hardly any effect of $W_6 < 0$. **Right** $N_f = 2$: For two flavors the increasingly negative W_6 drives the system from the Aoki phase to the Sharpe-Singleton scenario as can be seen by the formation of the discontinuity of the chiral condensate on a scale of $m \sim 1/V$.

The unquenched microscopic chiral condensate is obtained from the microscopic partition function by

$$\Sigma^{\nu}_{N_f}(\hat{m};\hat{a}_i) = \frac{1}{N_f}\frac{1}{Z^{\nu}_{N_f}}\frac{d}{d\hat{m}}Z^{\nu}_{N_f}(\hat{m};\hat{a}_i). \tag{33}$$

Specifically, for two mass degenerate flavors we have [11]

$$\begin{aligned}\Sigma^{\nu}_{N_f=2}(\hat{m},\hat{m};\hat{a}_i) \;=\; & \frac{1}{2}\frac{1}{Z^{\nu}_{N_f=2}(\hat{m};\hat{a}_i)} \\ & \times \int_{-\pi}^{\pi} d\theta_1 d\theta_2 |e^{i\theta_1} - e^{i\theta_2}|^2 e^{i\nu(\theta_1+\theta_2)}(\cos\theta_1 + \cos\theta_2) \\ & \times \exp\left[\hat{m}(\cos\theta_1 + \cos\theta_2) - 4\hat{a}_6^2(\cos\theta_1 + \cos\theta_2)^2 - 2\hat{a}_8^2(\cos(2\theta_1) + \cos(2\theta_2))\right]\end{aligned} \tag{34}$$

with

$$\begin{aligned}Z^{\nu}_{N_f=2}(\hat{m};\hat{a}_i) \;=\; & \int_{-\pi}^{\pi} d\theta_1 d\theta_2 |e^{i\theta_1} - e^{i\theta_2}|^2 e^{i\nu(\theta_1+\theta_2)} \\ & \times \exp\left[\hat{m}(\cos\theta_1 + \cos\theta_2) - 4\hat{a}_6^2(\cos\theta_1 + \cos\theta_2)^2 - 2\hat{a}_8^2(\cos(2\theta_1) + \cos(2\theta_2))\right].\end{aligned} \tag{35}$$

The quenched condensate was derived in [12, 22]

$$\Sigma^{\nu}_{N_f=0}(\hat{m};\hat{a}_i) \;=\; \int_{-\infty}^{\infty} ds \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\; \sin(\theta) e^{(i\theta-s)\nu} \exp[-\hat{m}\sin(\theta) - i\hat{m}\sinh(s) - \epsilon\cosh s$$
$$+4\hat{a}_6^2(-i\sin(\theta)+\sinh(s))^2 + 4\hat{a}_7^2(\cos(\theta)-\cosh(s))^2 + 2\hat{a}_8^2(\cos(2\theta)-\cosh(2s))]$$
$$\times\Big(-\frac{\hat{m}}{2}\sin(\theta) + i\frac{\hat{m}}{2}\sinh(s) - 4(\hat{a}_6^2+\hat{a}_7^2)(\sin^2(\theta)+\sinh^2(s))$$
$$+2\hat{a}_8^2(\cos(2\theta)+\cosh(2s)+e^{i\theta+s}+e^{-i\theta-s})+\frac{1}{2}\Big). \qquad (36)$$

In Refs. [12, 22], its imaginary part was studied since it is directly related to the real eigenvalues of D_W. Here we are after the quenched condensate itself which is given by its real part. Figure 4 compares the behavior of the quenched chiral condensate and the chiral condensate for $N_f = 2$ for three sets of values of W_6 and W_8. While the first order jump forms in the thermodynamic limit for the condensate with dynamical quarks when W_8+2W_6 turns negative the kink in the mass dependence of the quenched condensate remains. This directly verifies that the Sharpe-Singleton scenario is absent in quenched theory independent of the value of W_6.

Note that the authors of [6] concluded that both the Aoki phase and the Sharpe-Singleton scenario are possible in the quenched theory. They reached this conclusion because they worked in the large N_c limit in which W_6 and W_7 vanish, and because the constraint on the sign of W_8 was not known at the time.

VII. CONCLUSIONS

The first order scenario of Sharpe and Singleton for lattice QCD with Wilson fermions has been studied from the perspective of the eigenvalues of the Wilson Dirac operator. The behavior of the Wilson Dirac eigenvalues not only gives constraints on the additional low energy parameters of Wilson chiral perturbation theory ($W_6 < 0$, $W_7 < 0$ and $W_8 > 0$), it also allows us to explain the way in which the first order discontinuity of the chiral condensate is realized. In particular, we have shown that the associated collective jump of the spectrum of the Wilson Dirac operator only occurs in the theory with dynamical quarks. The Sharpe-Singleton scenario is therefore not realized in the quenched theory which enters in the Aoki phase at sufficiently small quark mass. By a direct computation of the quenched microscopic chiral condensate we verified that the second order phase transition occurs in the quenched theory even if $W_8 + 2W_6 < 0$. This explains the puzzle why the Aoki phase

dominates in the chiral limit of quenched lattice simulations while both the Aoki phase and the Sharpe-Singleton scenario have been observed in lattice QCD with dynamical Wilson fermions.

The above conclusion was made possible by the computation of the exact analytical result for the microscopic spectral density of the Wilson Dirac operator in lattice QCD with two dynamical flavors. The explicit form of the microscopic expression allowed us to compute the mean field eigenvalue density and in turn make a direct connection to the original leading order p-regime results of Sharpe and Singleton.

It would be most interesting to test the predictions presented in this paper against dynamical lattice QCD simulations. Since the effects of W_6 and W_8 on the spectrum of D_W in the unquenched theory are drastically different this offers a direct way to determine the values of these low energy constants. An early lattice study of the Wilson Dirac eigenvalues in dynamical simulations with light quarks appeared in [53].

Finally, since the additional low energy constants of Wilson chiral perturbation theory parameterize the discretization errors, it is also most interesting to consider the effects of improvements of the lattice action on the unquenched spectrum of the Wilson Dirac operator [54].

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Appendix A. WILSON RANDOM MATRIX THEORY

In order to derive the microscopic spectral density of D_W it is convenient to use Wilson chiral random matrix theory introduced in [11].

The partition function of Wilson chiral random matrix theory is defined as

$$\tilde{Z}^{\nu}_{N_f} = \int dA dB dW \prod_{f=1}^{N_f} \det(\tilde{D}_W + \tilde{m}_f)\ \mathcal{P}(A,B,W). \tag{37}$$

The matrix integrals are over the complex Haar measure.

The random matrix analogue of the Wilson Dirac operator is

$$\tilde{D}_W = \begin{pmatrix} \tilde{a}A & iW \\ iW^\dagger & \tilde{a}B \end{pmatrix}, \tag{38}$$

where

$$A = A^\dagger \quad \text{and} \quad B^\dagger = B \tag{39}$$

are $(n+\nu)\times(n+\nu)$ and $n\times n$ complex matrices, respectively, and W is an arbitrary complex $(n+\nu)\times n$ matrix. Finally, the weight is

$$\mathcal{P}(A,B,W) \equiv \exp\left[-\frac{N}{4}\mathrm{Tr}[A^2+B^2] - \frac{N}{2}\mathrm{Tr}[WW^\dagger]\right], \tag{40}$$

where $N = 2n+\nu$.

As was shown in Ref. [12], the Wilson random matrix partition function matches the microscopic partition function of Wilson chiral perturbation theory in the limit $N\to\infty$ with $N\tilde{m}$ and $N\tilde{a}^2$ fixed provided that we identify

$$N\tilde{m} = m\Sigma V, \qquad \frac{N\tilde{a}^2}{4} = a^2 W_8 V. \tag{41}$$

An eigenvalue representation of the partition function was derived in [44]

$$\tilde{Z}^\nu_{N_f} \;=\; \int dZ\; \Delta_{2n+\nu}(Z)\; \prod_{a=1}^{n}(z_{ar}-m)^{N_f} \prod_{b=1}^{n+\nu}(z_{bl}-m)^{N_f} \prod_{a=1}^{n} g_2(z_{al},z_{ar}) \prod_{b=1}^{\nu} z_{bl}^{b-1} g_1(z_{bl}) \tag{42}$$

where $Z = (z_{1r}\ldots,z_{nr},z_{1l},\ldots,z_{n+\nu,l})$ are the $2n+\nu$ eigenvalues of D_W and

$$g_1(z) = \sqrt{\frac{n}{2\pi\tilde{a}^2}}\exp\left[-\frac{n}{2\tilde{a}^2}x^2\right]\delta(y), \tag{43}$$

and

$$\begin{aligned} g_2(z_1,z_2) \;=\; & \sqrt{\frac{n^3}{4\pi\tilde{a}^2(1+a^2)}}\frac{z_1^*-z_2^*}{|z_1-z_2|} \\ & \times \left[\exp\left[-\frac{n(x_1+x_2)^2}{4\tilde{a}^2} - \frac{n(y_1-y_2)^2}{4}\right]\delta^{(2)}(z_1-z_2^*)\right. \\ & + \left.\frac{1}{2}\exp\left[-\frac{n}{4\tilde{a}^2}(x_1+x_2)^2 + \frac{n}{4}(x_1-x_2)^2\right]\right. \\ & \times \left.\mathrm{erfc}\left[\frac{\sqrt{n(1+\tilde{a}^2)}}{2\tilde{a}}|x_1-x_2|\right]\delta(y_1)\delta(y_2)\right]. \end{aligned} \tag{44}$$

Finally, $\Delta(Z)$ is the Vandermonde determinant of the $2n+\nu$ eigenvalues.

In section V we use this eigenvalue representation to derive the general form of the unquenched spectral density of D_W.

Appendix B. SIMPLIFICATION OF THE PARTITION FUNCTION

In this appendix we express the general partition function with even N_f in terms of a Pfaffian of two flavor partition functions. This Pfaffian form was first given in [46]. Here we give a proof in terms of chiral Lagrangians rather than random matrix theories. In particular, we explicitly express the four flavor partition function entering Eq. (16) in terms of two flavor partition functions.

We start from the general N_f microscopic partition function, Eq. (5), with $\hat{a}_6 = \hat{a}_7 = 0$ and make use of the identity

$$\begin{aligned} \exp\left[\hat{a}_8^2 \mathrm{Tr}(U^2 + U^{-2})\right] &= \exp\left[2N_f\hat{a}_8^2 + \hat{a}_8^2 \mathrm{Tr}(U - U^{-1})^2\right], \\ &= ce^{2N_f\hat{a}_8^2} \int d\sigma \exp\left[\frac{\mathrm{Tr}\sigma^2}{16\hat{a}_8^2} + \frac{i}{2}\mathrm{Tr}\sigma(U - U^{-1})\right], \end{aligned} \tag{45}$$

where σ is an $N_f \times N_f$ anti-Hermitian matrix and c a normalization constant. After a shift of σ by $\mathcal{M}$ we obtain

$$Z^{\nu}_{N_f}(\mathcal{M}; \hat{a}_8) = ce^{2N_f\hat{a}_8^2} \int d\sigma \int dU \, \det{}^{\nu}(iU) \exp\left[\frac{\mathrm{Tr}(\sigma - \mathcal{M})^2}{16\hat{a}_8^2} + \frac{i}{2}\mathrm{Tr}\,\sigma(U - U^{-1})\right]. \tag{46}$$

The next step is to decompose $\sigma = uSu^{-1}$ with S a diagonal matrix and perform the integration over u by the Itzykson-Zuber integral. We find

$$\begin{aligned} Z^{\nu}_{N_f}(\mathcal{M}; \hat{a}_8) &= \frac{e^{2N_f\hat{a}_8^2}}{(16\pi\hat{a}_8^2)^{N_f/2}} \int ds \frac{\Delta(S)}{\Delta(\mathcal{M})} \exp\left[\frac{\mathrm{Tr}(S - \mathcal{M})^2}{16\hat{a}_8^2}\right] \\ &\quad \times \prod_k (is_k)^{\nu} \tilde{Z}^{\nu}_{N_f}\left(\{is_k\}; \hat{a}_8 = 0\right). \end{aligned} \tag{47}$$

The Vandermonde determinant is defined by

$$\Delta(x_1, \cdots, x_p) = \prod_{k>l}^{p} (x_k - x_l), \tag{48}$$

and an explicit expression for the partition function at $\hat{a}_8 = 0$ is given by

$$\begin{aligned} &\tilde{Z}^{\nu}_{N_f}(x_1, \cdots, x_{N_f}; \hat{a}_8 = 0) \\ &\quad = c \left(\frac{1}{\prod_{k=1}^{N_f} x_k}\right)^{\nu} \frac{\det[(x_k)^{l-1} I_{\nu+l-1}(x_k)]}{\Delta(x_1^2, \cdots, x_{N_f}^2)}. \end{aligned} \tag{49}$$

We have that

$$\Delta(x_k) \prod_k (x_k)^{\nu} \tilde{Z}^{\nu}_{N_f}(x_k; \hat{a}_8 = 0) = \frac{\Delta(x_k)}{\Delta(x_k^2)} \det x_k^{l-1} I_{\nu+l-1}(x_k), \tag{50}$$

which we will denote by the symbol D. We now express D as a Pfaffian.

By using recursion relations for Bessel functions, D can be rewritten as

$$D \equiv \frac{\Delta(x_k)}{\Delta(x_k^2)} \det x_k^{l-1} I_{\nu+P(l-1)}(x_k), \tag{51}$$

where $P(k) = (1-(-1)^k)/2$. Writing the determinant as a sum over permutations and splitting the permutations into permutations of odd integers, π^o, even integers, π^e, and the mixed permutations of even and odd integers, π^{eo}, we obtain

$$D \equiv \frac{\Delta(x_k)}{\Delta(x_k^2)} \sum_{\pi^{eo}} (-1)^{\sigma^{eo}} \sum_{\pi^e} \sum_{\pi^o} (-1)^{\sigma^e+\sigma^o} \prod_{l=0 \text{ odd}}^{n-1} x_{\pi^o(l)}^{2l} I_\nu(x_{\pi^o(l)}) \prod_{l=0 \text{ even}}^{n-1} x_{\pi^e(l)}^{2l+1} I_{\nu+1}(x_{\pi^e(l)}). \tag{52}$$

The permutation over the even and odd integers can be resummed into a Vandermonde determinant

$$\begin{aligned} \sum_{\pi^o} \prod_{l=0 \text{ odd}}^{n-1} (-1)^{\sigma^o} x_{\pi^o(l)}^{2l} I_\nu(x_{\pi^o(l)}) &= \Delta(x_{k^o}^2) \prod_{k^o \; odd} I_\nu(x_{k^o}) \\ \sum_{\pi^e} \prod_{l=0 \text{ even}}^{n-1} (-1)^{\sigma^e} x_{\pi^e(l)}^{2l+1} I_{\nu+1}(x_{\pi^e(l)}) &= \Delta(x_{k^e}^2) \prod_{k^e \; even} I_{\nu+1}(x_{k^e}). \end{aligned} \tag{53}$$

Next we combine the Vandermonde determinants as

$$\begin{aligned} \frac{\Delta(x_{k^o}^2)\Delta(x_{k^e}^2)\Delta(x_k)}{\Delta(x_k^2)} &= \frac{\Delta(x_{k^e})\Delta(x_{k^o})\Gamma(x_{k^o}, x_{k^e})}{\Gamma(x_{k^o}^2, x_{k^e}^2)} \\ &= \frac{\Delta(x_{k^e})\Delta(x_{k^o})}{\Gamma(x_{k^o}, -x_{k^e})} \\ &= \det \frac{1}{x_{k^o} + x_{l^e}} \end{aligned} \tag{54}$$

with

$$\Gamma(x_k, y_k) = \prod_{k,l} (x_k - y_l). \tag{55}$$

The combination D can thus be written as

$$D = \sum_{\pi^{eo}} (-1)^{\sigma^{eo}} \det \frac{I_\nu(x_{k^o}) x_{l^e} I_{\nu+1}(x_{l^e})}{x_{k^o} + x_{l^e}}. \tag{56}$$

The determinant is a sum over permutations of even and odd integers which together with π^{eo} can be combined into a sum over all permutations

$$D = \sum_{\pi} (-1)^{\sigma} \frac{I_\nu(x_{\pi(k)}) x_{\pi(l)} I_{\nu+1}(x_{\pi(l)})}{x_{\pi(k)} + x_{\pi(l)}}, \tag{57}$$

which is equal to the Pfaffian

$$D = \mathrm{Pf}\left[\frac{I_\nu(x_k)x_l I_{\nu+1}(x_l) - I_\nu(x_l)x_k I_{\nu+1}(x_k)}{x_k + x_l}\right], \tag{58}$$

where we have recovered the Pfaffian structure of [46]. This leads to [46]

$$Z^\nu_{N_f}(\mathcal{M};\hat{a}_8) = \frac{1}{\Delta(\mathcal{M})}\mathrm{Pf}[(\hat{m}_j - \hat{m}_i)Z^\nu_{N_f=2}(\hat{m}_j,\hat{m}_i;\hat{a}_8)]_{j,i=1,\ldots,N_f}. \tag{59}$$

The alternative proof given here shows that the result is manifestly universal.

A. The four flavor partition function

For the four flavor partition function entering Eq. (16) the Pfaffian structure yields

$$\begin{aligned} Z^\nu_{N_f=4}(\hat{z},\hat{z}^*,\hat{m}_3,\hat{m}_4;\hat{a}_8) \;=\; & \frac{Z^\nu_2(\hat{z},\hat{z}^*;\hat{a}_8)Z^\nu_2(\hat{m}_3,\hat{m}_4;\hat{a}_8)}{(\hat{z}-\hat{m}_3)(\hat{z}-\hat{m}_4)(\hat{z}^*-\hat{m}_3)(\hat{z}^*-\hat{m}_4)} \\ & -\frac{Z^\nu_2(\hat{z},\hat{m}_3;\hat{a}_8)Z^\nu_2(\hat{z}^*,\hat{m}_4;\hat{a}_8)}{(\hat{z}-\hat{z}^*)(\hat{z}-\hat{m}_4)(\hat{z}^*-\hat{m}_3)(\hat{m}_3-\hat{m}_4)} \\ & +\frac{Z^\nu_2(\hat{z}^*,\hat{m}_3;\hat{a}_8)Z^\nu_2(\hat{z},\hat{m}_4;\hat{a}_8)}{(\hat{z}-\hat{z}^*)(\hat{z}-\hat{m}_3)(\hat{z}^*-\hat{m}_4)(\hat{m}_3-\hat{m}_4)}. \end{aligned} \tag{60}$$

The latter two terms form a derivative in the limit $\hat{m}_3 \to \hat{m}_4 = \hat{m}$

$$\begin{aligned} Z^\nu_4(\hat{z},\hat{z}^*,\hat{m},\hat{m};\hat{a}_8) \;=\; & \frac{Z^\nu_2(\hat{z},\hat{z}^*;\hat{a}_8)Z^\nu_2(\hat{m},\hat{m};\hat{a}_8)}{(\hat{z}-\hat{m})^2(\hat{z}^*-\hat{m})^2} \\ & -\frac{\partial_{\hat{m}}[\hat{Z}^\nu_2(\hat{z},\hat{m};\hat{a}_8)]\hat{Z}^\nu_2(\hat{z}^*,\hat{m};\hat{a}_8) - \hat{Z}^\nu_2(\hat{z},\hat{m};\hat{a}_8)\partial_{\hat{m}}[\hat{Z}^\nu_2(\hat{z}^*,\hat{m};\hat{a}_8)]}{(\hat{z}-\hat{z}^*)(\hat{z}-\hat{m})^2(\hat{z}^*-\hat{m})^2}. \end{aligned} \tag{61}$$

With this we have succeeded in expressing the four flavor partition function in terms of the two flavor partition function. This form inserted in Eq. (16) leads to Eq. (17).

Appendix C. MEAN FIELD INCLUDING FLUCTUATIONS

Here we compute the mean field eigenvalue density of D_W including the fluctuations about the saddle points. In Appendix C A we derive the mean field limit of the two flavor partition function. A mean field approximation for the four flavor partition function that enters in the spectral density, (21), is given in Appendix C B, and the mean field result for the spectral density is derived in Appendix C C. We discuss the explicit dependence on the low energy constants W_6 and W_8 and give the result both for the Aoki phase and the Sharpe-Singleton scenario. As explained in section IV we have $W_6 < 0$ and $W_8 > 0$.

A. The two flavor partition function

We consider the two-flavor partition function

$$
\begin{aligned}
Z_2^\nu(\hat{m};\hat{a}_6,\hat{a}_8) &= \int\limits_{\mathrm{U}(2)} \exp\left[\frac{\hat{m}}{2}\mathrm{Tr}(U+U^{-1}) + |\hat{a}_6|^2[\mathrm{Tr}(U+U^{-1})]^2 - \hat{a}_8^2\mathrm{Tr}(U^2+U^{-2})\right] \\
&\times \det{}^\nu U d\mu(U) \\
&= \frac{1}{2\pi^2}\int\limits_{[0,2\pi]^2} \exp\left[\hat{m}(\cos\varphi_1+\cos\varphi_2) + 4|\hat{a}_6|^2(\cos\varphi_1+\cos\varphi_2)^2\right] \\
&\times \exp\left[-4\hat{a}_8^2(\cos^2\varphi_1+\cos^2\varphi_2) + 4\hat{a}_8^2\right] e^{\imath\nu(\varphi_1+\varphi_2)}\sin^2\left(\frac{\varphi_1-\varphi_2}{2}\right) d[\varphi] \\
&= \frac{1}{2\pi^2}\int\limits_{[0,2\pi]^2} \exp\left[-2(\hat{a}_8^2-2|\hat{a}_6|^2)\left(\cos\varphi_1+\cos\varphi_2 - \frac{\hat{m}}{4(\hat{a}_8^2-2|\hat{a}_6|^2)}\right)^2\right] e^{\imath\nu(\varphi_1+\varphi_2)} \\
&\times \exp\left[-2\hat{a}_8^2(\cos\varphi_1-\cos\varphi_2)^2 + 4\hat{a}_8^2 + \frac{\hat{m}^2}{8(\hat{a}_8^2-2|\hat{a}_6|^2)}\right]\sin^2\left(\frac{\varphi_1-\varphi_2}{2}\right) d[\varphi].
\end{aligned}
\tag{62}
$$

From the exponent we recognize that in the mean field limit we always have

$$
\cos\varphi_1 = \cos\varphi_2. \tag{63}
$$

For $\hat{a}_8^2 + 2\hat{a}_6^2 < 0$ the solution of

$$
\cos\varphi_1 + \cos\varphi_2 = \frac{\hat{m}}{4(\hat{a}_8^2-2|\hat{a}_6|^2)} \tag{64}
$$

is a minimum and does not contribute in the mean field limit (this is the case of the Sharpe-Singleton scenario). Therefore the maxima can only come from

$$
\sin\varphi_1 = \sin\varphi_2 = 0. \tag{65}
$$

In combination with Eq. (63) this yields the two solutions $\cos\varphi_1 = \cos\varphi_2 = \pm 1$.

We make the following expansion

$$
\begin{aligned}
\varphi_{1/2}^{(+)} &= \delta\varphi_{1/2}, \quad \cos\varphi_{1/2}^{(+)} = 1 - \frac{1}{2}\delta\varphi_{1/2}^2, \\
\varphi_{1/2}^{(-)} &= \pi + \delta\varphi_{1/2}, \quad \cos\varphi_{1/2}^{(-)} = -1 + \frac{1}{2}\delta\varphi_{1/2}^2.
\end{aligned}
\tag{66}
$$

The maximum of the two points is at $\cos\varphi_{1/2} = \mathrm{sign}\,\hat{m}$. Thus we obtain the two flavor

partition function

$$
\begin{aligned}
Z_2^{\mathrm{MF}}(\hat{m};\hat{a}_6,\hat{a}_8) &= \frac{1}{8\pi^2}\exp\left[2|\hat{m}|+16|\hat{a}_6|^2-4\hat{a}_8^2\right] \\
&\times \int\limits_{\mathbb{R}^2}\exp\left[-\left(\frac{|\hat{m}|}{2}+8|\hat{a}_6|^2-4\hat{a}_8^2\right)(\delta\varphi_1^2+\delta\varphi_2^2)\right](\delta\varphi_1-\delta\varphi_2)^2 d[\delta\varphi] \\
&= \frac{\exp\left[2|\hat{m}|+16|\hat{a}_6|^2-4\hat{a}_8^2\right]}{2\pi(|\hat{m}|+16|\hat{a}_6|^2-8\hat{a}_8^2)^2}
\end{aligned}
\tag{67}
$$

for $\hat{a}_8^2+2\hat{a}_6^2<0$.

For $\hat{a}_8^2+2\hat{a}_6^2>0$ (i.e. in the Aoki phase) the saddlepoint given in Eq. (64), is a maximum. Hence we have to take it into account in the saddlepoint analysis if the right hand side of Eq. (64) is in the interval $[-2,2]$. Thereby we recognize that there are actually four saddlepoints fulfilling both conditions (63) and (64). The two angles may have the same sign or the opposite one. Those with the same sign are algebraically suppressed by the $\sin^2$ in the measure.

Let $\varphi_0=\arccos(\hat{m}/(8\hat{a}_8^2-16|\hat{a}_6|^2))$. The expansion about $\pm\varphi_0$ is given by

$$
\begin{aligned}
\varphi_{1/2}^{(+)} &= \pm\varphi_0+\delta\varphi_{1/2}, \quad \cos\varphi_{1/2}^{(+)}=\frac{\hat{m}}{8\hat{a}_8^2-16|\hat{a}_6|^2}\mp\sin\varphi_0\delta\varphi_{1/2}, \\
\varphi_{1/2}^{(-)} &= \mp\varphi_0+\delta\varphi_{1/2}, \quad \cos\varphi_{1/2}^{(-)}=\frac{\hat{m}}{8\hat{a}_8^2-16|\hat{a}_6|^2}\pm\sin\varphi_0\delta\varphi_{1/2}.
\end{aligned}
\tag{68}
$$

The simplified integral which we have to solve is

$$
\begin{aligned}
&\frac{1}{2\pi^2}\int\limits_{\mathbb{R}^2}\exp\left[-2(\hat{a}_8^2-2|\hat{a}_6|^2)\sin^2\varphi_0\,(\delta\varphi_1-\delta\varphi_2)^2\right]e^{\imath\nu(\varphi_1+\varphi_2)} \\
&\times\ \exp\left[-2\hat{a}_8^2\sin^2\varphi_0(\delta\varphi_1+\delta\varphi_2)^2+4\hat{a}_8^2+\frac{\hat{m}^2}{8(\hat{a}_8^2-2|\hat{a}_6|^2)}\right]\sin^2\varphi_0 d[\delta\varphi] \\
&= \frac{1}{8\pi\sqrt{\hat{a}_8^2(\hat{a}_8^2-2|\hat{a}_6|^2)}}\exp\left[4\hat{a}_8^2+\frac{\hat{m}^2}{8(\hat{a}_8^2-2|\hat{a}_6|^2)}\right].
\end{aligned}
\tag{69}
$$

Hence the two flavor partion function is given by

$$
\begin{aligned}
Z_2^{\mathrm{MF}}(\hat{m};\hat{a}_6,\hat{a}_8) &= \frac{\exp\left[2|\hat{m}|+16|\hat{a}_6|^2-4\hat{a}_8^2\right]}{2\pi(|\hat{m}|+16|\hat{a}_6|^2-8\hat{a}_8^2)^2} \\
&+ \frac{1}{4\pi\sqrt{\hat{a}_8^2(\hat{a}_8^2-2|\hat{a}_6|^2)}}\exp\left[4\hat{a}_8^2+\frac{\hat{m}^2}{8(\hat{a}_8^2-2|\hat{a}_6|^2)}\right]\theta(8\hat{a}_8^2-16|\hat{a}_6|^2-|\hat{m}|)
\end{aligned}
\tag{70}
$$

for $\hat{a}_8^2+2\hat{a}_6^2>0$. Please notice that the second term results from two saddlepoints at $\pm\varphi_0$ and only appears in a certain range of the quark mass. Moreover the result (70) is also valid for $\hat{a}_8^2+2\hat{a}_6^2<0$ since the Heavyside distribution vanishes in this regime.

B. The modified four flavor partition function

We consider the four flavor partition function which for $W_6 < 0$ can be written as

$$\begin{aligned}
\widetilde{Z}_4^\nu(\hat{z},\hat{z}^*,\hat{m};\hat{a}_6,\hat{a}_8) &= \frac{\sqrt{\hat{a}_8^2+2|\hat{a}_6|^2}}{4\sqrt{\pi}|\hat{a}_6|\hat{a}_8}|\hat{y}||\hat{z}-\hat{m}|^4 \int\limits_{\mathbb{R}} dy_6 \exp\left[-\frac{y_6^2}{16|\hat{a}_6|^2}-\frac{(\hat{x}-y_6)^2}{8\hat{a}_8^2}-4\hat{a}_8^2\right] \\
\times\ & Z_4^\nu(\hat{z}-y_6,\hat{z}^*-y_6,\hat{m}-y_6;\hat{a}_8) \qquad (71)\\
=\ & |\hat{y}||\hat{z}-\hat{m}|^4 \int\limits_{\mathrm{U}(4)} \exp\left[\frac{1}{2}\mathrm{Tr}\,\mathrm{diag}(\hat{m},\hat{m},\hat{z},\hat{z}^*)(U+U^{-1})-\hat{a}_8^2\mathrm{Tr}(U^2+U^{-2})\right] \\
\times\ & \exp\left[\frac{4|\hat{a}_6|^2\hat{a}_8^2}{\hat{a}_8^2+2|\hat{a}_6|^2}\left(\frac{1}{2}\mathrm{Tr}(U+U^{-1})-\frac{\hat{x}}{4\hat{a}_8^2}\right)^2-\frac{\hat{x}^2}{8\hat{a}_8^2}-4\hat{a}_8^2\right]\det{}^\nu U\,d\mu(U) \\
=\ & \imath\frac{64}{\pi^4}\mathrm{sign}(\hat{y}) \int\limits_{[0,2\pi]^4} d[\varphi] \prod_{1\le i<j\le 4}\sin^2\left(\frac{\varphi_i-\varphi_j}{2}\right)\exp\left[-4\hat{a}_8^2\sum_{j=1}^4\cos^2\varphi_j+8\hat{a}_8^2\right] \\
\times\ & \exp\left[\frac{4|\hat{a}_6|^2\hat{a}_8^2}{\hat{a}_8^2+2|\hat{a}_6|^2}\left(\sum_{j=1}^4\cos\varphi_j-\frac{x}{4\hat{a}_8^2}\right)^2-\frac{\hat{x}^2}{8\hat{a}_8^2}-4\hat{a}_8^2+\imath\nu\sum_{j=1}^4\varphi_j\right] \\
\times\ & \frac{\det\left[\exp[\hat{m}\cos\varphi_j],\cos\varphi_j\exp[\hat{m}\cos\varphi_j],\exp[\hat{z}\cos\varphi_j],\exp[\hat{z}^*\cos\varphi_j]\right]}{\prod_{1\le i<j\le 4}(\cos\varphi_i-\cos\varphi_j)} \\
=\ & \frac{32}{\pi^4}\exp[4\hat{a}_8^2] \int\limits_{[0,2\pi]^4} d[\varphi] \prod_{1\le i<j\le 4}\sin^2\left(\frac{\varphi_i-\varphi_j}{2}\right) e^{\imath\nu(\varphi_1+\varphi_2+\varphi_3+\varphi_4)} \sum_{\omega\in\mathbf{S}(4)} \\
\times\ & \frac{\exp\left[-2\hat{a}_8^2(\cos\varphi_{\omega(1)}-\cos\varphi_{\omega(2)})^2-2\hat{a}_8^2(\cos\varphi_{\omega(3)}-\cos\varphi_{\omega(4)})^2\right]}{(\cos\varphi_{\omega(1)}-\cos\varphi_{\omega(3)})(\cos\varphi_{\omega(1)}-\cos\varphi_{\omega(4)})(\cos\varphi_{\omega(2)}-\cos\varphi_{\omega(3)})(\cos\varphi_{\omega(2)}-\cos\varphi_{\omega(4)})} \\
\times\ & \frac{\sin\left[|\hat{y}|(\cos\varphi_{\omega(3)}-\cos\varphi_{\omega(4)})\right]}{\cos\varphi_{\omega(3)}-\cos\varphi_{\omega(4)}} \\
\times\ & \exp\left[(4|\hat{a}_6|^2-2\hat{a}_8^2)(\cos\varphi_{\omega(1)}+\cos\varphi_{\omega(2)})^2+\hat{m}(\cos\varphi_{\omega(1)}+\cos\varphi_{\omega(2)})\right] \\
\times\ & \exp\left[-\frac{1}{8(\hat{a}_8^2+2|\hat{a}_6|^2)}[\hat{x}+8|\hat{a}_6|^2(\cos\varphi_{\omega(1)}+\cos\varphi_{\omega(2)})-4\hat{a}_8^2(\cos\varphi_{\omega(3)}+\cos\varphi_{\omega(4)})]^2\right].
\end{aligned}$$

The permutation group of four elements is denoted by $\mathbf{S}(4)$.

In the mean field limit we have to expand the partition function about the maxima of the exponent. Omitting the permutations we identify two imediate conditions,

$$\cos\varphi_1^{(0)}=\cos\varphi_2^{(0)} \quad \text{and} \quad \cos\varphi_3^{(0)}=\cos\varphi_4^{(0)}. \qquad (72)$$

This is solved by

$$\varphi_1^{(0)}=-\varphi_2^{(0)} \quad \text{and} \quad \varphi_3^{(0)}=-\varphi_4^{(0)}. \qquad (73)$$

Other choices are supressed by the Vandermonde determinant. Hence we have to maximize the function

$$\begin{aligned} f(x,\varphi_1) &= \exp\left[8(2|\hat{a}_6|^2-\hat{a}_8^2)\cos^2\varphi_1+2\hat{m}\cos\varphi_1\right] \\ &\times \exp\left[-\frac{1}{8(\hat{a}_8^2+2|\hat{a}_6|^2)}[\hat{x}+16|\hat{a}_6|^2\cos\varphi_1-8\hat{a}_8^2\cos\varphi_3]^2\right]. \end{aligned} \tag{74}$$

We consider the case $\hat{a}_8^2+2\hat{a}_6^2<0$ (the Sharpe-Singleton scenario). Therefore the extremum for $\cos\varphi_1$ is a minimum and not a maximum. The situation would be completely different for $\hat{a}_8^2+2\hat{a}_6^2>0$, see discussion after Eq. (83).

The maximum of $f(x,\varphi_1)$ for all x is given by

$$\max_{x\in\mathbb{R}} f(x,\varphi_1)=\exp\left[8(2|\hat{a}_6|^2-\hat{a}_8^2)\cos^2\varphi_1+2\hat{m}\cos\varphi_1\right]. \tag{75}$$

This result takes its maximum at $\cos\varphi_1^{(0)}=\operatorname{sign} m$ yielding

$$\max_{x\in\mathbb{R},\varphi_1\in[0,2\pi]} f(x,\varphi_1)=\exp\left[16|\hat{a}_6|^2-8\hat{a}_8^2+2|\hat{m}|\right]. \tag{76}$$

In the integral (71) this maximum should be inside the interval

$$\hat{x}\in\left[-8\hat{a}_8^2-16|\hat{a}_6|^2\operatorname{sign}\hat{m},8\hat{a}_8^2-16|\hat{a}_6|^2\operatorname{sign}\hat{m}\right]. \tag{77}$$

The condition for the second integral is then

$$\cos\varphi_3^{(0)}=\frac{\hat{x}+16|\hat{a}_6|^2\operatorname{sign}\hat{m}}{8\hat{a}_8^2}. \tag{78}$$

We make the following expansion

$$\begin{aligned} \varphi_1 &= \frac{1-\operatorname{sign}\hat{m}}{2}\pi+\delta\varphi_1,\quad \cos\varphi_1=\operatorname{sign}\hat{m}-\frac{\operatorname{sign}\hat{m}}{2}\delta\varphi_1^2, \\ \varphi_2 &= -\frac{1-\operatorname{sign}\hat{m}}{2}\pi+\delta\varphi_2,\quad \cos\varphi_2=\operatorname{sign}\hat{m}-\frac{\operatorname{sign}\hat{m}}{2}\delta\varphi_2^2, \\ \varphi_3 &= \varphi_3^{(0)}+\delta\varphi_3,\quad \cos\varphi_3=\cos\varphi_3^{(0)}-\sin\varphi_3^{(0)}\delta\varphi_3, \\ \varphi_4 &= -\varphi_3^{(0)}+\delta\varphi_4,\quad \cos\varphi_4=\cos\varphi_3^{(0)}+\sin\varphi_3^{(0)}\delta\varphi_4. \end{aligned} \tag{79}$$

This expansion is substituted into Eq. (71) and we omit the sum since each term gives the

same contribution and the degeneracy of the maximum,

$$\widetilde{Z}_4^{\rm MF}(\hat{z},\hat{z}^*,\hat{m};\hat{a}_6,\hat{a}_8) = 24\left(\frac{2}{\pi}\right)^4 \exp[4\hat{a}_8^2]\int\limits_{\mathbb{R}^4} d[\delta\varphi]\sin^2\varphi_3^{(0)}\sin^8\left(\frac{1-\operatorname{sign}\hat{m}}{4}\pi-\frac{\varphi_3^{(0)}}{2}\right)$$
$$\times\ (\delta\varphi_1-\delta\varphi_2)^2\,\frac{\exp\left[-2\hat{a}_8^2\sin^2\varphi_3^{(0)}(\delta\varphi_3+\delta\varphi_4)^2\right]}{(\operatorname{sign}\hat{m}-(\hat{x}+16|\hat{a}_6|^2\operatorname{sign}\hat{m})/8\hat{a}_8^2)^4}\,\frac{\sin\left[|\hat{y}|\sin\varphi_3^{(0)}(\delta\varphi_3+\delta\varphi_4)\right]}{\sin\varphi_3^{(0)}(\delta\varphi_3+\delta\varphi_4)} \qquad (80)$$
$$\times\ \exp\left[-\left(\frac{|\hat{m}|}{2}+8|\hat{a}_6|^2-4\hat{a}_8^2\right)(\delta\varphi_1^2+\delta\varphi_2^2)+2|\hat{m}|+16|\hat{a}_6|^2-8\hat{a}_8^2\right]$$
$$\times\ \exp\left[-\frac{2\hat{a}_8^4}{(\hat{a}_8^2+2|\hat{a}_6|^2)}\sin^2\varphi_3^{(0)}[\delta\varphi_3-\delta\varphi_4]^2\right]\theta(8\hat{a}_8^2-|\hat{x}+16|\hat{a}_6|^2\operatorname{sign}\hat{m}|).$$

This integral decouples into two two-fold integrals. We need the following integral for large $|y|$,

$$\int\limits_{\mathbb{R}}\exp[-2\hat{a}_8^2\lambda^2]\frac{\sin(|\hat{y}|\lambda)}{\lambda}d\lambda \;=\; \pi\mathrm{erf}\left[\frac{|\hat{y}|}{\sqrt{8}\hat{a}_8}\right]\overset{|\hat{y}|\gg 1}{=}\pi, \qquad (81)$$

where erf is the error function and use the identity

$$\begin{aligned}\sin^8\left(\frac{1-\operatorname{sign}\hat{m}}{4}\pi-\frac{\varphi_3^{(0)}}{2}\right) &= \frac{1}{2}\left(1-\cos\left(\frac{1-\operatorname{sign}\hat{m}}{2}\pi-\varphi_3^{(0)}\right)\right)^4\\ &= \frac{1}{16}\left(1-\operatorname{sign}\hat{m}\cos\varphi_3^{(0)}\right)^4\\ &= \frac{1}{16}(\operatorname{sign}\hat{m}-(\hat{x}+16|\hat{a}_6|^2\operatorname{sign}\hat{m})/8\hat{a}_8^2)^4. \qquad (82)\end{aligned}$$

Then the final result for the partition function is given by

$$\begin{aligned}\widetilde{Z}_4^{\rm MF}(\hat{z},\hat{z}^*,\hat{m};\hat{a}_6,\hat{a}_8) &= 3\left(\frac{2}{\pi}\right)^{3/2}\frac{\sqrt{\hat{a}_8^2+2|\hat{a}_6|^2}}{\hat{a}_8^2}\frac{\exp\left[2|\hat{m}|+16|\hat{a}_6|^2-4\hat{a}_8^2\right]}{(|\hat{m}|/2+8|\hat{a}_6|^2-4\hat{a}_8^2)^2}\\ &\times\ \theta(8\hat{a}_8^2-|\hat{x}+16|\hat{a}_6|^2\operatorname{sign}\hat{m}|) \qquad (83)\end{aligned}$$

for $\hat{a}_8^2+2\hat{a}_6^2<0$ (in the Sharpe Singleton scenario).

In the Aoki phase, $\hat{a}_8^2+2\hat{a}_6^2>0$, the extremum for $\cos\varphi_1$ is a maximum, cf. Eq. (74). However it will only contribute if

$$|\hat{m}|\le 8\hat{a}_8^2-16|\hat{a}_6|^2 \qquad (84)$$

and

$$\left|\hat{x}+\frac{2|\hat{a}_6|^2\hat{m}}{\hat{a}_8^2-2|\hat{a}_6|^2}\right|\le 8\hat{a}_8^2. \qquad (85)$$

Then the saddlepoint changes to

$$\begin{aligned}\cos\varphi_1^{(0)} &= \frac{\hat{m}}{8\hat{a}_8^2-16|\hat{a}_6|^2},\\ \cos\varphi_3^{(0)} &= \frac{\hat{x}}{8\hat{a}_8^2}+\frac{|\hat{a}_6|^2\hat{m}}{4\hat{a}_8^2(\hat{a}_8^2-2|\hat{a}_6|^2)}.\end{aligned} \tag{86}$$

Hence the expansion about the saddle points is given by

$$\begin{aligned}\varphi_1 &= \varphi_1^{(0)}+\delta\varphi_1, \quad \cos\varphi_1=\cos\varphi_1^{(0)}-\sin\varphi_1^{(0)}\delta\varphi_2,\\ \varphi_2 &= -\varphi_1^{(0)}+\delta\varphi_2, \quad \cos\varphi_2=\cos\varphi_1^{(0)}+\sin\varphi_1^{(0)}\delta\varphi_2,\\ \varphi_3 &= \varphi_3^{(0)}+\delta\varphi_3, \quad \cos\varphi_3=\cos\varphi_3^{(0)}-\sin\varphi_3^{(0)}\delta\varphi_3,\\ \varphi_4 &= -\varphi_3^{(0)}+\delta\varphi_4, \quad \cos\varphi_4=\cos\varphi_3^{(0)}+\sin\varphi_3^{(0)}\delta\varphi_4.\end{aligned} \tag{87}$$

The degeneracy of this maximum is four which results in the integral

$$\begin{aligned}&3\frac{2^{10}}{\pi^4}\exp[4\hat{a}_8^2]\int\limits_{\mathbb{R}^4} d[\delta\varphi]\sin^2\varphi_1^{(0)}\sin^2\varphi_3^{(0)}\sin^4\left(\frac{\varphi_1^{(0)}-\varphi_3^{(0)}}{2}\right)\sin^4\left(\frac{\varphi_1^{(0)}+\varphi_3^{(0)}}{2}\right)\\ &\times\ \frac{\exp\left[-2\hat{a}_8^2\sin^2\varphi_1^{(0)}(\delta\varphi_1+\delta\varphi_2)^2-2\hat{a}_8^2\sin^2\varphi_3^{(0)}(\delta\varphi_3+\delta\varphi_4)^2\right]}{(\cos\varphi_1^{(0)}-\cos\varphi_3^{(0)})^4}\\ &\times\ \frac{\sin\left[|\hat{y}|\sin\varphi_3^{(0)}(\delta\varphi_3+\delta\varphi_4)\right]}{\sin\varphi_3^{(0)}(\delta\varphi_3+\delta\varphi_4)}\\ &\times\ \exp\left[(4|\hat{a}_6|^2-2\hat{a}_8^2)\sin^2\varphi_1^{(0)}(\delta\varphi_1+\delta\varphi_2)^2+\frac{\hat{m}^2}{8\hat{a}_8^2-16|\hat{a}_6|^2}\right]\\ &\times\ \exp\left[-\frac{2}{\hat{a}_8^2+2|\hat{a}_6|^2}\left[2|\hat{a}_6|^2\sin\varphi_1^{(0)}(\delta\varphi_1-\delta\varphi_2)-\hat{a}_8^2\sin\varphi_3^{(0)}(\delta\varphi_3-\delta\varphi_4)\right]^2\right]\\ &=\ 6\left(\frac{2}{\pi}\right)^{3/2}\frac{1}{\hat{a}_8^2}\sqrt{\frac{\hat{a}_8^2+2|\hat{a}_6|^2}{\hat{a}_8^2-2|\hat{a}_6|^2}}\exp\left[\frac{\hat{m}^2}{8\hat{a}_8^2-16|\hat{a}_6|^2}+4\hat{a}_8^2\right].\end{aligned} \tag{88}$$

Combining this with the result (83) for $\hat{a}_8^2+2\hat{a}_6^2<0$ we find

$$\begin{aligned}\widetilde{Z}_4^{\mathrm{MF}}(\hat{z},\hat{z}^*,\hat{m};\hat{a}_6,\hat{a}_8) &= 3\left(\frac{2}{\pi}\right)^{3/2}\frac{\sqrt{\hat{a}_8^2+2|\hat{a}_6|^2}}{\hat{a}_8^2}\frac{\exp\left[2|\hat{m}|+16|\hat{a}_6|^2-4\hat{a}_8^2\right]}{(|\hat{m}|/2+8|\hat{a}_6|^2-4\hat{a}_8^2)^2}\\ &\times\ \theta(8\hat{a}_8^2-|\hat{x}+16|\hat{a}_6|^2\mathrm{sign}\,\hat{m}|)+6\left(\frac{2}{\pi}\right)^{3/2}\frac{1}{\hat{a}_8^3}\sqrt{\frac{\hat{a}_8^2+2|\hat{a}_6|^2}{\hat{a}_8^2-2|\hat{a}_6|^2}}\\ &\times\ \exp\left[\frac{\hat{m}^2}{8\hat{a}_8^2-16|\hat{a}_6|^2}+4\hat{a}_8^2\right]\theta(8\hat{a}_8^2-16|\hat{a}_6|^2-|\hat{m}|)\theta\left(8\hat{a}_8^2-\left|\hat{x}+\frac{2|\hat{a}_6|^2\hat{m}}{(\hat{a}_8^2-2|\hat{a}_6|^2)}\right|\right).\end{aligned} \tag{89}$$

This formula applies to both scenarios since the Heavyside distribution puts the second term to zero in the Sharpe-Singleton scenario.

C. The unquenched level density

Combining the mean field limit of the numerator and denominator of Eq. (21) given by Eq. (70) and Eq. (89), respectively, we obtain

$$\begin{aligned}\rho^{\rm MF}_{c,N_f=2}(\hat{x},\hat{m};\hat{a}_6,\hat{a}_8) &= \frac{1}{32(2\pi)^{5/2}\sqrt{\hat{a}_8^2+2|\hat{a}_6|^2}}\frac{\widetilde{Z}_4^{\rm MF}(\hat{z},\hat{z}^*,\hat{m};\hat{a}_6,\hat{a}_8)}{Z_2^{\rm MF}(\hat{m};\hat{a}_6,\hat{a}_8)} \\ &= \frac{3}{(2\pi)^3}\frac{1}{\hat{a}_8^2}\left[\theta(2|\hat{a}_6|^2-\hat{a}_8^2)\theta(8\hat{a}_8^2-|\hat{x}+16|\hat{a}_6|^2{\rm sign}\,\hat{m}|)\right. \\ &+ \left.\theta(8\hat{a}_8^2-16|\hat{a}_6|^2-|\hat{m}|)\theta\left(8\hat{a}_8^2-\left|\hat{x}+\frac{2|\hat{a}_6|^2\hat{m}}{(\hat{a}_8^2-2|\hat{a}_6|^2)}\right|\right)\right]\end{aligned} \tag{90}$$

independent of the value of W_6. The first term will drop out if we are in the Aoki phase whereas the second term vanishes in the Sharpe-Singleton scenario. However the reason for this mechanism is quite different in the two cases. In the Aoki phase the first term is exponentially supressed in comparison to the second one which results from the extremum (86). In the Sharpe-Singleton scenario the saddlepoint is a minimum and enters *a priori* not in the saddlepoint analysis. Hence we have to look at the boundaries of the four dimensional box spanned by the four cosinus, see the discussion in Appendix C B.

This mechanism explains why we find a second order transition in the Aoki phase and a first order transition in the Sharpe-Singleton scenario. The extremum (86) can cross the four dimensional box with varying quark mass $\hat{m}$ and eigenvalue $\hat{x}$. Hence we have a continuous process from one boundary to the other in the Aoki scenario. When this extremum is excluded as in the Sharpe-Singleton scenario, the maximum has to jump from one boundary to the other. This manifests itself in the sign of the mass in the Heavyside distribution of the first term and the mass itself in the other one.

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[55] Note that we use the convention of [11, 12] for the low energy constants W_6, W_7 and W_8. In [4] these constants are denoted by $-W_6'$, $-W_7'$ and $-W_8'$ respectively.

[56] This duality of the Wilson and iWilson fermion lattice theories is due to the fact that the two are related by an axial transformation and an interchange $m \leftrightarrow i\zeta$ and $\zeta \leftrightarrow im$: The axial transformation $R = \exp(i\pi/4)$ and $L = \exp(-i\pi/4)$ takes $D_W + m + \zeta\gamma_5 \rightarrow D_{iW} + i\zeta + im\gamma_5$. The corresponding transformation on the Goldstone field is $U \rightarrow RUL^\dagger = iU$.

Mixing of orthogonal and skew-orthogonal polynomials and its relation to Wilson RMT

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Abstract. The unitary Wilson random matrix theory is an interpolation between the chiral Gaussian unitary ensemble and the Gaussian unitary ensemble. This new way of interpolation is also reflected in the orthogonal polynomials corresponding to such a random matrix ensemble. Although the chiral Gaussian unitary ensemble as well as the Gaussian unitary ensemble are associated to the Dyson index $\beta = 2$ the intermediate ensembles exhibit a mixing of orthogonal polynomials and skew-orthogonal polynomials. We consider the Hermitian as well as the non-Hermitian Wilson random matrix and derive the corresponding polynomials, their recursion relations, Christoffel-Darboux-like formulas, Rodrigues formulas and representations as random matrix averages in a unifying way. With help of these results we derive the unquenched k-point correlation function of the Hermitian and the non-Hermitian Wilson random matrix in terms of two-flavor partition functions only. This representation is due to a Pfaffian factorization. It drastically simplifies the expressions which can be easily numerically evaluated. It also serves as a good starting point for studying the Wilson-Dirac operator in the ϵ-regime of lattice quantum chromodynamics.

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1. Introduction

In the microscopic limit chiral random matrix theory (χRMT) can be directly mapped to the ϵ-regime of quantum chromodynamics (QCD) and is successfully applied to it since the 90's [1, 2]. Both theories share the same universality class which is the reason for the existence of this equivalence. χRMT was also extended to a non-zero chemical potential by adding a scalar proportional to γ_0 [3, 4, 5, 6, 7]. In the last decade a second approach was pursued. A second chiral random matrix was introduced yielding the chiral analogue of the Ginibre ensembles [8, 9, 10, 11, 12, 13, 14, 15]. A quantitative analysis of the sign problem in Monte-Carlo simulations was quite elusive until it was

solved in χRMT [6, 7, 16, 17]. The hope is now to extend these new insights to QCD at non-zero lattice spacing.

Recently, random matrix theories for lattice QCD became the focus of interest. The idea is to derive analytical results of lattice artefacts in the data. One important realization of lattice QCD is by means of staggered fermions. In Refs. [18, 19], a χRMT was considered which is equivalent to the ϵ-regime of these fermions. Unfortunately, this model is highly involved due to the high number of low energy constants and, hence, of the coupling constants in the random matrix model.

The Wilson Dirac operator is another realization of lattice QCD. It proved that the corresponding random matrix model [20, 21, 22, 23, 19, 24] is much better accessible for analytical calculations than the one of the staggered fermions. The Wilson term which is given by a Laplace operator [25, 26] explicitly breaks chiral symmetry and is Hermitian. Thus the main idea was to add on the diagonal of χRMT two Hermitian matrices to simulate the same effect [20] and it proved to be in the same universality class as the Wilson Dirac operator in the ϵ-regime [27, 28, 29, 30]. Actually one can consider a Hermitian version [20, 21, 22, 23, 31, 32] of this random matrix ensemble which is numerically cheaper in lattice simulations. However only the non-Hermitian version [19, 24, 32] is directly related to the chiral symmetry breaking by a finite lattice spacing. The Hermitian version can also be considered as an interpolation between a chiral Gaussian unitary ensemble (χGUE) and a Gaussian unitary ensemble (GUE). The coupling constant is then the lattice spacing.

Quite recently this new kind of random matrix model has given new insights on the signs of the low energy constants in the chiral Lagrangian of the Wilson Dirac operator [21, 33, 34]. These signs are controversial since they are crucial to decide if an Aoki phase [35] exists or not. Such a phase is a pure lattice artefact and has no analogue in continuum QCD. Therefore a large analytical [27, 36, 33, 34] as well as numerical [37, 38, 39, 31, 32] effort was made to determine the low energy constants.

Orthogonal polynomial theory [40, 41, 42, 43, 44, 45, 46, 47] was as successfully applied to RMT as the supersymmetry method [48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. In particular the combination of both methods with the recently developed method of an algebraical rearrangement of the joint probability density with quotients of characteristic polynomials [58] are quite efficient to find compact and simple analytical results of the spectral correlations of random matrix ensembles. In this work we address the k-point correlation functions of the Hermitian as well as the non-Hermitian version and make use of such a combination.

An interesting point of view of Wilson RMT appears when we study it with help of orthogonal polynomial theory. In Ref. [23] the authors considered the Hermitian Wilson RMT and found that the construction of the skew-orthogonal polynomials strongly depend on the index ν of the random matrix which is the number of zero modes in the continuum limit. They only explicitly constructed these polynomials for $\nu = 0, 1$. In this article we construct these polynomials for an arbitrary index and for both version of Wilson RMT in a unifying way. We also successfully look for

a recursion relation, Christoffel Darboux-like formulas, Rodrigues formulas and their explicit expression as random matrix integrals. By this study we get a complete picture what these orthogonal and skew-orthogonal polynomials are and how they are related to the orthogonal polynomials of some limits, in particular the continuum limit and the limit of a large lattice spacing.

First we specify what are the conditions the orthogonal and skew-orthogonal polynomials have to fulfill. Thereby we recognize that the corresponding weight has to satisfy a particular property, too. Luckily we are able to modify the weight in the joint probability densities without changing the partition functions and the k-point correlation functions such that this property can be achieved. In the second step we construct the polynomials with help of Pfaffians whose anti-symmetry under permutations of rows and columns proves quite useful.

After we show some useful properties of the orthogonal and skew-orthogonal polynomials we derive a Pfaffian factorization of the k-point correlation functions. This factorization is for numerical evaluations advantageous because it reduces the complexity of the integrand to an average over two characteristic polynomials. In combination with the supersymmetry method [52, 54, 55, 57] one may simplify the whole problem to two-fold integrals. Factorizations to determinants and Pfaffians were found for many random matrix ensembles of completely different symmetries [40, 41, 42, 59, 60, 61, 62, 44, 46, 63, 58]. A Pfaffian for the eigenvalue correlations of the Hermitian Wilson random matrix ensemble was already shown in Ref. [23]. We prove the existence of such a structure for the non-Hermitian version, too. Furthermore we identify the kernels of both Pfaffians with two-flavor partition functions. The identification as well as the structure carry over to the microscopic limit which makes them also applicable to the chiral Lagrangian of the Wilson-Dirac operator.

We consider unquenched Wilson RMT, i.e. a finite number of fermionic flavors. Recently, the partition function with one fermionic flavor and the corresponding microscopic level density was studied in Ref. [64]. In our calculations the number of fermions may be arbitrary. Nevertheless all eigenvalue correlations, also the one of the unquenched theory, can be expressed by two-flavor partition functions because of the Pfaffian factorization.

Moreover such a Pfaffian determinant of the k-point correlation functions comes in handy when calculating the individual eigenvalue distributions. The authors of Ref. [65] were able to express the gap probability of the eigenvalues of the Hermitian Wilson random matrix ensemble as a Fredholm-Pfaffian only due to this structure. Hence a similar simplification is highly desirable for the non-Hermitian Wilson random matrix ensemble.

The outline of this article is as follows. In Sec. 2 we briefly introduce the Wilson random matrix model and its two kinds of joint probability densities corresponding to the Hermitian and the non-Hermitian version. With help of the k-point correlation function we propose the problem. In particular we will list the conditions the polynomials have to fulfill. In Sec. 3 we construct the orthogonal and skew-

orthogonal polynomials. Thereby we derive a recursion relation which is helpful to proof a Christoffel Darboux-like formula. Moreover we show a representation of the polynomials and the Christoffel Darboux-like formula as random matrix averages. Such a representation is useful to study the microscopic limit of the random matrix ensemble by means of the supersymmetry method. Hence we calculate the asymptotics of the polynomials and the Christoffel Darboux-like formula. In Sec. 4 we apply the derived results to the k-point correlation functions of the Hermitian and the non-Hermitian version of Wilson RMT and identify the kernels of the Pfaffian with two-flavor partition functions. Readers only interested in the k-point correlation functions of the Wilson random matrix ensemble can jump to this section because it contains the main results which can be mostly understood without the technical details in Sec. 3 due to the identification of the kernels with two-flavor partition functions. The conclusions are made in Sec. 5 and the details of the calculations are given in the appendices.

2. Two joint probability densities for one random matrix theory

The models we want to consider are motivated by the Wilson Dirac operator in lattice QCD [20]. The corresponding random matrix theory consists of the matrix

$$D_{\mathrm{W}} = \begin{pmatrix} A & W \\ -W^{\dagger} & B \end{pmatrix} \tag{2.1}$$

distributed by the Gaussian

$$\begin{aligned} P(D_{\mathrm{W}}) = & \left(\frac{n}{2\pi a^2}\right)^{[n^2+(n+\nu)^2]/2} \left(-\frac{n}{2\pi}\right)^{n(n+\nu)} \exp\left[-\frac{a^2}{2}\left(\mu_{\mathrm{r}}^2 + \frac{n+\nu}{n}\mu_{\mathrm{l}}^2\right)\right] \\ & \times \exp\left[-\frac{n}{2a^2}(\operatorname{tr} A^2 + \operatorname{tr} B^2) - n \operatorname{tr} WW^{\dagger} + \mu_{\mathrm{r}} \operatorname{tr} A + \mu_{\mathrm{l}} \operatorname{tr} B\right]. \end{aligned} \tag{2.2}$$

The Hermitian matrices A and B have the dimensions $n \times n$ and $(n+\nu) \times (n+\nu)$ and explicitly break chiral symmetry,

$$\gamma_5 \, D_{\mathrm{W}}|_{m=0} \, \gamma_5 \neq - \, D_{\mathrm{W}}|_{m=0} \quad \text{with} \quad \gamma_5 = \operatorname{diag}(\mathbb{1}_n, -\mathbb{1}_{n+\nu}). \tag{2.3}$$

The matrix W is a $n \times (n+\nu)$ complex matrix with independent entries. The variable a plays the role of the lattice spacing and the Gaussian of A and B yields one low energy constant known as W_8 [20, 21, 22, 19, 24, 34]. The variables $\mu_{\mathrm{r}\,/\mathrm{l}}$ might be also considered as Gaussian distributed random variables and generate two additional low energy constants, W_6 and W_7 [21, 22, 34], in chiral perturbation theory of the Wilson Dirac operator [27, 28, 29, 30]. Here we consider them as fixed constants to keep the calculation as simple as possible but the model is general enough to introduce also the Gaussian integrals for $\mu_{\mathrm{r}\,/\mathrm{l}}$ at the end of the day. They originate from a shift of the matrices A and B by mass terms. The case when we do not integrate over $\mu_{\mathrm{r}\,/\mathrm{l}}$ by Gaussians and keep them as constants corresponds to the low energy constants $W_6 = W_7 = 0$.

The parameter ν is called the index of the Dirac operator and is the number of the generic real modes of $D_{\rm W}$. Since $D_{\rm W}$ is γ_5-Hermitian, i.e. $D_{\rm W}^\dagger = \gamma_5 D_{\rm W} \gamma_5$, the matrix

$$D_5 = \gamma_5 D_{\rm W} \tag{2.4}$$

is Hermitian. Moreover the complex eigenvalues of $D_{\rm W}$ come in complex conjugated pairs only. The number of these pairs, l, varies from 0 to n.

The matrix $D_5 = VxV^{-1}$ can be diagonalized by a unitary matrix $V \in {\rm U}(2n+\nu)$ whereas the matrix $D_{\rm W} = UZ_lU^{-1}$ can only be quasi diagonalized by a matrix in the non-compact unitary group $U \in {\rm U}(n, n+\nu)$, i.e. $U^{-1} = \gamma_5 U^\dagger \gamma_5$. The diagonal matrix $x = {\rm diag}\,(x_1, \ldots, x_{2n+\nu})$ consists of real eigenvalues, only. The quasi-diagonal matrix

$$Z_l = \left(\begin{array}{cc|cc} x^{(1)} & 0 & 0 & 0 \\ 0 & x^{(2)} & y^{(2)} & 0 \\ \hline 0 & -y^{(2)} & x^{(2)} & 0 \\ 0 & 0 & 0 & x^{(3)} \end{array}\right), \tag{2.5}$$

depends on the real diagonal matrices $x^{(1)} = {\rm diag}\,(x_1^{(1)}, \ldots, x_{n-l}^{(1)})$, $x^{(2)} = {\rm diag}\,(x_1^{(2)}, \ldots, x_l^{(2)})$, $y^{(2)} = {\rm diag}\,(y_1^{(2)}, \ldots, y_l^{(2)})$ and $x^{(3)} = {\rm diag}\,(x_1^{(3)}, \ldots, x_{n+\nu-l}^{(3)})$ with the dimensions $n-l$, l, l and $n+\nu-l$, respectively. Then the complex conjugated eigenvalue pairs of $D_{\rm W}$ are $(z^{(2)}, z^{*(2)}) = (x^{(2)} + \imath y^{(2)}, x^{(2)} - \imath y^{(2)})$. The $n+1$ different sectors of different numbers of the complex conjugated pairs are labelled by l.

The joint probability density is one of the best quantities for analyzing the eigenvalue correlations of random matrices. It is also the starting point of our discussions in the ensuing sections. The Hermitian, D_5, and the non-Hermitian, $D_{\rm W}$, Wilson random matrix have different joint probability densities. Though these densities have a completely different form, we will see that their orthogonal and skew-orthogonal polynomials have much in common, see Sec. 3.

The joint probability density of D_5 is [23]

$$\begin{aligned} p_5(x)d[x] = c_-(1-a^2)^{-n(n+\nu-1/2)} a^{-n-\nu^2} \exp\left[-\frac{a^2}{2}\left(\mu_{\rm r}^2 + \left(1+\frac{\nu}{n}\right)\mu_{\rm I}^2\right) + \frac{n\widehat{m}_{6-}^2}{8\widehat{a}_-^2}\right]\Delta_{2n+\nu}(x) \\ \times {\rm Pf}\begin{bmatrix} \left\{g_2^{(-)}(x_i,x_j)\right\}_{1\le i,j\le 2n+\nu} & \{x_i^{j-1} g_1^{(-)}(x_i)\}_{\substack{1\le i\le 2n+\nu \\ 1\le j\le \nu}} \\ \{-x_j^{i-1} g_1^{(-)}(x_j)\}_{\substack{1\le i\le \nu \\ 1\le j\le 2n+\nu}} & 0 \end{bmatrix} \prod_{j=1}^{2n+\nu} dx_j, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} g_2^{(-)}(x_1,x_2) = {} & \exp\left[-\frac{n}{4a^2}(x_1+x_2)^2 - \frac{n}{4}(x_1-x_2)^2 + \frac{n\widehat{\lambda}_{7-}}{4\widehat{a}_-^2}(x_1+x_2)\right] \\ & \times {\rm erf}\left[\frac{1}{\sqrt{8}\widehat{a}_-}\left(n(x_2-x_1) - \widehat{m}_{6-}\right), \frac{1}{\sqrt{8}\widehat{a}_-}\left(n(x_1-x_2) - \widehat{m}_{6-}\right)\right], \end{aligned} \tag{2.7}$$

$$g_1^{(\pm)}(x) = \exp\left[-\frac{n}{2a^2}x^2 \pm \mu_{\rm I}\, x\right]. \tag{2.8}$$

We define the constants

$$\widehat{a}_\pm = \sqrt{\frac{na^2}{2(1\pm a^2)}}, \tag{2.9}$$

$$\widehat{m}_{6\pm} = \frac{2\widehat{a}_\pm^2}{n}(\mu_{\rm r}+\mu_{\rm l}), \tag{2.10}$$

$$\widehat{\lambda}_{7\pm} = \frac{2\widehat{a}_\pm^2}{n}(\mu_{\rm r}-\mu_{\rm l}), \tag{2.11}$$

$$\frac{1}{c_-} = \left(\frac{16\pi}{n}\right)^{n/2}(2\pi)^{\nu/2}n^{-\nu^2/2-n(n+\nu)}(2n+\nu)!\prod_{j=0}^{n-1}j!\prod_{j=0}^{n+\nu-1}j!\,. \tag{2.12}$$

The notation of $\widehat{m}_{6\pm}$ and $\widehat{\lambda}_{7\pm}$ reflects the nature of their symmetries. The constant $\widehat{m}_{6\pm}$ acts as an effective mass and $\widehat{\lambda}_{7\pm}$ as an effective axial mass, i.e. a source term proportional to γ_5. They refer to the low-energy constants W_6 and W_7 which are found in the microscopic limit [20, 21, 22, 19, 24, 34], i.e. $n\to\infty$, $\widehat{a}=\sqrt{n}a/\sqrt{2}=\widehat{a}_\pm=$ const., $\widehat{m}_6=\widehat{m}_{6\pm}=$ const., $\widehat{\lambda}_7=\widehat{\lambda}_{7\pm}=$ const. and $\widehat{z}=2nz=$ const. We emphasize that we have to integrate over $\widehat{m}_6$ and $\widehat{\lambda}_7$ to obtain the low energy constants W_6 and W_7, respectively. Please notice that our notation differs from the one in Refs. [20, 21] where the source terms proportional to γ_5 are denoted by z. To avoid confusion with the complex eigenvalues of $D_{\rm W}$ we denote these variables by λ in the present article. Furthermore we renamed the variables y_6 and y_7 to $\widehat{m}_6$ and $\widehat{\lambda}_7$, respectively, since the former notation can create a confusion with the imaginary parts of the complex eigenvalues of $D_{\rm W}$.

The Vandermonde determinant is given by

$$\Delta_{2n+\nu}(x)=\prod_{1\le i<j\le 2n+\nu}(x_i-x_j)=(-1)^{(2n+\nu)(2n+\nu-1)/2}\det\left[x_i^{j-1}\right]_{1\le i,j\le 2n+\nu}. \tag{2.13}$$

The function $\mathrm{erf}(x_1,x_2)=\mathrm{erf}(x_2)-\mathrm{erf}(x_1)$ is the generalized error function.

The Pfaffian in p_5, see Eq. (2.6), is due to the symmetrization of the eigenvalues. The two-point weight $g_2^{(-)}$ is anti-symmetric and is a strong interaction of two different eigenvalues. In the continuum limit, $a\to 0$, $g_2^{(-)}$ generates a Dirac delta function enforcing that we have always an eigenvalue pair $(\lambda,-\lambda)$ of the Dirac operator if $\lambda\neq 0$. The two off-diagonal blocks are reminiscent of Vandermonde determinants and are artefacts of the zero modes at $a=0$.

The joint probability density of $D_{\rm W}$ is

$$\begin{aligned}p_{\rm W}(Z)d[Z] =& c_+(1+a^2)^{-n(n+\nu-1/2)}a^{-n-\nu^2}\exp\left[-\frac{a^2}{2}\left(\mu_{\rm r}^2+\left(1+\frac{\nu}{n}\right)\mu_{\rm l}^2\right)+\frac{n\widehat{\lambda}_{7+}^2}{8\widehat{a}_+^2}\right]\Delta_{2n+\nu}(Z)\\ &\times\ \det\begin{bmatrix}\{g_2^{(+)}(z_i^{(\rm r)},z_j^{(\rm l)})\}_{\substack{1\le i\le n\\ 1\le j\le n+\nu}}\\ \{(x_j^{(\rm l)})^{i-1}g_1^{(+)}(x_j^{(\rm l)})\delta(y_j^{(\rm l)})\}_{\substack{1\le i\le \nu\\ 1\le j\le n+\nu}}\end{bmatrix}\prod_{j=1}^{n}dx_j^{(\rm r)}dy_j^{(\rm r)}\prod_{j=1}^{n+\nu}dx_j^{(\rm l)}dy_j^{(\rm l)},\end{aligned} \tag{2.14}$$

where

$$g_2^{(+)}(z_1,z_2) = g_{\rm r}(x_1,x_2)\delta(y_1)\delta(y_2) + g_{\rm c}(z_1)\delta(x_1-x_2)\delta(y_1+y_2), \tag{2.15}$$

$$\begin{aligned} g_{\rm r}(x_1,x_2) &= \exp\left[-\frac{n}{4a^2}(x_1+x_2)^2 + \frac{n}{4}(x_1-x_1)^2 + \frac{n\widehat{m}_{6+}}{4\widehat{a}_+^2}(x_1+x_2)\right] \\ &\times \left(\frac{x_1-x_2}{|x_1-x_2|} - \mathrm{erf}\left[\frac{1}{\sqrt{8}\widehat{a}_+}\left(n(x_1-x_2) - \widehat{\lambda}_{7+}\right)\right]\right), \end{aligned} \tag{2.16}$$

$$g_{\rm c}(z) = -2\imath\frac{y}{|y|}\exp\left[-\frac{n}{a^2}x^2 - ny^2 + \frac{n\widehat{m}_{6+}}{2\widehat{a}_+^2}x\right], \tag{2.17}$$

$$\frac{1}{c_+} = (-1)^{\nu(\nu-1)/2+n(n-1)/2}\left(\frac{16\pi}{n}\right)^{n/2}(2\pi)^{\nu/2}n^{-\nu^2/2-n(n+\nu)}\prod_{j=0}^{n}j!\prod_{j=0}^{n+\nu}j!. \tag{2.18}$$

Note that the one point weight $g_1^{(\pm)}$ of D_5 and of $D_{\rm W}$ is apart from the sign of the linear shift in the exponent the same. Also the other distributions show similarities with each other.

Comparing $p_{\rm W}$ with p_5 we recognize the major difference is the determinant which replaces the Pfaffian. The reason is a broken permutation symmetry in the eigenvalues of $D_{\rm W}$. We have to symmetrize over the eigenvalues $z^{(\rm r)}$ and $z^{(\rm l)}$ separately. Since the two-point weight $g_2^{(+)}$ only couples $z^{(\rm r)}$ with $z^{(\rm l)}$ but not two eigenvalues of one and the same set the symmetrization yields a determinant. Another crucial difference of $p_{\rm W}$ to p_5 is the distinction of real and complex eigenvalues reflecting the non-Hermiticity of $D_{\rm W}$. Interestingly the complex conjugated pairs only enter the two-point weight $g_2^{(+)}$. In the continuum limit the interaction of a pair of real eigenvalues, $g_{\rm r}$, is suppressed and the term for the complex eigenvalues, $g_{\rm c}$, enforces the pairing of non-zero eigenvalues, $(\imath\lambda, -\imath\lambda)$, along the imaginary axis. Again a block resembling the Vandermonde determinant appears and is again a relict of the former zero modes.

In the next two subsection we motivate the polynomials constructed in Sec. 3. For this we consider the k-point correlation functions of $D_{\rm W}$ and D_5.

2.1. The k-point correlation function of D_5

First, we consider the fermionic partition function of D_5 with $N_{\rm f}$ axial masses (characteristic polynomials of D_5), $\lambda = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_{N_{\rm f}})$,

$$Z_{N_{\rm f}}^{(n,\nu,-)}(\lambda) \propto \int d[D_{\rm W}]P(D_{\rm W})\prod_{j=1}^{N_{\rm f}}\det(D_5+\lambda_j\mathbb{1}_{2n+\nu}). \tag{2.19}$$

The unit matrix of dimension $2n+\nu$ is denoted by $\mathbb{1}_{2n+\nu}$. In the microscopic limit this partition function corresponds to the integral [20, 21]

$$\begin{aligned} &Z_{N_{\rm f}}^{(n,\nu,-)}\left(\frac{\lambda}{2n}\right) \overset{n\gg 1}{\propto} \int\limits_{\mathrm{U}\,(N_{\rm f})} d\mu(U)\det{}^{\nu}U \\ &\times \exp\left[\frac{\widehat{m}_6}{2}\,\mathrm{tr}(U+U^{-1}) + \frac{1}{2}\,\mathrm{tr}(\widehat{\lambda}_7\mathbb{1}_{N_{\rm f}}+\widehat{\lambda})(U-U^{-1}) - \widehat{a}^2\,\mathrm{tr}(U^2+U^{-2})\right]. \end{aligned} \tag{2.20}$$

This is the effective Lagrangian of the Wilson-Dirac operator of the partition function with N_f fermionic quarks with a degenerate quark mass $\widehat{m}_6$ and N_f source terms $\widehat{\lambda}_7 \mathbb{1}_{N_\mathrm{f}} + \widehat{\lambda}$ proportional to γ_5, cf. Refs. [27, 28, 29, 30]. An integration over the variables $\widehat{m}_6$ and $\widehat{\lambda}_7$ weighted by two additional Gaussian will yield the two low energy constants W_6 and W_7 proportional to two squared trace terms [20, 21, 34]. Here we will not consider these integrals.

Employing the joint probability density p_5, see Eq. (2.6), we combine the Vandermonde determinant and the characteristic polynomials to a quotient of two Vandermonde determinants. Then we rewrite the finite n partition function (2.19) as

$$\begin{aligned} Z_{N_\mathrm{f}}^{(n,\nu,-)}(\lambda) \quad &\propto \int\limits_{\mathbb{R}^{2n+\nu}} d[x] \frac{\Delta_{2n+\nu+N_\mathrm{f}}(x,-\lambda)}{\Delta_{N_\mathrm{f}}(\lambda)} \\ &\times \mathrm{Pf} \begin{bmatrix} \left\{ g_2^{(-)}(x_i,x_j) \right\}_{1\le i,j\le 2n+\nu} & \{x_i^{j-1} g_1^{(-)}(x_i)\}_{\substack{1\le i\le 2n+\nu \\ 1\le j\le \nu}} \\ \{-x_j^{i-1} g_1^{(-)}(x_j)\}_{\substack{1\le i\le \nu \\ 1\le j\le 2n+\nu}} & 0 \end{bmatrix}. \end{aligned} \tag{2.21}$$

We want to consider a little bit more than the partition function namely the k-point correlation function. For this purpose we only integrate over $2n+\nu-k$ variables, $\widetilde{x} = \mathrm{diag}\,(x_{k+1},\ldots,x_{2n+\nu})$. The remaining variables $x' = \mathrm{diag}\,(x_1,\ldots,x_k)$ are the k levels we look at, i.e.

$$\begin{aligned} R_{N_\mathrm{f},k}^{(n,\nu,-)}(x',\lambda) \;&\propto \int\limits_{\mathbb{R}^{2n+\nu-k}} \prod_{j=k+1}^{2n+\nu} dx_j \frac{\Delta_{2n+\nu+N_\mathrm{f}}(x,-\lambda)}{\Delta_{N_\mathrm{f}}(\lambda)} \\ &\times \mathrm{Pf} \begin{bmatrix} \left\{ g_2^{(-)}(x_i,x_j) \right\}_{1\le i,j\le 2n+\nu} & \{x_i^{j-1} g_1^{(-)}(x_i)\}_{\substack{1\le i\le 2n+\nu \\ 1\le j\le \nu}} \\ \{-x_j^{i-1} g_1^{(-)}(x_j)\}_{\substack{1\le i\le \nu \\ 1\le j\le 2n+\nu}} & 0 \end{bmatrix}. \end{aligned} \tag{2.22}$$

The idea is the following. In the Vandermonde determinant of the numerator we can build an arbitrary basis of polynomials from order 0 to order $2n+\nu+N_\mathrm{f}-1$,

$$\begin{aligned} \Delta_{2n+\nu+N_\mathrm{f}}(x,-\lambda) &= (-1)^{(2n+\nu+N_\mathrm{f})(2n+\nu+N_\mathrm{f}-1)/2} \\ &\times \det \begin{bmatrix} \left\{ p_j^{(-)}(x_i) \right\}_{\substack{1\le i\le 2n+\nu \\ 0\le j\le \nu-1}} & \left\{ q_{\nu+j}^{(-)}(x_i) \right\}_{\substack{1\le i\le 2n+\nu \\ 0\le j\le 2n+N_\mathrm{f}-1}} \\ \left\{ p_j^{(-)}(-\lambda_i) \right\}_{\substack{1\le i\le N_\mathrm{f} \\ 0\le j\le \nu-1}} & \left\{ q_{\nu+j}^{(-)}(-\lambda_i) \right\}_{\substack{1\le i\le N_\mathrm{f} \\ 0\le j\le 2n+N_\mathrm{f}-1}} \end{bmatrix}. \end{aligned} \tag{2.23}$$

Also the entries of the Pfaffian can be transformed by adding rows and columns with each other,

$$\mathrm{Pf} \begin{bmatrix} \left\{ g_2^{(-)}(x_i,x_j) \right\}_{1\le i,j\le 2n+\nu} & \{x_i^{j-1} g_1^{(-)}(x_i)\}_{\substack{1\le i\le 2n+\nu \\ 1\le j\le \nu}} \\ \{-x_j^{i-1} g_1^{(-)}(x_j)\}_{\substack{1\le i\le \nu \\ 1\le j\le 2n+\nu}} & 0 \end{bmatrix}$$

$$= \mathrm{Pf} \begin{bmatrix} \left\{ G_2^{(-)}(x_i, x_j) \right\}_{1 \le i,j \le 2n+\nu} & \{ p_j^{(-)}(x_i) g_1^{(-)}(x_i) \}_{\substack{1 \le i \le 2n+\nu \\ 0 \le j \le \nu-1}} \\ \{ -p_i^{(-)}(x_j) g_1^{(-)}(x_j) \}_{\substack{0 \le i \le \nu-1 \\ 1 \le j \le 2n+\nu}} & 0 \end{bmatrix}, \tag{2.24}$$

where we change the basis of the monomials to the polynomials $p_j^{(-)}$ and the two-point weight $g_2^{(-)}$ to $G_2^{(-)}$.

To shorten the notation we define the scalar product of two integrable functions f_1 and f_2 with the one-point weight $g_1^{(\pm)}$

$$\langle f_1 | f_2 \rangle_{g_1^{(\pm)}} = \int_{\mathbb{C}} d[z] f_1(z) f_2(z) g_1^{(\pm)}(x) \delta(y) \tag{2.25}$$

with $d[z] = dxdy$. The same can be done for the two-point weight $G_2^{(-)}$. We define the anti-symmetric product

$$(f_1 | f_2)_{G_2^{(-)}} = \frac{1}{2} \int_{\mathbb{C}^2} d[z_1] d[z_2] \det \begin{bmatrix} f_1(z_1) & f_2(z_1) \\ f_1(z_2) & f_2(z_2) \end{bmatrix} G_2^{(-)}(z_1, z_2) \delta(y_1) \delta(y_2). \tag{2.26}$$

Both definitions are extended to the complex plane by Dirac delta functions because we want to discuss the situation for both random matrices D_5 and D_{W} in a unifying way.

In the next step we employ the de Bruijn-like integration theorem derived in Appendix A.1 which yields

$$R_{N_\mathrm{f},k}^{(n,\nu,-)}(x', \lambda) \propto \frac{1}{\Delta_{N_\mathrm{f}}(\lambda)} \mathrm{Pf} \begin{bmatrix} M_-^{(11)} & M_-^{(12)} \\ -(M_-^{(12)})^T & M_-^{(22)} \end{bmatrix} \tag{2.27}$$

for the k-point correlation function. The matrices in the Pfaffian determinant are

$$M_-^{(11)} = \left[\begin{array}{c|c|c} \left(p_i^{(-)} | p_j^{(-)} \right)_{G_2^{(-)}} & \left(p_i^{(-)} | q_{\nu+j}^{(-)} \right)_{G_2^{(-)}} & \langle p_i^{(-)} | p_j^{(-)} \rangle_{g_1^{(-)}} \\ \hline \left(q_{\nu+i}^{(-)} | p_j^{(-)} \right)_{G_2^{(-)}} & \left(q_{\nu+i}^{(-)} | q_{\nu+j}^{(-)} \right)_{G_2^{(-)}} & \langle q_{\nu+i}^{(-)} | p_j^{(-)} \rangle_{g_1^{(-)}} \\ \hline -\langle p_i^{(-)} | p_j^{(-)} \rangle_{g_1^{(-)}} & -\langle p_i^{(-)} | q_{\nu+j}^{(-)} \rangle_{g_1^{(-)}} & 0 \end{array} \right], \tag{2.28}$$

$$M_-^{(12)} = \left[\begin{array}{c|c|c} \int\limits_{\mathbb{R}} d\widetilde{x} p_i^{(-)}(\widetilde{x}) G_2^{(-)}(\widetilde{x}, x_j) & p_i^{(-)}(x_j) & p_i^{(-)}(-\lambda_j) \\ \hline \int\limits_{\mathbb{R}} d\widetilde{x} q_{\nu+i}^{(-)}(\widetilde{x}) G_2^{(-)}(\widetilde{x}, x_j) & q_{\nu+i}^{(-)}(x_j) & q_{\nu+i}^{(-)}(-\lambda_j) \\ \hline -p_i^{(-)}(x_j) g_1^{(-)}(x_j) & 0 & 0 \end{array} \right], \tag{2.29}$$

$$M_-^{(22)} = \left[\begin{array}{c|c|c} G_2^{(-)}(x_i, x_j) & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right]. \tag{2.30}$$

In the Pfaffian (2.27) the indices i and j of the rows and columns are $(0 \dots \nu-1, 0 \dots 2n+N_\mathrm{f}-1, 0 \dots \nu-1, 1 \dots k, 1 \dots k, 1 \dots N_\mathrm{f})$ from top to bottom and left to right. Please notice that regardless what the polynomials $p_l^{(-)}$ and $q_{\nu+l}^{(-)}$ and the modified two-point weight $G_2^{(-)}$ are Eq. (2.27) tells us that the joint probability density p_5 can also be written as a single Pfaffian. We have only to choose $k = 2n + \nu$ to see that this statement is

true. However the representation (2.27) is quite cumbersome. A more compact one is given in subsection 4.1.

The aim is now to choose $q^{(-)}_{\nu+i}$, $p^{(-)}_i$ and $G^{(-)}_2$ such that the matrix $M^{(11)}_-$ becomes quasi-diagonal since we want to invert this matrix. A quasi-diagonal structure is equivalent to the conditions

$$\langle p^{(-)}_i | p^{(-)}_j \rangle_{g^{(-)}_1} = h^{(-)}_j \delta_{ij}, \quad \text{for} \quad 0 \le i,j \le \nu-1, \tag{2.31}$$

$$\langle p^{(-)}_i | q^{(-)}_{\nu+j} \rangle_{g^{(-)}_1} = 0, \quad \text{for} \quad 0 \le i \le \nu-1, 0 \le j \le 2n+N_{\rm f}-1, \tag{2.32}$$

$$\left(p^{(-)}_i | p^{(-)}_j\right)_{G^{(-)}_2} = 0, \quad \text{for} \quad 0 \le i,j \le \nu-1, \tag{2.33}$$

$$\left(p^{(-)}_i | q^{(-)}_{\nu+j}\right)_{G^{(-)}_2} = 0, \quad \text{for} \quad 0 \le i \le \nu-1, 0 \le j \le 2n+N_{\rm f}-1, \tag{2.34}$$

$$\left(q^{(-)}_{\nu+2i+1} | q^{(-)}_{\nu+2j+1}\right)_{G^{(-)}_2} = 0, \quad \text{for} \quad 0 \le i,j \le 2n+N_{\rm f}-1, \tag{2.35}$$

$$\left(q^{(-)}_{\nu+2i} | q^{(-)}_{\nu+2j}\right)_{G^{(-)}_2} = 0, \quad \text{for} \quad 0 \le i,j \le 2n+N_{\rm f}-1, \tag{2.36}$$

$$\left(q^{(-)}_{\nu+2i} | q^{(-)}_{\nu+2j+1}\right)_{G^{(-)}_2} = o^{(-)}_j \delta_{ij}, \quad \text{for} \quad 0 \le i,j \le 2n+N_{\rm f}-1. \tag{2.37}$$

The constants $h^{(-)}_j$ and $o^{(-)}_j$ are the normalization constants of the polynomials. In Sec. 3 we will see that this system of equations have indeed a solution. We will give an explicit construction of them.

Please note that the solution of the odd skew-orthogonal polynomials, $q^{(-)}_{\nu+2j+1}$, exhibits an ambiguity. The polynomials $q^{(-)}_{\nu+2j+1} + c_j q^{(-)}_{\nu+2j}$ are also a solution of the Eqs. (2.31-2.37) with arbitrary constants $c_j \in \mathbb{C}$ as it was already found in Ref. [66] for pure skew-orthogonal polynomials.

2.2. The $(k_{\rm r}, k_{\rm l})$-point correlation function of $D_{\rm W}$

The next case we want to consider is the fermionic partition function of $D_{\rm W}$ with $N_{\rm f}$ quark masses, $m = \mathrm{diag}\,(m_1, \ldots, m_{N_{\rm f}})$,

$$Z^{(n,\nu,+)}_{N_{\rm f}}(m) \propto \int d[D_{\rm W}] P(D_{\rm W}) \prod_{j=1}^{N_{\rm f}} \det(D_{\rm W} + m_j \mathbb{1}_{2n+\nu}). \tag{2.38}$$

In the microscopic limit it corresponds to [20, 21]

$$Z^{(n,\nu,+)}_{N_{\rm f}}\left(\frac{m}{2n}\right) \overset{n\gg 1}{\propto} \int\limits_{\mathrm{U}\,(N_{\rm f})} d\mu(U) \det{}^{\nu} U \tag{2.39}$$

$$\times \exp\left[\frac{1}{2}\,\mathrm{tr}(\widehat{m}_6 \mathbb{1}_{N_{\rm f}} + \widehat{m})(U + U^{-1}) + \frac{\widehat{\lambda}_7}{2}\,\mathrm{tr}(U - U^{-1}) - \widehat{a}^2\,\mathrm{tr}(U^2 + U^{-2})\right].$$

This is the effective Lagrangian of the Wilson-Dirac operator of the partition function with $N_{\rm f}$ fermionic quarks with non-degenerate quark masses $\widehat{m}_6 \mathbb{1}_{N_{\rm f}} + \widehat{m}$ and one source term $\widehat{\lambda}_7$ proportional to γ_5 [27, 28, 29, 30]. Again one can integrate over the two variables

$\widehat{m}_6$ and $\widehat{\lambda}_7$ weighted by Gaussians to obtain the two low energy constants W_6 and W_7 but we will consider $Z_{N_\mathrm{f}}^{(n,\nu,+)}$ without these integrals.

The partition function with the joint probability density p_W reads

$$Z_{N_\mathrm{f}}^{(n,\nu,+)}(m) \quad \propto \int\limits_{\mathbb{C}^{2n+\nu}} d[Z] \frac{\Delta_{2n+\nu+N_\mathrm{f}}(Z,-m)}{\Delta_{N_\mathrm{f}}(m)} \times \det \begin{bmatrix} \{g_2^{(+)}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{l})})\}_{\substack{1\le i\le n \\ 1\le j\le n+\nu}} \\ \{(x_j^{(\mathrm{l})})^{i-1} g_1^{(+)}(x_j^{(\mathrm{l})})\delta(y_j^{(\mathrm{l})})\}_{\substack{1\le i\le \nu \\ 1\le j\le n+\nu}} \end{bmatrix}. \tag{2.40}$$

Since the permutation symmetry in the eigenvalues of D_W is broken we have to consider a two parameter set of eigenvalue correlation functions. The number of eigenvalues $z^{(\mathrm{r})}$ is independent of the number for the eigenvalues $z^{(\mathrm{l})}$. Let $Z' = \mathrm{diag}\,(z_1^{(\mathrm{r})}, \ldots, z_{k_\mathrm{r}}^{(\mathrm{r})}, z_1^{(\mathrm{l})}, \ldots, z_{k_\mathrm{l}}^{(\mathrm{l})})$. Hence we define the $(k_\mathrm{r}\,, k_\mathrm{l})$-point correlation function,

$$R_{N_\mathrm{f},k_\mathrm{r},k_\mathrm{l}}^{(n,\nu,+)}(Z',m) \propto \int\limits_{\mathbb{C}^{2n+\nu-k_\mathrm{r}-k_\mathrm{l}}} \prod_{j=k_\mathrm{r}+1}^{n} d[z_j^{(\mathrm{r})}] \prod_{j=k_\mathrm{l}+1}^{n+\nu} d[z_j^{(\mathrm{l})}] \frac{\Delta_{2n+\nu+N_\mathrm{f}}(Z,-m)}{\Delta_{N_\mathrm{f}}(m)} \times \det \begin{bmatrix} \{g_2^{(+)}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{l})})\}_{\substack{1\le i\le n \\ 1\le j\le n+\nu}} \\ \{(x_j^{(\mathrm{l})})^{i-1} g_1^{(+)}(x_j^{(\mathrm{l})})\delta(y_j^{(\mathrm{l})})\}_{\substack{1\le i\le \nu \\ 1\le j\le n+\nu}} \end{bmatrix}. \tag{2.41}$$

As in subsection 2.1 we construct an arbitrary basis of polynomials in the Vandermonde determinant to inflict some conditions on them later on,

$$\Delta_{2n+\nu+N_\mathrm{f}}(Z,-m) = (-1)^{(2n+\nu+N_\mathrm{f})(2n+\nu+N_\mathrm{f}-1)/2} \times \det \begin{bmatrix} \left\{p_j^{(+)}(z_i^{(\mathrm{r})})\right\}_{\substack{1\le i\le n \\ 0\le j\le \nu-1}} & \left\{q_{\nu+j}^{(+)}(z_i^{(\mathrm{r})})\right\}_{\substack{1\le i\le n \\ 0\le j\le 2n+N_\mathrm{f}-1}} \\ \left\{p_j^{(+)}(z_i^{(\mathrm{l})})\right\}_{\substack{1\le i\le n+\nu \\ 0\le j\le \nu-1}} & \left\{q_{\nu+j}^{(+)}(z_i^{(\mathrm{l})})\right\}_{\substack{1\le i\le n+\nu \\ 0\le j\le 2n+N_\mathrm{f}-1}} \\ \left\{p_j^{(+)}(-m_i)\right\}_{\substack{1\le i\le N_\mathrm{f} \\ 0\le j\le \nu-1}} & \left\{q_{\nu+j}^{(+)}(-m_i)\right\}_{\substack{1\le i\le N_\mathrm{f} \\ 0\le j\le 2n+N_\mathrm{f}-1}} \end{bmatrix}. \tag{2.42}$$

Also the other determinant in the numerator can be transformed,

$$\det \begin{bmatrix} \{g_2^{(+)}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{l})})\}_{\substack{1\le i\le n \\ 1\le j\le n+\nu}} \\ \{(x_j^{(\mathrm{l})})^{i-1} g_1^{(+)}(x_j^{(\mathrm{l})})\delta(y_j^{(\mathrm{l})})\}_{\substack{1\le i\le \nu \\ 1\le j\le n+\nu}} \end{bmatrix} = \det \begin{bmatrix} \{G_2^{(+)}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{l})})\}_{\substack{1\le i\le n \\ 1\le j\le n+\nu}} \\ \{p_i^{(+)}(x_j^{(\mathrm{l})}) g_1^{(+)}(x_j^{(\mathrm{l})})\delta(y_j^{(\mathrm{l})})\}_{\substack{0\le i\le \nu-1 \\ 1\le j\le n+\nu}} \end{bmatrix}. \tag{2.43}$$

The whole procedure works analogous to the one for D_5, cf. subsection 2.1.

Let

$$(f_1|f_2)_{G_2^{(+)}} = \frac{1}{2} \int\limits_{\mathbb{C}^2} d[z^{(\mathrm{r})}] d[z^{(\mathrm{l})}] \det \begin{bmatrix} f_1(z^{(\mathrm{r})}) & f_2(z^{(\mathrm{r})}) \\ f_1(z^{(\mathrm{l})}) & f_2(z^{(\mathrm{l})}) \end{bmatrix}$$

$$\times \left(G_2^{(+)}(z^{(\mathrm{r})}, z^{(\mathrm{l})}) - G_2^{(+)}(z^{(\mathrm{l})}, z^{(\mathrm{r})}) \right) \tag{2.44}$$

be the anti-symmetric scalar product of two integrable functions f_1 and f_2 with respect to the two-point weight $G_2^{(+)}$. Notice that $G_2^{(+)}$ as well as $g_2^{(+)}$ are not anti-symmetric under a permutation of their entries whereas the two-point weight $G_2^{(-)}$ is anti-symmetric. The reason for this is again the breaking of the permutation symmetry in the eigenvalues of D_{W}.

Considering the $(k_{\mathrm{r}}, k_{\mathrm{l}})$-point correlation function we apply the de Bruijn-like integration theorem derived in Appendix A.2 to the partition function (2.40) and find

$$R^{(n,\nu,+)}_{N_{\mathrm{f}},k_{\mathrm{r}},k_{\mathrm{l}}}(Z', m) \propto \frac{1}{\Delta_{N_{\mathrm{f}}}(m)} \mathrm{Pf} \begin{bmatrix} M_+^{(11)} & M_+^{(12)} \\ -(M_+^{(12)})^T & M_+^{(22)} \end{bmatrix}, \tag{2.45}$$

where the three matrices are

$$M_+^{(11)} = \left[\begin{array}{c|c|c} \left(p_i^{(+)}|p_j^{(+)}\right)_{G_2^{(+)}} & \left(p_i^{(+)}|q_{\nu+j}^{(+)}\right)_{G_2^{(+)}} & \langle p_i^{(+)}|p_j^{(+)}\rangle_{g_1^{(+)}} \\ \hline \left(q_{\nu+i}^{(+)}|p_j^{(+)}\right)_{G_2^{(+)}} & \left(q_{\nu+i}^{(+)}|q_{\nu+j}^{(+)}\right)_{G_2^{(+)}} & \langle q_{\nu+i}^{(+)}|p_j^{(+)}\rangle_{g_1^{(+)}} \\ \hline -\langle p_i^{(+)}|p_j^{(+)}\rangle_{g_1^{(+)}} & -\langle p_i^{(+)}|q_{\nu+j}^{(+)}\rangle_{g_1^{(+)}} & 0 \end{array}\right], \tag{2.46}$$

$$(M_+^{(12)})^T = \left[\begin{array}{c|c|c} \int\limits_{\mathbb{C}} d[\widetilde{z}] p_i^{(+)}(\widetilde{z}) G_2^{(+)}(z_j^{(\mathrm{r})}, \widetilde{z}) & \int\limits_{\mathbb{C}} d[\widetilde{z}] q_{\nu+i}^{(+)}(\widetilde{z}) G_2^{(+)}(z_j^{(\mathrm{r})}, \widetilde{z}) & 0 \\ \hline \int\limits_{\mathbb{C}} d[\widetilde{z}] p_i^{(+)}(\widetilde{z}) G_2^{(+)}(\widetilde{z}, z_j^{(\mathrm{l})}) & \int\limits_{\mathbb{C}} d[\widetilde{z}] q_{\nu+i}^{(+)}(\widetilde{z}) G_2^{(+)}(\widetilde{z}, z_j^{(\mathrm{l})}) & -p_i^{(+)}(x_j^{(\mathrm{l})}) g_1^{(+)}(x_j^{(\mathrm{l})}) \delta(y_j^{(\mathrm{l})}) \\ \hline p_i^{(+)}(z_j^{(\mathrm{r})}) & q_{\nu+i}^{(+)}(z_j^{(\mathrm{r})}) & 0 \\ \hline p_i^{(+)}(z_j^{(\mathrm{l})}) & q_{\nu+i}^{(+)}(z_j^{(\mathrm{l})}) & 0 \\ \hline p_i^{(+)}(-\lambda_j) & q_{\nu+i}^{(+)}(-\lambda_j) & 0 \end{array}\right], \tag{2.47}$$

$$M_+^{(22)} = \left[\begin{array}{c|c|c|c|c} 0 & -G_2^{(+)}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{l})}) & 0 & 0 & 0 \\ \hline G_2^{(+)}(z_j^{(\mathrm{r})}, z_i^{(\mathrm{l})}) & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.48}$$

In the Pfaffian (2.45) the indices i and j are in the range $(0 \dots \nu-1, 0 \dots 2n+N_{\mathrm{f}}-1, 0 \dots \nu-1, 1 \dots k_{\mathrm{r}}, 1 \dots k_{\mathrm{l}}, 1 \dots k_{\mathrm{r}}, 1 \dots k_{\mathrm{l}}, 1 \dots N_{\mathrm{f}})$ from top to bottom and from left to right.

Please notice the similarity of Eq. (2.45) with Eq. (2.27). If $k_{\mathrm{r}} = n$ and $k_{\mathrm{l}} = n+\nu$ the correlation function is equal to the joint probability density p_{W}. Hence p_{W} can also be written as a single Pfaffian which can be cast into a more compact form, see subsection 4.2.

As in subsection 2.1 we want to invert and, thus, quasi-diagonalize the matrix $M_+^{(11)}$. This yields the following system of equations

$$\langle p_i^{(+)}|p_j^{(+)}\rangle_{g_1^{(+)}} = h_i^{(+)}\delta_{ij}, \quad \text{for} \quad 0 \le i,j \le \nu-1, \tag{2.49}$$

$$\langle p_i^{(+)}|q_{\nu+j}^{(+)}\rangle_{g_1^{(+)}} = 0, \quad \text{for} \quad 0 \le i \le \nu-1, 0 \le j \le 2n+N_{\mathrm{f}}-1, \tag{2.50}$$

$$\left(p_i^{(+)}|p_j^{(+)}\right)_{G_2^{(+)}} = 0, \quad \text{for} \quad 0 \le i,j \le \nu-1, \tag{2.51}$$

$$\left(p_i^{(+)}|q_{\nu+j}^{(+)}\right)_{G_2^{(+)}} = 0, \quad \text{for} \quad 0 \le i \le \nu-1, 0 \le j \le 2n+N_{\rm f}-1, \tag{2.52}$$

$$\left(q_{\nu+2i+1}^{(+)}|q_{\nu+2j+1}^{(+)}\right)_{G_2^{(+)}} = 0, \quad \text{for} \quad 0 \le i,j \le 2n+N_{\rm f}-1, \tag{2.53}$$

$$\left(q_{\nu+2i}^{(+)}|q_{\nu+2j}^{(+)}\right)_{G_2^{(+)}} = 0, \quad \text{for} \quad 0 \le i,j \le 2n+N_{\rm f}-1, \tag{2.54}$$

$$\left(q_{\nu+2i}^{(+)}|q_{\nu+2j+1}^{(+)}\right)_{G_2^{(+)}} = o_i^{(+)}\delta_{ij}, \quad \text{for} \quad 0 \le i,j \le 2n+N_{\rm f}-1 \tag{2.55}$$

with the normalization constants $h_j^{(+)}$ and $o_j^{(+)}$. Comparing this system of equations with the one of D_5 we recognize that they are of the same form. Hence, if we solve them in a general setting we solve them for both random matrices, D_5 and $D_{\rm W}$.

As for D_5 the odd skew-orthogonal polynomials, $q_{\nu+2j+1}^{(-)}$, can be added by the polynomials $c_j q_{\nu+2j}^{(-)}$ with arbitrary constants $c_j \in \mathbb{C}$. They solve the same set of equations.

3. Construction of the polynomials and some of their properties

In subsection 3.1 we construct the orthogonal polynomials as well as the skew-orthogonal ones starting from the conditions (2.31-2.37) and (2.49-2.55). Furthermore we give explicit expressions of the modified two-point weights, $G_2^{(\pm)}$, and specify the constants h_l and $o_{\nu+2l}^{(\pm)}$. Recursion relations of the polynomials are shown in subsection 3.2. With aid of these relations we derive the Christoffel Darboux-like formula, in subsection 3.3. In subsection 3.4, we rewrite the polynomials as well as the Christoffel Darboux-like formula to random matrix averages and take the microscopic limit of them in subsection 3.5.

3.1. The polynomials

The starting point of the construction are the monomials

$$\mathfrak{m}_j(z) = z^j \quad \text{with} \quad j \in \mathbb{N}_0. \tag{3.1}$$

With help of the general formula for the orthogonal polynomials of the one-point weight $g_1^{(\pm)}$ [44] as a quotient of determinants we find

$$p_l^{(\pm)}(z) = \det{}^{-1}\left[\langle \mathfrak{m}_i|\mathfrak{m}_j\rangle_{g_1^{(\pm)}}\right]_{1\le i,j\le l-1} \det\begin{bmatrix} \left\{\langle \mathfrak{m}_i|\mathfrak{m}_j\rangle_{g_1^{(\pm)}}\right\}_{\substack{0\le i\le l-1\\ 0\le j\le l}} \\ \{\mathfrak{m}_j(z)\}_{0\le j\le l} \end{bmatrix} \tag{3.2}$$

in monic normalization, i.e. $p_l^{(\pm)}(z) = z^l + \ldots$ Since g_1 is a shifted Gaussian the polynomials $p_l^{(\pm)}$ are shifted Hermite polynomials, H_l, in monic normalization,

$$p_l^{(\pm)}(z) = \left(\frac{a^2}{n}\right)^{l/2} H_l\left(\sqrt{\frac{n}{a^2}}z \mp \sqrt{\frac{a^2}{n}}\mu_l\right). \tag{3.3}$$

This agrees with Refs. [20, 21, 22, 23] where a mixing of the eigenvalue statistics with a finite dimensional GUE was found. The normalization constant is

$$h_l^{(\pm)} = h_l = \sqrt{2\pi}\left(\frac{a^2}{n}\right)^{l+1/2} l!\exp\left[\frac{a^2\mu_1^2}{2n}\right]. \tag{3.4}$$

Thus the normalization constants of the orthogonal polynomials are the same for D_{W} and for D_5.

Starting from the orthogonal polynomials $p_l^{(\pm)}$ we want to construct the polynomials $q_{\nu+l}^{(-)}$ and $q_{\nu+l}^{(+)}$ fulfilling the orthogonality conditions (2.32) and (2.50), respectively. Let N_{f} be even for simplicity. If it is odd the anti-symmetric matrices $M_{\pm}^{(11)}$ are never invertible because their dimensions are odd. In such a case we extend the partition function by one fermionic flavor and remove it at the end of the day by sending its mass to infinity.

As for the orthogonal polynomials we begin with an intuitive definition,

$$q_{\nu+2l}^{(\pm)}(z) = \mathrm{Pf}^{-1}\left[\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right]_{\nu\le i,j\le\nu+2l-1} \tag{3.5}$$
$$\times\,\mathrm{Pf}\begin{bmatrix}\left\{\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right\}_{\nu\le i,j\le\nu+2l} & \left\{p_i^{(\pm)}(z)\right\}_{\nu\le i\le\nu+2l}\\ \left\{-p_j^{(\pm)}(z)\right\}_{\nu\le j\le\nu+2l} & 0\end{bmatrix},$$

$$q_{\nu+2l+1}^{(\pm)}(z) = \mathrm{Pf}^{-1}\left[\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right]_{\nu\le i,j\le\nu+2l-1} \tag{3.6}$$
$$\times\mathrm{Pf}\begin{bmatrix}\left\{\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right\}_{\nu\le i,j\le\nu+2l-1} & \left\{\left(p_i^{(\pm)}|p_{\nu+2l+1}^{(\pm)}\right)_{g_2^{(\pm)}}\right\}_{\nu\le i\le\nu+2l-1} & \left\{p_i^{(\pm)}(z)\right\}_{\nu\le i\le\nu+2l-1}\\ \left\{\left(p_{\nu+2l+1}^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right\}_{\nu\le j\le\nu+2l-1} & 0 & p_{\nu+2l+1}^{(\pm)}(z)\\ \left\{-p_j^{(\pm)}(z)\right\}_{\nu\le j\le\nu+2l-1} & -p_{\nu+2l+1}^{(\pm)}(z) & 0\end{bmatrix},$$

which is similar to the ansatz of the skew-orthogonal polynomials in χRMT with non-zero chemical potential and Dyson index $\beta = 1, 4$, see Ref. [15]. The anti-symmetric products of $g_2^{(-)}$ and $g_2^{(+)}$ are defined similar to Eqs. (2.26) and (2.44), respectively. One can readily prove that the orthogonality conditions (2.32) and (2.50) are fulfilled. The multi-linearity of the Pfaffian allows us to pull the scalar product into the Pfaffian. Then one row and one column is zero. For example, let $l \in \{0, 1, \ldots, \nu-1\}$. Then the orthogonality of the polynomials $p_l^{(\pm)}$ yields

$$\langle p_l^{(\pm)}|q_{\nu+2b}^{(\pm)}\rangle_{g_1^{(\pm)}} = \frac{\mathrm{Pf}\begin{bmatrix}\left\{\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right\}_{\nu\le i,j\le\nu+2b} & \left\{\langle p_l^{(\pm)}|p_i^{(\pm)}\rangle_{g_1^{(\pm)}}\right\}_{\nu\le i\le\nu+2b}\\ \left\{-\langle p_l^{(\pm)}|p_j^{(\pm)}\rangle_{g_1^{(\pm)}}\right\}_{\nu\le j\le\nu+2b} & 0\end{bmatrix}}{\mathrm{Pf}\left[\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right]_{\nu\le i,j\le\nu+2b-1}}$$

$$
\begin{aligned}
&\text{Pf}\left[\begin{array}{cc}\left\{\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right\}_{\nu\le i,j\le\nu+2b} & \{0\}_{\nu\le i\le\nu+2b}\\ \{0\}_{\nu\le j\le\nu+2b} & 0\end{array}\right]\\
&=\frac{}{\text{Pf}\left[\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right]_{\nu\le i,j\le\nu+2b-1}}\\
&=0. \qquad (3.7)
\end{aligned}
$$

In a similar way one can prove the other relations.

We underline that the odd skew-orthogonal polynomials (3.6) can be gauged by the even ones (3.5), i.e. $q^{(\pm)}_{\nu+2l+1}(z)\to q^{(\pm)}_{\nu+2l+1}(z)+c_l q^{(\pm)}_{\nu+2l}(z)$ with $c_l\in\mathbb{C}$ arbitrary. This gauge symmetry is similar to the one found for pure skew-orthogonal polynomials [66]. The fundamental reason is the anti-symmetry of the two-point weight $g_2^{(\pm)}$ which has always a non-trivial kernel.

The normalization constants in Eqs. (3.5) and (3.6) are finite since they are proportional to the constants in Eqs. (2.12) and (2.18), i.e.

$$
\text{Pf}\left[\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right]_{\nu\le i,j\le\nu+2l-1}\propto\frac{1}{c_\pm}. \qquad (3.8)
$$

It can be easily shown that the polynomials $q^{(\pm)}_{\nu+l}$ are in monic normalization, too. The constants (3.8) are related to the ones in Eqs. (2.37) and (2.55) by

$$
o_l^{(\pm)}=\frac{\text{Pf}\left[\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right]_{\nu\le i,j\le\nu+2l+1}}{\text{Pf}\left[\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right]_{\nu\le i,j\le\nu+2l-1}}. \qquad (3.9)
$$

Combining this identity with relation (3.8) the constants $o_l^{(\pm)}$ are mostly the quotient of two normalization constants of the joint probability density functions. Hence, the constants $o_l^{(\pm)}$ can be directly calculated by the two identities

$$
\begin{aligned}
&(-1)^n(2n+\nu)!\prod_{j=0}^{\nu-1}h_j\prod_{j=0}^{n-1}o_j^{(-)} \qquad (3.10)\\
&=\frac{1}{c_-}(1-a^2)^{n(n+\nu-1/2)}a^{n+\nu^2}\exp\left[\frac{a^2}{2}\left(\mu_{\text{r}}^2+\frac{n+\nu}{n}\mu_{\text{l}}^2\right)-\frac{n\widehat{m}_{6-}^2}{8\widehat{a}_-^2}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&(-1)^{\nu(\nu-1)/2+n(n+1)/2}n!(n+\nu)!\prod_{j=0}^{\nu-1}h_j\prod_{j=0}^{n-1}o_j^{(+)} \qquad (3.11)\\
&=\frac{1}{c_+}(1+a^2)^{n(n+\nu-1/2)}a^{n+\nu^2}\exp\left[\frac{a^2}{2}\left(\mu_{\text{r}}^2+\frac{n+\nu}{n}\mu_{\text{l}}^2\right)-\frac{n\widehat{\lambda}_{7+}^2}{8\widehat{a}_+^2}\right].
\end{aligned}
$$

These identities can be derived with aid of the de Bruijn-like integration theorems in Appendix A. With help of Eqs. (3.10) and (3.11) we conclude

$$
o_l^{(\pm)}=-4\,l!(l+\nu)!\sqrt{\frac{\pi}{n(1\pm a^2)}}\left(\frac{1\pm a^2}{n}\right)^{2l+\nu+1}a\exp\left[\frac{a^2}{4n}(\mu_{\text{r}}\pm\mu_{\text{l}})^2\pm\frac{a^4}{4n(1\pm a^2)}(\mu_{\text{r}}\mp\mu_{\text{l}})^2\right] \qquad (3.12)
$$

Hence the normalization constant is linear in a for small lattice spacing and is proportional to $n^{-2l-\nu-2}l!(l+\nu)!$ in the microscopic limit.

The polynomials $q^{(\pm)}_{\nu+2b}$ and $q^{(\pm)}_{\nu+2b+1}$ are also orthogonal to $p^{(\pm)}_{\nu+l}$, $l \in \{0,\dots,2b\}$ and $l \in \{0,\dots,2b-1,2b+1\}$, respectively, corresponding to the two-point weight $g_2^{(\pm)}$ since the rows and columns are not linearly independent anymore. For example

$$\begin{aligned}\left(p^{(\pm)}_{\nu+l}|q^{(\pm)}_{\nu+2b}\right)_{g_2^{(\pm)}} &= \frac{\text{Pf}\begin{bmatrix} \left\{\left(p^{(\pm)}_{\nu+i}|p^{(\pm)}_{\nu+j}\right)_{g_2^{(\pm)}}\right\}_{0\le i,j\le 2b} & \left\{\left(p^{(\pm)}_{\nu+l}|p^{(\pm)}_{\nu+i}\right)_{g_2^{(\pm)}}\right\}_{0\le i\le 2b} \\ \left\{-\left(p^{(\pm)}_{\nu+l}|p^{(\pm)}_{\nu+j}\right)_{g_2^{(\pm)}}\right\}_{0\le j\le 2b} & 0 \end{bmatrix}}{\text{Pf}\left[\left(p^{(\pm)}_{\nu+i}|p^{(\pm)}_{\nu+j}\right)_{g_2^{(\pm)}}\right]_{0\le i,j\le 2b-1}} \\ &= 0, \end{aligned} \tag{3.13}$$

the lth row and column and the last ones are the same. In the same way one can prove the skew-orthogonality of $p^{(\pm)}_{\nu+l}$, $l \in \{0,\dots,2b-1,2b+1\}$, with $q^{(\pm)}_{\nu+2b+1}$. Due to the definitions (3.5) and (3.6) the polynomials $q^{(\pm)}_{\nu+l}$ are a linear combination of $p^{(\pm)}_{\nu+l}$ with $0 \le l \le 2n+N_{\text{f}}-1$. Therefore the polynomials are indeed skew-orthogonal with respect to $g_2^{(\pm)}$. In particular they fulfill the conditions similar to the relations (2.35-2.37) and (2.53-2.55) by exchanging $G_2^{(\pm)} \to g_2^{(\pm)}$. However the remaining conditions (2.33), (2.34), (2.51) and (2.52) are not fulfilled. This is the reason for modifying the two-point weights.

The simplest way to enforce the remaining conditions is the projection of the measures $g_2^{(\pm)}$ onto the polynomials $q^{(\pm)}_{\nu+l}$ only. This means the polynomials $p_l^{(\pm)}$, $0 \le l \le \nu-1$, have to be in the kernel of $G_2^{(\pm)}$. We make the ansatz

$$\begin{aligned}(f_1,f_2)_{G_2^{(\pm)}} = (f_1,f_2)_{g_2^{(\pm)}} &- \sum_{j=0}^{\nu-1}\frac{1}{h_j}\left(\langle f_1|p_j^{(\pm)}\rangle_{g_1^{(\pm)}}(p_j^{(\pm)},f_2)_{g_2^{(\pm)}} + (f_1,p_j^{(\pm)})_{g_2^{(\pm)}}\langle p_j^{(\pm)}|f_2\rangle_{g_1^{(\pm)}}\right) \\ &+ \sum_{i,j=0}^{\nu-1}\frac{1}{h_jh_i}\langle f_1|p_i^{(\pm)}\rangle_{g_1^{(\pm)}}(p_i^{(\pm)},p_j^{(\pm)})_{g_2^{(\pm)}}\langle p_j^{(\pm)}|f_2\rangle_{g_1^{(\pm)}}. \end{aligned} \tag{3.14}$$

Indeed we have

$$\left(p_l^{(\pm)},f\right)_{G_2^{(-)}} = (p_l^{(\pm)},f)_{G_2^{(+)}} = 0, \quad \text{for all functions } f \text{ and } 0 \le l \le \nu-1, \tag{3.15}$$

$$\left(q_i^{(\pm)},q_j^{(\pm)}\right)_{G_2^{(\pm)}} = \left(q_i^{(\pm)},q_j^{(\pm)}\right)_{g_2^{(\pm)}}. \tag{3.16}$$

Thus all orthogonality conditions are fulfilled.

The explicit expressions of $G_2^{(-)}$ and $G_2^{(+)}$ are

$$\begin{aligned}G_2^{(-)}(x_1,x_2) = g_2^{(-)}(x_1,x_2) &- \sum_{j=0}^{\nu-1}\frac{1}{h_j}\left(\int_{\mathbb{R}}dx' p_j^{(-)}(x')g_2^{(-)}(x',x_2)p_j^{(-)}(x_1)g_1^{(-)}(x_1)\right. \\ &\left. + \int_{\mathbb{R}}dx' p_j^{(-)}(x')g_2^{(-)}(x_1,x')p_j^{(-)}(x_2)g_1^{(-)}(x_2)\right) + \sum_{i,j=0}^{\nu-1} \end{aligned} \tag{3.17}$$

$$\times \frac{1}{h_j h_i} \int\limits_{\mathbb{R}^2} d[x'] p_i^{(-)}(x_1') p_j^{(-)}(x_2') g_2^{(-)}(x_1', x_2') p_i^{(-)}(x_1) p_j^{(-)}(x_2) g_1^{(-)}(x_1) g_1^{(-)}(x_2),$$

$$\begin{aligned} G_2^{(+)}(z_1, z_2) &= g_2^{(+)}(z_1, z_2) \\ &- \sum_{j=0}^{\nu-1} \frac{1}{h_j} \int\limits_{\mathbb{C}} d[z'] p_j^{(+)}(z') \left(g_2^{(+)}(z_1, z') - g_2^{(+)}(z', z_1) \right) p_j^{(+)}(z_2) g_1^{(+)}(z_2) \\ &+ \sum_{i,j=0}^{\nu-1} \frac{1}{2 h_j h_i} \int\limits_{\mathbb{C}^2} d[z_1'] d[z_2'] p_i^{(+)}(z_1') p_j^{(+)}(z_2') \left(g_2^{(+)}(z_1', z_2') - g_2^{(+)}(z_2', z_1') \right) \\ &\times p_i^{(+)}(z_1) p_j^{(+)}(z_2) g_1^{(+)}(z_1) g_1^{(+)}(z_2). \end{aligned} \tag{3.18}$$

The change of the two-point measures is restricted by linear combinations with other rows and columns in the Pfaffian (2.24) and the determinant (2.42). Essentially we add the orthogonal polynomials $p_l^{(\pm)}$ to the weight. Thereby we have to recall that everything which is done with the rows has to be done with the columns in the Pfaffian. This is the reason why $G_2^{(-)}$ stays anti-symmetric whereas $G_2^{(+)}$ is asymmetric in the entries.

3.2. Recursion relations

The recursion relations of the orthogonal polynomials are

$$\frac{\partial p_l^{(\pm)}}{\partial x}(x) = l p_{l-1}^{(\pm)}(x), \tag{3.19}$$

$$x p_l^{(\pm)}(x) = p_{l+1}^{(\pm)}(x) \pm \frac{a^2 \mu_{\mathrm{l}}}{n} p_l^{(\pm)}(x) + \frac{l a^2}{n} p_{l-1}^{(\pm)}(x). \tag{3.20}$$

They result from the orthogonality relation (2.31) and the two identities

$$\langle D^{(\pm)} f_1 | f_2 \rangle_{g_1^{(\pm)}} = -\langle f_1 | D^{(\pm)} f_2 \rangle_{g_1^{(\pm)}}, \tag{3.21}$$

$$\langle \mathfrak{m}_1 f_1 | f_2 \rangle_{g_1^{(\pm)}} = \langle f_1 | \mathfrak{m}_1 f_2 \rangle_{g_1^{(\pm)}} \tag{3.22}$$

for two arbitrary integrable functions f_1 and f_2. The function $\mathfrak{m}_1$ is the monomial of order one and the differential operator $D^{(\pm)}$ is the creation operator of the harmonic oscillator corresponding to the measure $g_1^{(\pm)}$,

$$D^{(\pm)} = \frac{\partial}{\partial x} - \frac{n}{2a^2} x \pm \frac{\mu_{\mathrm{l}}}{2}. \tag{3.23}$$

Identity (3.22) cannot be extended to the measures $G_2^{(-)}$ and $G_2^{(+)}$ or equivalently $g_2^{(-)}$ and $g_2^{(+)}$, i.e.

$$(\mathfrak{m}_1 f_1 | f_2)_{g_2^{(\pm)}} \neq (f_1 | \mathfrak{m}_1 f_2)_{g_2^{(\pm)}}. \tag{3.24}$$

However Eq. (3.21) has an analogue. Defining the differential operator

$$\widetilde{D}^{(\pm)} = \frac{\partial}{\partial z} - \frac{n}{a^2} z + \frac{\mu_{\mathrm{r}} \pm \mu_{\mathrm{l}}}{2} \tag{3.25}$$

one can readily verify

$$\left(\widetilde{D}^{(\pm)} f_1 | f_2 \right)_{g_2^{(\pm)}} = -\left(f_1 | \widetilde{D}^{(\pm)} f_2 \right)_{g_2^{(\pm)}}. \tag{3.26}$$

The starting point of such a proof is the differential equation

$$\left[\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}+\frac{n}{a^2}(z_1+z_2)-(\mu_{\rm r}\pm\mu_{\rm l})\right]g_2^{(\pm)}(z_1,z_2)=0. \tag{3.27}$$

Notice that the differential operator is restricted to the real eigenvalues of D_5 and $D_{\rm W}$ and to the real part of the complex conjugated pair of $z=z_1=z_2^*$ of $D_{\rm W}$ due to the Dirac delta-functions.

Both operators $\widetilde{D}^{(\pm)}$ and $D^{(\pm)}$ are closely related with each other which is quite advantageous. For example the action of $\widetilde{D}^{(\pm)}$ in the scalar product (2.25) is

$$\langle\widetilde{D}^{(\pm)}f_1|f_2\rangle_{g_1^{(\pm)}}=-\langle f_1|f_2'\rangle_{g_1^{(\pm)}}+\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2}\langle f_1|f_2\rangle_{g_1^{(\pm)}}, \tag{3.28}$$

where f_2' is the first derivative of f_2.

We consider the action of $\widetilde{D}^{(\pm)}$ on the polynomials $p_l^{(\pm)}$ and $q_{\nu+l}^{(\pm)}$. The recursion relations (3.19) and (3.20) yield

$$\widetilde{D}^{(\pm)}p_l^{(\pm)}(z)=-\frac{n}{a^2}p_{l+1}^{(\pm)}(z)+\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2}p_l^{(\pm)}(z). \tag{3.29}$$

The polynomial $\widetilde{D}^{(\pm)}q_{\nu+l}^{(\pm)}$ can be expanded in the polynomials $\{p_j^{(\pm)},q_{\nu+j}^{(\pm)}\}$, i.e.

$$\widetilde{D}^{(\pm)}q_{\nu+l}^{(\pm)}(z)=\sum_{j=0}^{l+1}\alpha_{lj}^{(\pm)}q_{\nu+j}^{(\pm)}(z)+\sum_{j=0}^{\nu-1}\beta_{lj}^{(\pm)}p_j^{(\pm)}(z), \tag{3.30}$$

where $\alpha_{lj}^{(\pm)}$ and $\beta_{lj}^{(\pm)}$ are the coefficients which have to be found.

In Appendix B we derive the recursions

$$\widetilde{D}q_{\nu+2l}^{(\pm)}(z) = -\frac{n}{a^2}q_{\nu+2l+1}^{(\pm)}(z)+\widetilde{\epsilon}_l^{(\pm)}q_{\nu+2l}^{(\pm)}(z), \tag{3.31}$$

$$\widetilde{D}q_{\nu+2l+1}^{(\pm)}(z)=-\frac{n}{a^2}q_{\nu+2l+2}^{(\pm)}(z)-\widetilde{\epsilon}_l^{(\pm)}q_{\nu+2l+1}^{(\pm)}(z)+\epsilon_l^{(\pm)}q_{\nu+2l}^{(\pm)}(z)-\frac{n}{a^2}\frac{o_l^{(\pm)}}{o_{l-1}^{(\pm)}}q_{\nu+2l-2}^{(\pm)}(z), \tag{3.32}$$

with the coefficients

$$\widetilde{\epsilon}_l^{(\pm)}=(2l+1)\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2}, \tag{3.33}$$

$$\epsilon_l^{(\pm)}=\frac{n}{a^2}\left(\frac{\langle q_{\nu+2l+3}^{(\pm)}|p_{\nu+2l+1}\rangle_{g_1}}{h_{\nu+2l+1}}-\frac{\langle q_{\nu+2l+1}^{(\pm)}|p_{\nu+2l-1}\rangle_{g_1}}{h_{\nu+2l-1}}\right)-\frac{(l+1)^2(\mu_{\rm r}\mp\mu_{\rm l})^2a^2}{n}. \tag{3.34}$$

The recursion formula for $q_{\nu+2l+1}^{(\pm)}$, see Eq. (3.32), is restricted to $l\geq 1$. For $l=0$ we have to omit the last term, i.e. the constant $1/o_{-1}^{(\pm)}$ is zero, see Eq. (3.12) when replacing the factorial by Euler's Gamma-function. This formula is quite useful to find Christoffel Darboux-like formulas, see subsection 3.3.

3.3. A Christoffel Darboux-like formula

For the calculation of spectral correlations the Christoffel Darboux formula is quite useful. However searching for such a formula of skew-orthogonal polynomials proved as

a difficult task [69]. The same is true for the polynomials $q_{\nu+l}^{(\pm)}$ for which we want to simplify the sum

$$\Sigma_{n-1}^{(\pm)}(z_1,z_2)=\sum_{l=0}^{n-1}\frac{1}{o_l^{(\pm)}}\det\begin{bmatrix} q_{\nu+2l}^{(\pm)}(z_1) & q_{\nu+2l+1}^{(\pm)}(z_1)\\ q_{\nu+2l}^{(\pm)}(z_2) & q_{\nu+2l+1}^{(\pm)}(z_2)\end{bmatrix}. \tag{3.35}$$

For the orthogonal polynomials $p_l^{(\pm)}$ we already know such a result,

$$\sum_{l=0}^{\nu-1}\frac{1}{h_l}p_l^{(\pm)}(z_1)p_l^{(\pm)}(z_2)=\frac{1}{h_{\nu-1}}\frac{p_\nu^{(\pm)}(z_1)p_{\nu-1}^{(\pm)}(z_2)-p_\nu^{(\pm)}(z_2)p_{\nu-1}^{(\pm)}(z_1)}{z_1-z_2}. \tag{3.36}$$

Identity (3.36) is a direct consequence of the three term recursion relation (3.20). Hence we pursue the same idea for Eq. (3.35) which is done in Appendix C. The Christoffel Darboux-like formula for the skew-orthogonal polynomials is

$$\begin{aligned}\Sigma_{n-1}^{(\pm)}(z_1,z_2)=&\frac{n}{a^2o_{n-1}^{(\pm)}}\int_0^\infty d\widetilde{x}\exp\left[-\frac{n}{a^2}\widetilde{x}^2+\left(\mu_{\rm r}\pm\mu_{\rm l}-\frac{n}{a^2}(z_1+z_2)\right)\widetilde{x}\right]\\ &\times\ \det\begin{bmatrix} q_{\nu+2n-2}^{(\pm)}(z_1+\widetilde{x}) & q_{\nu+2n}^{(\pm)}(z_1+\widetilde{x})\\ q_{\nu+2n-2}^{(\pm)}(z_2+\widetilde{x}) & q_{\nu+2n}^{(\pm)}(z_2+\widetilde{x})\end{bmatrix}.\end{aligned} \tag{3.37}$$

This result only depends on a few polynomials as it is already well known for the original Christoffel Darboux formula, cf. Eq. (3.36).

3.4. Representation as random matrix averages

As we have already seen in subsection 3.2 all skew-orthogonal polynomials $q_{\nu+l}^{(\pm)}$ are easy to derive if we know a compact expression for l even. For this purpose we want to derive a representation as an integral over a random matrix. For the orthogonal polynomials the well known expression of this kind is

$$p_l^{(\pm)}(z)=\left(\frac{2\pi a^2}{n}\right)^{l^2}\int d[H]\det(z\mathbb{1}_l-H)\exp\left[-\frac{n}{2a^2}\operatorname{tr}\left(H\mp\frac{a^2}{n}\mu_{\rm l}\right)^2\right], \tag{3.38}$$

where H is a $l\times l$ Hermitian random matrix with the measure

$$d[H]=\prod_{i=1}^{l}dH_{ii}\prod_{1\le i<j\le l}2d\mathrm{Re}\ H_{ij}d\mathrm{Im}\ H_{ij}. \tag{3.39}$$

This random matrix integral can be drastically reduced to a small number of integration variables by the supersymmetry method [52, 54, 55, 57]. A famous representation of the Hermite polynomials can be derived in this way,

$$p_l^{(\pm)}(z)=\frac{l!}{2\pi}\int_0^{2\pi}d\varphi\exp\left[-\frac{a^2}{2n}e^{\imath 2\varphi}+\left(z\mp\frac{a^2}{n}\mu_{\rm l}\right)e^{\imath\varphi}\right]e^{-\imath l\varphi}. \tag{3.40}$$

The corresponding Rodrigues-formula for the Hermite polynomials is a simple lemma from this, i.e.

$$p_l^{(\pm)}(z)=\left(-\frac{a^2}{n}\right)^l\exp\left[\frac{n}{2a^2}z^2\mp\mu_{\rm l}z\right]\frac{\partial^l}{\partial z^l}\exp\left[-\frac{n}{2a^2}z^2\pm\mu_{\rm l}z\right] \tag{3.41}$$

$$= \left(\frac{a^2}{n}\right)^{l/2} H_l\left(\sqrt{\frac{n}{a^2}}z \mp \sqrt{\frac{a^2}{n}}\mu_l\right).$$

The aim is to find the formulas analogous to Eqs. (3.38), (3.40) and (3.41) for $q^{(\pm)}_{\nu+2l}$.

We compare the definition (3.5) with Eqs. (2.27) and (2.45) for $k = k_{\rm r} = k_{\rm l} = 0$, $n = l$ and $N_{\rm f} = 1$. Since we were free of choosing the two-point weight $G_2^{(\pm)}$ and the polynomials $q^{(\pm)}_{\nu+l}$ at this step of the calculation, Eqs. (2.27) and (2.45) are also valid when replacing $G_2^{(\pm)}$ by $g_2^{(\pm)}$ and $q^{(\pm)}_{\nu+l}$ by $p^{(\pm)}_{\nu+l}$. Moreover, $q^{(\pm)}_{\nu+2l}$ is also equal to

$$q^{(\pm)}_{\nu+2l}(z) = \frac{(-1)^{\nu(\nu+1)/2}}{\mathrm{Pf}\left[\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right]_{\nu\le i,j\le\nu+2l-1}\prod_{j=0}^{\nu-1}h_j} \tag{3.42}$$
$$\times\mathrm{Pf}\begin{bmatrix} \left\{\left(p_i^{(\pm)}|p_j^{(\pm)}\right)_{g_2^{(\pm)}}\right\}_{0\le i,j\le\nu+2l} & \left\{\langle p_i^{(\pm)}|p_j^{(\pm)}\rangle_{g_1^{(\pm)}}\right\}_{\substack{0\le i\le\nu+2l\\0\le j\le\nu-1}} & \left\{p_i^{(\pm)}(z)\right\}_{0\le i\le\nu+2l} \\ \left\{-\langle p_i^{(\pm)}|p_j^{(\pm)}\rangle_{g_1^{(\pm)}}\right\}_{\substack{0\le i\le\nu-1\\0\le j\le\nu+2l}} & 0 & 0 \\ \left\{-p_j(z)^{(\pm)}\right\}_{0\le j\le\nu+2l} & 0 & 0 \end{bmatrix}.$$

Indeed this equation coincides with the ansatz (3.5) since the scalar products in the second row and column are either zero or equal to the normalization constants h_j. An expansion in these rows and columns yields Eq. (3.5).

The polynomials $q^{(\pm)}_{\nu+2l}$ are the partition functions with one fermionic flavor,

$$q^{(\pm)}_{\nu+2l}(z) = (-1)^\nu \frac{Z_1^{(l,\nu,\pm)}(-z)}{Z_0^{(l,\nu,\pm)}}. \tag{3.43}$$

By means of the supersymmetry technique [52, 54, 55, 57] we find the result

$$q^{(\pm)}_{\nu+2l}(z) = \frac{(\pm1)^{l+\nu}l!(l+\nu)!}{(2\pi)^2}\int\limits_{[0,2\pi]^2} d\varphi_{\rm r}\,d\varphi_{\rm l}\,\exp\left[-\frac{a^2}{2n}(e^{\imath 2\varphi_{\rm r}}+e^{\imath 2\varphi_{\rm l}})+\frac{1}{n}e^{\imath(\varphi_{\rm r}+\varphi_{\rm l})}\right] \tag{3.44}$$
$$\times\ \exp\left[-\left(\frac{a^2\mu_{\rm r}}{n}-z\right)e^{\imath\varphi_{\rm r}}-\left(\frac{a^2\mu_{\rm l}}{n}\mp z\right)e^{\imath\varphi_{\rm l}}\right]e^{-\imath l\varphi_{\rm r}}\,e^{-\imath(l+\nu)\varphi_{\rm l}}$$

in Appendix D. Notice the similarity with Eq. (3.40).

Again we can ask for a Rodrigues formula and indeed it is a direct consequence of Eq. (3.44). We find

$$q^{(\pm)}_{\nu+2l}(z) = (\pm1)^{l+\nu}\left(\frac{a^2}{n}\right)^{(l+\nu)/2}\exp\left[\frac{a^2}{2n}\left(\mu_{\rm r}-\frac{n}{a^2}z\right)^2\right] \tag{3.45}$$
$$\times\ \left.\frac{\partial^l}{\partial\widetilde{x}^l}\right|_{\widetilde{x}=0}\exp\left[\frac{a^2}{2n}\left(\widetilde{x}+\mu_{\rm r}-\frac{n}{a^2}z\right)^2\right]H_{l+\nu}\left(\pm\sqrt{\frac{n}{a^2}}z-\sqrt{\frac{a^2}{n}}\mu_{\rm l}+\frac{\widetilde{x}}{\sqrt{na^2}}\right).$$

Performing the derivatives we find an explicit expression in terms of Hermite polynomials for the skew-orthogonal polynomials $q^{(\pm)}_{\nu+2l}$,

$$q^{(\pm)}_{\nu+2l}(z) = (\pm1)^{l+\nu}\left(\frac{1}{\sqrt{na^2}}\right)^l\left(\frac{a^2}{n}\right)^{(l+\nu)/2}\sum_{j=0}^{l}\frac{l!(l+\nu)!}{j!(l-j)!(\nu+j)!}a^{2j} \tag{3.46}$$

$$\times\, H_{\nu+j}\left(\pm\sqrt{\frac{n}{a^2}}z-\sqrt{\frac{a^2}{n}}\mu_{\rm l}\right) H_j\left(\sqrt{\frac{n}{a^2}}z-\sqrt{\frac{a^2}{n}}\mu_{\rm r}\right).$$

The polynomials for $q^{(\pm)}_{\nu+2l+1}$ can be readily obtained with help of relation (3.31). Remarkably the prefactors of the single summands are exactly the same as the ones of the modified Laguerre polynomials, $L_l^{(\nu)}$, when replacing the Hermite polynomials by monomials.

The limit $a \to 0$ yields the generalized Laguerre polynomials,

$$\begin{aligned} q^{(\pm)}_{\nu+2l}(z) &\overset{a\to 0}{=} \left(\pm\frac{1}{n}\right)^l z^\nu L_l^{(\nu)}(\mp n z^2), \\ q^{(\pm)}_{\nu+2l+1}(z) &= \left(\frac{a^2\widetilde{\epsilon}^{(\pm)}}{n}-\frac{a^2}{n}\widetilde{D}\right) q^{(\pm)}_{\nu+2l}(z) \overset{a\to 0}{=} \left(\pm\frac{1}{n}\right)^l z^{\nu+1} L_l^{(\nu)}(\mp n z^2), \end{aligned} \tag{3.47}$$

which is in agreement with Ref. [67]. The large a limit with fixed variables $\sqrt{n/a^2}z$ and $\sqrt{a^2/n}\mu_{\rm l\,/\,r}$ is a product of two Hermite polynomials

$$q^{(\pm)}_{\nu+2l}(z) \overset{a\gg 1}{=} \left(\frac{a^2}{n}\right)^{(2l+\nu)/2} H_{\nu+l}\left(\sqrt{\frac{n}{a^2}}z\mp\sqrt{\frac{a^2}{n}}\mu_{\rm l}\right) H_l\left(\sqrt{\frac{n}{a^2}}z-\sqrt{\frac{a^2}{n}}\mu_{\rm r}\right). \tag{3.48}$$

Both limits can already be directly derived from the random matrix model, cf. Eqs. (2.1) and (2.2). For $a = 0$ we have a χGUE whose orthogonal polynomials are the Laguerre polynomials. Recently it was shown that the χGUE has also a non-trivial Pfaffian factorization whose skew-orthogonal polynomials of even order are the orthogonal polynomials itself. Hence the limit (3.47) agrees with the observation in Ref. [67].

In the large a limit the off-diagonal blocks W and $W^\dagger$, see (2.1), are suppressed. Therefore we end up with two decoupled GUE's. One is of dimension l and the other one of dimension $l + \nu$. This indeed yields a product of two Hermite polynomials, cf. Eq. (3.48).

A particular case of the polynomials can be obtained for the random matrix D_5. Let $\mu_{\rm r} = -\mu_{\rm l} = \mu$ and $a = 1$. Then we have a $2n + \nu$ dimensional GUE, cf. Eq. (2.2). Indeed we also get the corresponding Hermite polynomials. Equation (3.44) simplifies to

$$\begin{aligned} q^{(-)}_{\nu+2l}(z) &= \frac{(\pm 1)^{l+\nu} l!(l+\nu)!}{(2\pi)^2} \int\limits_{[0,2\pi]^2} d\varphi_{\rm r}\, d\varphi_{\rm l}\, \exp\left[-\frac{1}{2n}(e^{\imath\varphi_{\rm r}}-e^{\imath\varphi_{\rm l}})^2\right] \\ &\quad\times\, \exp\left[\left(z-\frac{\mu}{n}\right)(e^{\imath\varphi_{\rm r}}-e^{\imath\varphi_{\rm l}})\right] e^{-\imath l\varphi_{\rm r}} e^{-\imath(l+\nu)\varphi_{\rm l}} \\ &= n^{-(\nu+2l)/2} H_{\nu+2l}\left(\sqrt{n}z-\frac{\mu}{\sqrt{n}}\right) \end{aligned} \tag{3.49}$$

for the even polynomials and

$$\begin{aligned} q^{(-)}_{\nu+2l+1}(z) &= \left(-\frac{\mu}{n}+z-\frac{1}{n}\frac{\partial}{\partial z}\right) q^{(-)}_{\nu+2l}(z) \\ &= n^{-(\nu+2l+1)/2} H_{\nu+2l+1}\left(\sqrt{n}z-\frac{\mu}{\sqrt{n}}\right) \end{aligned} \tag{3.50}$$

for the odd ones. Therefore all polynomials are given by Hermite polynomials corresponding to the same Gaussian distribution.

Another useful random matrix integral representation would be the one for the Christoffel Darboux-like formula (3.37). For the orthogonal polynomials $p_l^{(\pm)}$ such a representation is well known,

$$\begin{aligned}&\frac{p_\nu^{(\pm)}(z_1)p_{\nu-1}^{(\pm)}(z_2)-p_\nu^{(\pm)}(z_2)p_{\nu-1}^{(\pm)}(z_1)}{z_1-z_2}\\ &=\left(\frac{2\pi a^2}{n}\right)^{(\nu-1)^2}\int d[H]\det(z_1\mathbb{1}_{\nu-1}-H)\det(z_2\mathbb{1}_{\nu-1}-H)\exp\left[-\frac{n}{2a^2}\operatorname{tr}\left(H\mp\frac{a^2}{n}\mu_{\mathrm{l}}\right)^2\right]\end{aligned}\tag{3.51}$$

with a $(\nu-1)\times(\nu-1)$ Hermitian matrix H. With the supersymmetry method [52, 54, 55, 57] one can also find the representation

$$\begin{aligned}&\frac{p_\nu^{(\pm)}(z_1)p_{\nu-1}^{(\pm)}(z_2)-p_\nu^{(\pm)}(z_2)p_{\nu-1}^{(\pm)}(z_1)}{z_1-z_2}\\ &=\nu!(\nu-1)!\int\limits_{\mathrm{U}(2)}d\mu(U)\exp\left[-\frac{a^2}{2n}\operatorname{tr}U^2+\operatorname{tr}\left(\operatorname{diag}(z_1,z_2)\mp\frac{a^2\mu_{\mathrm{l}}}{n}\mathbb{1}_2\right)U\right]\det{}^{-\nu+1}U,\end{aligned}\tag{3.52}$$

where $d\mu(U)$ is the normalized Haar measure of the unitary group $\mathrm{U}(2)$.

In Appendix E we show that the Christoffel Darboux-like formula (3.37) is essentially the partition function with two fermionic flavors, i.e.

$$\Sigma_n^{(\pm)}(z_1,z_2)=(z_1-z_2)\frac{Z_2^{(n,\nu,\pm)}(-z_1,-z_2)}{o_n^{(\pm)}Z_0^{(n,\nu,\pm)}}.\tag{3.53}$$

Also the two-flavor partition function can be mapped to an integral over unitary groups by performing the same calculation as for the one-flavor partition function, see the discussion in Appendix D. Therefore $\Sigma_n^{(\pm)}$ is an integral over a compact set,

$$\begin{aligned}\Sigma_n^{(\pm)}(z_1,z_2)=&-\frac{(n+1)!(n+\nu+1)!}{4}\sqrt{\frac{n(1\pm a^2)}{\pi}}\left(\frac{n}{1\pm a^2}\right)^{2n+\nu+1}\frac{1}{a}\\ &\times\exp\left[-\frac{a^2}{4n}(\mu_{\mathrm{r}}\pm\mu_{\mathrm{l}})^2\mp\frac{a^4}{4n(1\pm a^2)}(\mu_{\mathrm{r}}\mp\mu_{\mathrm{l}})^2\right](z_1-z_2)\\ &\times\int\limits_{\mathrm{U}(2)\times\mathrm{U}(2)}d\mu(U_{\mathrm{r}})d\mu(U_{\mathrm{l}})\exp\left[-\frac{a^2}{2n}(\operatorname{tr}U_{\mathrm{r}}^2+\operatorname{tr}U_{\mathrm{l}}^2)+\frac{1}{n}\operatorname{tr}U_{\mathrm{r}}U_{\mathrm{l}}\right]\\ &\times\exp\left[-\operatorname{tr}\left(\frac{a^2\mu_{\mathrm{r}}}{n}\mathbb{1}_2-\operatorname{diag}(z_1,z_2)\right)U_{\mathrm{r}}\right]\det{}^{-n}U_{\mathrm{r}}\\ &\times\exp\left[-\operatorname{tr}\left(\frac{a^2\mu_{\mathrm{l}}}{n}\mathbb{1}_2\mp\operatorname{diag}(z_1,z_2)\right)U_{\mathrm{l}}\right]\det{}^{-n-\nu}U_{\mathrm{l}}.\end{aligned}\tag{3.54}$$

Equations (3.44) and (3.54) are suitable for discussing the asymptotic behavior as it is done in subsection 3.5.

3.5. Asymptotics

The microscopic limit ($n \to \infty$, see discussion after Eq. (2.12)) directly relates chiral random matrix theory with QCD in the ϵ-regime. Hence we want to know the expressions of the polynomials $q^{(\pm)}_{\nu+2l}$ as well as the one of the Christoffel Darboux-like formula $\Sigma^{(\pm)}_{n-1}$ in this limit.

For an arbitrary function f which is n-independent and smooth on the group $\mathrm{U}(k) \times \mathrm{U}(k)$ the following asymptotic result exists

$$\int\limits_{\mathrm{U}(k)\times\mathrm{U}(k)} d\mu(U_\mathrm{r}) d\mu(U_\mathrm{l}) f(U_\mathrm{r}, U_\mathrm{l}) \exp\left[n \operatorname{tr} U_\mathrm{r} U_\mathrm{l} - n \operatorname{tr} \ln U_\mathrm{r} U_\mathrm{l}\right] \tag{3.55}$$

$$\stackrel{n\gg 1}{=} (2\pi)^{-k/2} n^{-k^2/2} e^{nk} \prod_{j=0}^{k-1} j! \int\limits_{\mathrm{U}(k)} d\mu(U) f(U, U^{-1}).$$

This identity can be readily proven by a shift of the unitary matrix $U_\mathrm{l} \to U_\mathrm{r}^{-1} U_\mathrm{l}$. Then the exponent only depends on U_l. The saddlepoint approximation yields an expansion of U_l about the unit matrix yielding Eq. (3.55).

Equations (3.44) and (3.54) are particular cases of identity (3.55). Hence we have

$$q^{(\pm)}_{\nu+2n}\left(\frac{\widehat{z}}{2n}\right) \stackrel{n\gg 1}{=} \frac{(-1)^\nu (\pm 1)^{n+\nu} \sqrt{n} e^{-n}}{\sqrt{2\pi}} \int\limits_{[0,2\pi]} d\varphi \exp\left[-\widehat{a}^2 (e^{\imath 2\varphi} + e^{-\imath 2\varphi})\right] \tag{3.56}$$

$$\times \quad \exp\left[\frac{1}{2}\left(\widehat{m}_6 + \widehat{\lambda}_7 - \widehat{z}\right) e^{\imath\varphi} + \frac{1}{2}\left(\widehat{m}_6 - \widehat{\lambda}_7 \mp \widehat{z}\right) e^{-\imath\varphi}\right] e^{\imath\nu\varphi}.$$

for the polynomials which is the one-flavor partition function derived in Refs. [20] and

$$\Sigma^{(\pm)}_n\left(\frac{\widehat{z}_1}{2n}, \frac{\widehat{z}_2}{2n}\right) \stackrel{n\gg 1}{=} -\frac{1}{8\sqrt{2\pi}} \frac{n^2}{\widehat{a}} \exp\left[-\frac{\widehat{a}^2}{2n^2}(\mu_\mathrm{r} \pm \mu_\mathrm{l})^2 \mp 4\widehat{a}^2\right] (\widehat{z}_1 - \widehat{z}_2) \tag{3.57}$$

$$\times \int\limits_{\mathrm{U}(2)} d\mu(U) \exp\left[\frac{1}{2} \operatorname{tr}\left((\widehat{m}_6 - \widehat{\lambda}_7)\mathbb{1}_2 \mp \operatorname{diag}(\widehat{z}_1, \widehat{z}_2)\right) U^{-1}\right] \det{}^\nu U$$

$$\times \quad \exp\left[-\widehat{a}^2 \operatorname{tr}(U^2 + U^{-2}) + \frac{1}{2} \operatorname{tr}\left((\widehat{m}_6 + \widehat{\lambda}_7)\mathbb{1}_2 - \operatorname{diag}(\widehat{z}_1, \widehat{z}_2)\right) U\right]$$

for the Christoffel Darboux-like formula, cf. Eqs. (2.20) and (2.39). In both equation we applied Stirling's formula to the factorials.

In the case of the polynomials $q^{(\pm)}_{\nu+2n}(z)$ we are able to integrate over the domain,

$$q^{(\pm)}_{\nu+2n}(z) \stackrel{n\gg 1}{=} \frac{(-1)^\nu (\pm 1)^{n+\nu} \sqrt{n} e^{-n-\nu}}{(2\pi)^{3/2}} \sum_{j=-\infty}^{\infty} \int\limits_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \exp\left[-\widehat{a}^2 (e^{\imath\varphi_1} + e^{-\imath\varphi_1})\right] \tag{3.58}$$

$$\times \quad \exp\left[\frac{1}{2}\left(\widehat{m}_6 + \widehat{\lambda}_7 - \widehat{z}\right) e^{\imath\varphi_2} + \frac{1}{2}\left(\widehat{m}_6 - \widehat{\lambda}_7 \mp \widehat{z}\right) e^{-\imath\varphi_2}\right] e^{\imath\nu\varphi_2} e^{\imath j(\varphi_1 - 2\varphi_2)}$$

$$= \quad (-1)^\nu (\pm 1)^{n+\nu} \sqrt{2\pi n} e^{-n-\nu} \sum_{j=-\infty}^{\infty} \left(\frac{\widehat{m}_6 - \widehat{\lambda}_7 \mp \widehat{z}}{\widehat{m}_6 + \widehat{\lambda}_7 - \widehat{z}}\right)^{\nu/2+j}$$

$$\times \quad I_{\nu+2j}\left(\sqrt{(\widehat{m}_6+\widehat{\lambda}_7-\widehat{z})(\widehat{m}_6-\widehat{\lambda}_7\mp\widehat{z})}\right) I_j(-\widehat{a}^2).$$

Due to the modified Bessel functions of the second kind $I_l(z) \propto (ze/2|l|)^{|l|}/\sqrt{2\pi|l|} \propto (z/2)^{|l|}/|l|!$, for $|l| \gg 1$, the series rapidly converges and is numerically more stable than the integral (3.56) in simulations.

Unfortunately it is much harder to find such a sum for the Christoffel Darboux-like formula. However we can diagonalize the unitary matrix U and find

$$\begin{aligned}\Sigma_n^{(+)}\left(\frac{\widehat{z}_1}{2n},\frac{\widehat{z}_2}{2n}\right) \overset{n\gg 1}{=} & -\frac{n^2}{4(2\pi)^{5/2}\widehat{a}}\int\limits_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1-\varphi_2}{2}\right] \\ & \times \quad \exp\left[\sum_{j=1}^{2}\left(-\left(2\widehat{a}\cos\varphi_j+\frac{\widehat{m}_6}{4\widehat{a}}\right)^2+\imath\nu\varphi_j-\imath\widehat{\lambda}_7\sin\varphi_j\right)\right] \\ & \times \quad \frac{\exp\left[\widehat{z}_1\cos\varphi_1+\widehat{z}_2\cos\varphi_2\right]-\exp\left[\widehat{z}_2\cos\varphi_1+\widehat{z}_1\cos\varphi_2\right]}{\cos\varphi_1-\cos\varphi_2}\end{aligned} \tag{3.59}$$

for D_{W} and

$$\begin{aligned}\Sigma_n^{(-)}\left(\frac{\widehat{z}_1}{2n},\frac{\widehat{z}_2}{2n}\right) \overset{n\gg 1}{=} & \frac{\imath n^2}{4(2\pi)^{5/2}\widehat{a}}\int\limits_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1-\varphi_2}{2}\right] \\ & \times \quad \exp\left[\sum_{j=1}^{2}\left(\left(2\widehat{a}\sin\varphi_j-\frac{\imath\widehat{\lambda}_7}{4\widehat{a}}\right)^2+\imath\nu\varphi_j-\imath\widehat{m}_6\cos\varphi_j\right)\right] \\ & \times \quad \frac{\exp\left[\imath\widehat{z}_1\sin\varphi_1+\imath\widehat{z}_2\sin\varphi_2\right]-\exp\left[\imath\widehat{z}_2\sin\varphi_1+\imath\widehat{z}_1\sin\varphi_2\right]}{\sin\varphi_1-\sin\varphi_2}\end{aligned} \tag{3.60}$$

for D_5. These two formulas are quite suitable for the applications discussed in Sec. 4. Both Christoffel-Darboux formulas are mostly two-flavor partition functions. In Ref. [22] these functions are expressed as non-compact integrals over Bessel functions.

4. Application to Wilson RMT

The results of the previous sections are helpful to simplify the k-point functions of D_5 as well as of D_{W}. A Pfaffian factorization of the eigenvalue correlations of D_5 was already given in Ref. [23]. We obtain this structure in Sec. 4.1, too. Moreover we express the kernels of the Pfaffian in terms of two-flavor partition functions which has proven fruitful in other random matrix ensembles, see Ref. [58] and the references therein. The unquenched $(k_{\mathrm{r}}\,,k_{\mathrm{l}})$-point correlation function of D_{W} is shown in Sec. 4.2 which is a completely new result. Also this result displays a Pfaffian factorization whose entries are two-flavor partition functions.

4.1. The Hermitian Wilson random matrix ensemble

In the k-point correlation function (2.27) we encounter an integral transform of the orthogonal and skew-orthogonal polynomials, cf. Eq. (2.29). Thus we define the integral

transform of the skew-orthogonal polynomials,

$$\begin{aligned}\widetilde{q}^{(-)}_{\nu+l}(x) &= \int\limits_{\mathbb{R}} d\widetilde{x} q^{(-)}_{\nu+l}(\widetilde{x}) G^{(-)}_2(\widetilde{x},x) \\ &= \int\limits_{\mathbb{R}} d\widetilde{x} q^{(-)}_{\nu+l}(\widetilde{x}) g^{(-)}_2(\widetilde{x},x) - \sum_{j=0}^{\nu-1} \frac{(q^{(-)}_{\nu+l}|p^{(-)}_j)_{g^{(-)}_2}}{h_j} p^{(-)}_j(x) g^{(-)}_1(x) \\ &= -\int\limits_{\mathbb{R}} d\widetilde{x} q^{(-)}_{\nu+l}(\widetilde{x}) G^{(-)}_2(x,\widetilde{x}).\end{aligned} \tag{4.1}$$

The same integral transform for the orthogonal polynomials $p^{(-)}_l$, $0 \le l \le \nu-1$, vanishes, i.e.

$$\int\limits_{\mathbb{R}} d\widetilde{x} p^{(-)}_l(\widetilde{x}) G^{(-)}_2(\widetilde{x},x) = 0, \tag{4.2}$$

cf. Eq. (3.17).

Using the identity

$$\mathrm{Pf}\begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} = \mathrm{Pf}\, A \mathrm{Pf}\,[C + B^T A^{-1} B], \tag{4.3}$$

where B and C are arbitrary and A is invertible, the k-point correlation function with an even number of fermionic flavors $N_\mathrm{f} = 2n_\mathrm{f}$, see Eq. (2.27), is

$$\begin{aligned} R^{(n,\nu,-)}_{2n_\mathrm{f},k}(x') &= \frac{(-1)^{k(k+1)/2}}{\mathrm{Pf}\,[K^{(-,n+n_\mathrm{f})}_3(-\lambda_i,-\lambda_j)]_{1\le i,j\le 2n_\mathrm{f}}} \\ &\times \mathrm{Pf}\left[\begin{array}{c|c|c} K^{(-,n+n_\mathrm{f})}_1(x_i,x_j) & -K^{(-,n+n_\mathrm{f})}_2(x_j,x_i) & -K^{(-,n+n_\mathrm{f})}_2(-\lambda_j,x_i) \\ \hline K^{(-,n+n_\mathrm{f})}_2(x_i,x_j) & K^{(-)}_3(x_i,x_j) & K^{(-,n+n_\mathrm{f})}_3(x_i,-\lambda_j) \\ \hline K^{(-,n+n_\mathrm{f})}_2(-\lambda_i,x_j) & K^{(-)}_3(-\lambda_i,x_j) & K^{(-,n+n_\mathrm{f})}_3(-\lambda_i,-\lambda_j) \end{array}\right] \end{aligned} \tag{4.4}$$

which is the main result for the Hermitian Wilson random matrix D_5. The indices i and j of the Pfaffian in the denominator take the values $(1,\ldots,k,1,\ldots,k,1,\ldots,2n_\mathrm{f})$. The functions in the entries are

$$K^{(-,n+n_\mathrm{f})}_1(x_1,x_2) = G^{(-)}_2(x_1,x_2) + \sum_{l=0}^{n+n_\mathrm{f}-1} \frac{1}{o^{(-)}_l} \det\begin{bmatrix} \widetilde{q}^{(-)}_{\nu+2l+1}(x_1) & \widetilde{q}^{(-)}_{\nu+2l}(x_1) \\ \widetilde{q}^{(-)}_{\nu+2l+1}(x_2) & \widetilde{q}^{(-)}_{\nu+2l}(x_2) \end{bmatrix}, \tag{4.5}$$

$$\begin{aligned} K^{(-,n+n_\mathrm{f})}_2(x_1,x_2) &= \sum_{l=0}^{\nu-1} \frac{1}{h_l} p^{(-)}_l(x_1) p^{(-)}_l(x_2) g^{(-)}_1(x_2) \\ &+ \sum_{l=0}^{n+n_\mathrm{f}-1} \frac{1}{o^{(-)}_l} \det\begin{bmatrix} q^{(-)}_{\nu+2l+1}(x_1) & q^{(-)}_{\nu+2l}(x_1) \\ \widetilde{q}^{(-)}_{\nu+2l+1}(x_2) & \widetilde{q}^{(-)}_{\nu+2l}(x_2) \end{bmatrix}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} K^{(-,n+n_\mathrm{f})}_3(x_1,x_2) &= \sum_{l=0}^{n+n_\mathrm{f}-1} \frac{1}{o^{(-)}_l} \det\begin{bmatrix} q^{(-)}_{\nu+2l+1}(x_1) & q^{(-)}_{\nu+2l}(x_1) \\ q^{(-)}_{\nu+2l+1}(x_2) & q^{(-)}_{\nu+2l}(x_2) \end{bmatrix} \\ &= -\Sigma^{(-)}_{n+n_\mathrm{f}-1}(x_1,x_2). \end{aligned} \tag{4.7}$$

The k-point correlation function for an odd number of flavors can be derived by shifting one of the axial masses λ to infinity. Then we get the skew-orthogonal polynomial $q^{(-)}_{\nu+2(n+n_{\rm f}-1)}$ and its integral transform $\widetilde{q}^{(-)}_{\nu+2(n+n_{\rm f}-1)}$ in one row and one column of the numerator and the denominator of Eq. (4.4).

The case $k = 0$ is the normalization. For $k = 2n + \nu$ we have a compact representation of the joint probability density p_5 as a single Pfaffian determinant.

The representation (4.5-4.7) in terms of the Hermite polynomials $p^{(-)}_l$ and the skew-orthogonal polynomials $q^{(-)}_{\nu+l}$ can be easily interpreted. The ν former zero modes are broadened by a GUE of dimension ν. The skew-orthogonal polynomials can be identified with the remaining modes and describe the spectral density thereof. Both spectra, the one of the GUE and the one of the remaining modes, are coupled by the sum in Eq. (4.1). They manifest the repulsion of the former zero modes with the remaining modes which is given by the Vandermonde determinant in the joint probability density (2.6).

Not only the kernel $K_3^{(-,n+n_{\rm f})}$ can be expressed in terms of two-flavor partition functions, note that the Christoffel-Darboux-like formula, $\Sigma^{(-)}_{n+n_{\rm f}-1}$, is mostly such a partition function. Also the kernels $K_1^{(-,n+n_{\rm f})}$ and $K_2^{(-,n+n_{\rm f})}$ can be traced back to partition functions. In Appendix F we derive the following results

$$K_1^{(-,n)}(x_1,x_2) = \frac{o_n^{(-)}}{\pi^2}(x_1-x_2)\operatorname*{Im}_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} \tag{4.8}$$
$$\times\left\langle \frac{1}{\det(D_5-(x_1+\imath\varepsilon_1)\mathbb{1}_{2n+\nu+2})\det(D_5-(x_2+\imath\varepsilon_2)\mathbb{1}_{2n+\nu+2})}\right\rangle_{n+1,\nu},$$
$$K_2^{(-,n)}(x_1,x_2) = \frac{1}{\pi}\frac{1}{x_2-x_1}\operatorname*{Im}_{\varepsilon\to 0}\left\langle \frac{\det(D_5-x_1\mathbb{1}_{2n+\nu})}{\det(D_5-(x_2+\imath\varepsilon)\mathbb{1}_{2n+\nu})}\right\rangle_{n,\nu}. \tag{4.9}$$

We employ the notations

$$\operatorname*{Im}_{\varepsilon\to 0}\int dx\frac{f(x)}{x-\imath\varepsilon} = \lim_{\varepsilon\to 0}\int dx\frac{\varepsilon f(x)}{x^2+\varepsilon^2} = \pi f(0) \tag{4.10}$$

and

$$\langle F(D_{\rm W})\rangle_{N,\nu} = \langle F(\gamma_5 D_5)\rangle_{N,\nu} = \int d[D_{\rm W}]F(D_{\rm W})P(D_{\rm W}) \tag{4.11}$$

for two arbitrary sufficiently integrable functions f and F and the definition of the probability density P in Eq. (2.2). The random matrix on the right hand side of Eq. (4.11) has the dimension $(2N+\nu)\times(2N+\nu)$ with index ν. Hence we have to take the averages (4.8) and (4.9) over a Wilson random matrix with $N = n + n_{\rm f}$.

Considering Eqs. (3.53), (4.7), (4.8) and (4.9) we traced the unquenched k-point correlation functions of D_5 back to partition functions with two fermionic, two bosonic and one fermionic and one bosonic determinant. Hence the structure of the eigenvalue correlations of D_5 is in the same class of matrix ensembles as the $\beta = 1$ and $\beta = 4$ standard ensembles, e.g. GOE, GSE, the real and quaternion Ginibre ensemble, the chiral GOE and the chiral GSE, see Ref. [58] and the references therein. When taking the continuum limit, $a \to 0$, the Pfaffian determinant will persist though we have then

chiral GUE. This observation agrees with the result found in Ref. [67]. Therein a non-trivial Pfaffian was derived for all random matrix ensembles corresponding to orthogonal polynomials. Exactly this structure carries over to the finite lattice spacing result (4.4).

The kernel $K_2^{(-,n)}(x,x)$ is equal to the quenched one point function of D_5, denoted by $\rho_5(x)$ in Refs. [20, 21]. Due to the prefactor $1/(x_1-x_2)$, see Eq. (4.9), we have to apply l'Hospital's rule which exactly agrees with the common definition of ρ_5.

The Pfaffian factorization (4.4) was already discovered in Ref. [23] but we made the connection to two-flavor partition functions. Furthermore the structure as well as the expression in two-flavor partition functions carry over to the microscopic limit. In this limit Wilson random matrix theory is directly related to the ϵ-regime of Wilson fermions in lattice QCD [27, 28, 29, 30, 39]. Hence we found a neat representation which drastically simplifies the numerical realization of the k-point correlation functions.

The microscopic limit of the kernel $K_3^{(-,n)}$ is shown in Sec. 3.5, see Eq. (3.60). A derivation of the other kernels as well as a qualitative discussion of the results will be done elsewhere [68].

In the notation of Refs. [20, 21, 22] the kernels are proportional to the two-flavor partition functions of the chiral Lagrangian,

$$K_1^{(-,\infty)}(x_1,x_2) \propto (\widehat{x}_1-\widehat{x}_2) \operatorname*{Im}_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} Z^\nu_{0/2}(\widehat{m}_6,\widehat{m}_6;\widehat{\lambda}_7-\widehat{x}_1,\widehat{\lambda}_7-\widehat{x}_2;\widehat{a}_8,\widehat{a}_{6/7}=0), \tag{4.12}$$

$$K_2^{(-,\infty)}(x_1,x_2) \propto \operatorname*{Im}_{\varepsilon\to 0} \frac{Z^\nu_{1/1}(\widehat{m}_6,\widehat{m}_6;\widehat{\lambda}_7-\widehat{x}_1,\widehat{\lambda}_7-\widehat{x}_2;\widehat{a}_8,\widehat{a}_{6/7}=0)}{\widehat{x}_2-\widehat{x}_1}, \tag{4.13}$$

$$K_3^{(-,\infty)}(x_1,x_2) \propto (\widehat{x}_1-\widehat{x}_2) Z^\nu_{2/0}(\widehat{m}_6,\widehat{m}_6;\widehat{\lambda}_7-\widehat{x}_1,\widehat{\lambda}_7-\widehat{x}_2;\widehat{a}_8,\widehat{a}_{6/7}=0), \tag{4.14}$$

in the microscopic limit. Please recall that $\widehat{x}=2nx$ is fixed. The constants $\widehat{a}_i$ are essentially the product of the lattice spacing a times the square roots of the low energy constants, $\sqrt{W_i}$, [20, 21, 22]. We get the case $\widehat{a}_{6/7}\neq 0$ when we multiply the expression (4.4) with the partition function of N_f fermionic flavors cancelling the Pfaffian in the denominator. Then we have to integrate over Gaussian distributions of $\widehat{m}_6$ and $\widehat{\lambda}_7$. Finally we divide the result by the partition function of N_f fermionic flavors with $\widehat{a}_{6/7}\neq 0$ which is also the two Gaussian integrals over $\widehat{m}_6$ and $\widehat{\lambda}_7$ of the partition function with $\widehat{a}_{6/7}=0$, cf. Ref. [34]. Please notice that we will lose the Pfaffian factorization when going from $\widehat{a}_{6/7}=0$ to $\widehat{a}_{6/7}\neq 0$.

4.2. The non-Hermitian Wilson random matrix ensemble

As in the Hermitian version we define the integral transform of the skew-orthogonal polynomials $q^{(+)}_{\nu+l}$. However we have to distinguish between left and right transformation because $G_2^{(+)}$ is not anti-symmetric anymore,

$$\widetilde{q}^{(1,+)}_{\nu+l}(z) = \int_{\mathbb{C}} d[\widetilde{z}] q^{(+)}_{\nu+l}(\widetilde{z}) G_2^{(+)}(\widetilde{z},z) \tag{4.15}$$

$$= \int_{\mathbb{C}} d[\widetilde{z}] q_{\nu+l}^{(+)}(\widetilde{z}) g_2^{(+)}(\widetilde{z}, z) - \sum_{j=0}^{\nu-1} \frac{(q_{\nu+l}^{(+)} | p_j^{(+)})_{g_2^{(+)}}}{h_j} p_j^{(+)}(z) g_1(z),$$

$$\widetilde{q}_{\nu+l}^{(\mathrm{r},+)}(z) = \int_{\mathbb{C}} d[\widetilde{z}] q_{\nu+l}^{(+)}(\widetilde{z}) G_2^{(+)}(z, \widetilde{z}) \tag{4.16}$$

$$= \int_{\mathbb{C}} d[\widetilde{z}] q_{\nu+l}^{(+)}(\widetilde{z}) g_2^{(+)}(z, \widetilde{z}).$$

Another difference to the Hermitian case is a non-vanishing integral transform of the orthogonal polynomials

$$\widetilde{p}_l(z) = \int_{\mathbb{C}} d[\widetilde{z}] p_l^{(+)}(\widetilde{z}) G_2^{(+)}(\widetilde{z}, z) = \int_{\mathbb{C}} d[\widetilde{z}] p_l^{(+)}(\widetilde{z}) G_2^{(+)}(z, \widetilde{z}) \tag{4.17}$$

for $0 \leq l \leq \nu - 1$ due to Eq. (3.18).

Again we consider an even number of fermionic flavors. Then we arrive at our main result for the non-Hermitian Wilson random matrix which is the $(k_\mathrm{r}, k_\mathrm{l})$-point correlation function (2.45),

$$R_{2n_\mathrm{f},k_\mathrm{r},k_\mathrm{l}}^{(n,\nu,+)}(Z', -m) = \frac{1}{\mathrm{Pf}\,[K_6^{(+,n+n_\mathrm{f})}(m_i, m_j)]_{1\leq i,j\leq 2n_\mathrm{f}}} \tag{4.18}$$

$$\times \mathrm{Pf} \left[\begin{array}{c|c|c} \widehat{K}_1^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{r})}) & \widehat{K}_3^{(+,n+n_\mathrm{f})}(z_j^{(\mathrm{l})}, z_i^{(\mathrm{r})}) & \widehat{K}_4^{(+,n+n_\mathrm{f})}(m_j, z_i^{(\mathrm{r})}) \\ \hline -\widehat{K}_3^{(+,n+n_\mathrm{f})T}(z_i^{(\mathrm{l})}, z_j^{(\mathrm{r})}) & \widehat{K}_2^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{l})}, z_j^{(\mathrm{l})}) & \widehat{K}_5^{(+,n+n_\mathrm{f})}(m_j, z_i^{(\mathrm{l})}) \\ \hline -\widehat{K}_4^{(+,n+n_\mathrm{f})T}(m_i, z_j^{(\mathrm{r})}) & -\widehat{K}_5^{(+,n+n_\mathrm{f})T}(m_i, z_j^{(\mathrm{l})}) & K_6^{(+,n+n_\mathrm{f})}(m_i, m_j) \end{array}\right],$$

with

$$\widehat{K}_1^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{r})}) = \begin{bmatrix} K_1^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{r})}) & -K_3^{(+,n+n_\mathrm{f})}(z_j^{(\mathrm{r})}, z_i^{(\mathrm{r})}) \\ K_3^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{r})}) & K_6^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{r})}) \end{bmatrix}, \tag{4.19}$$

$$\widehat{K}_2^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{l})}, z_j^{(\mathrm{l})}) = \begin{bmatrix} K_4^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{l})}, z_j^{(\mathrm{l})}) & K_5^{(+,n+n_\mathrm{f})}(z_j^{(\mathrm{l})}, z_i^{(\mathrm{l})}) \\ -K_5^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{l})}, z_j^{(\mathrm{l})}) & K_6^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{l})}, z_j^{(\mathrm{l})}) \end{bmatrix}, \tag{4.20}$$

$$\widehat{K}_3^{(+,n+n_\mathrm{f})}(z_j^{(\mathrm{l})}, z_i^{(\mathrm{r})}) = \begin{bmatrix} K_2^{(+,n+n_\mathrm{f})}(z_j^{(\mathrm{l})}, z_i^{(\mathrm{r})}) & -K_3^{(+,n+n_\mathrm{f})}(z_j^{(\mathrm{l})}, z_i^{(\mathrm{r})}) \\ -K_5^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{l})}) & K_6^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{r})}, z_j^{(\mathrm{l})}) \end{bmatrix}, \tag{4.21}$$

$$\widehat{K}_4^{(+,n+n_\mathrm{f})}(m_j, z_i^{(\mathrm{r})}) = \begin{bmatrix} -K_3^{(+,n+n_\mathrm{f})}(m_j, z_i^{(\mathrm{r})}) \\ K_6^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{r})}, m_j) \end{bmatrix}, \tag{4.22}$$

$$\widehat{K}_5^{(+,n+n_\mathrm{f})}(m_j, z_i^{(\mathrm{l})}) = \begin{bmatrix} K_5^{(+,n+n_\mathrm{f})}(m_j, z_i^{(\mathrm{l})}) \\ K_6^{(+,n+n_\mathrm{f})}(z_i^{(\mathrm{l})}, m_j) \end{bmatrix}, \tag{4.23}$$

where the indices i and j take the values $(1, \ldots, k_\mathrm{r}, 1, \ldots, k_\mathrm{l}, 1, \ldots, 2n_\mathrm{f})$ from left to right and top to bottom. The functions are given by

$$K_1^{(+,n+n_\mathrm{f})}(z_1, z_2) = \sum_{l=0}^{n+n_\mathrm{f}-1} \frac{1}{o_l^{(+)}} \det \begin{bmatrix} \widetilde{q}_{\nu+2l+1}^{(\mathrm{r},+)}(z_1) & \widetilde{q}_{\nu+2l}^{(\mathrm{r},+)}(z_1) \\ \widetilde{q}_{\nu+2l+1}^{(\mathrm{r},+)}(z_2) & \widetilde{q}_{\nu+2l}^{(\mathrm{r},+)}(z_2) \end{bmatrix}$$

$$= -\int_{\mathbb{C}^2} d[\widetilde{z}_1]d[\widetilde{z}_2]\Sigma^{(+)}_{n+n_{\rm f}-1}(\widetilde{z}_1,\widetilde{z}_2)g_2^{(+)}(z_1,\widetilde{z}_1)g_2^{(+)}(z_2,\widetilde{z}_2), \tag{4.24}$$

$$K_2^{(+,n+n_{\rm f})}(z_1,z_2) = G_2^{(+)}(z_2,z_1) + \sum_{l=0}^{n+n_{\rm f}-1}\frac{1}{o_l^{(+)}}\det\begin{bmatrix} \widetilde{q}^{(\rm l,+)}_{\nu+2l+1}(z_1) & \widetilde{q}^{(\rm l,+)}_{\nu+2l}(z_1) \\ \widetilde{q}^{(\rm r,+)}_{\nu+2l+1}(z_2) & \widetilde{q}^{(\rm r,+)}_{\nu+2l}(z_2)\end{bmatrix}$$
$$- \sum_{l=0}^{\nu-1}\frac{1}{h_l}p_l^{(+)}(x_1)\widetilde{p}_l(z_2)g_1^{(+)}(x_1)\delta(y_1), \tag{4.25}$$

$$K_3^{(+,n+n_{\rm f})}(z_1,z_2) = \sum_{l=0}^{n+n_{\rm f}-1}\frac{1}{o_l^{(+)}}\det\begin{bmatrix} q^{(+)}_{\nu+2l+1}(z_1) & q^{(+)}_{\nu+2l}(z_1) \\ \widetilde{q}^{(\rm r,+)}_{\nu+2l+1}(z_2) & \widetilde{q}^{(\rm r,+)}_{\nu+2l}(z_2)\end{bmatrix}$$
$$= -\int_{\mathbb{C}} d[\widetilde{z}]\Sigma^{(+)}_{n+n_{\rm f}-1}(z_1,\widetilde{z})g_2^{(+)}(z_2,\widetilde{z}), \tag{4.26}$$

$$K_4^{(+,n+n_{\rm f})}(z_1,z_2) = \sum_{l=0}^{n+n_{\rm f}-1}\frac{1}{o_l^{(+)}}\det\begin{bmatrix} \widetilde{q}^{(\rm l,+)}_{\nu+2l+1}(z_1) & \widetilde{q}^{(\rm l,+)}_{\nu+2l}(z_1) \\ \widetilde{q}^{(\rm l,+)}_{\nu+2l+1}(z_2) & \widetilde{q}^{(\rm l,+)}_{\nu+2l}(z_2)\end{bmatrix}$$
$$+ \sum_{l=0}^{\nu-1}\frac{1}{h_l}\det\begin{bmatrix} \widetilde{p}_l(z_1) & p_l^{(+)}(x_1)g_1^{(+)}(x_1)\delta(y_1) \\ \widetilde{p}_l(z_2) & p_l^{(+)}(x_2)g_1^{(+)}(x_2)\delta(y_2)\end{bmatrix}, \tag{4.27}$$

$$K_5^{(+,n+n_{\rm f})}(z_1,z_2) = \sum_{l=0}^{\nu-1}\frac{1}{h_l}p_l^{(+)}(z_1)p_l^{(+)}(x_2)g_1^{(+)}(x_2)\delta(y_2)$$
$$+ \sum_{l=0}^{n+n_{\rm f}-1}\frac{1}{o_l^{(+)}}\det\begin{bmatrix} q^{(+)}_{\nu+2l+1}(z_1) & q^{(+)}_{\nu+2l}(z_1) \\ \widetilde{q}^{(\rm l,+)}_{\nu+2l+1}(z_2) & \widetilde{q}^{(\rm l,+)}_{\nu+2l}(z_2)\end{bmatrix}, \tag{4.28}$$

$$K_6^{(+,n+n_{\rm f})}(z_1,z_2) = \sum_{l=0}^{n+n_{\rm f}-1}\frac{1}{o_l^{(+)}}\det\begin{bmatrix} q^{(+)}_{\nu+2l+1}(z_1) & q^{(+)}_{\nu+2l}(z_1) \\ q^{(+)}_{\nu+2l+1}(z_2) & q^{(+)}_{\nu+2l}(z_2)\end{bmatrix}$$
$$= -\Sigma^{(+)}_{n+n_{\rm f}-1}(z_1,z_2). \tag{4.29}$$

Note that although some of the sums seem to look identical they slightly differ by the integral transforms which have to be taken.

The result for an odd number of flavors can again be obtained by taking the limit of one mass to infinity. Then one row and one column only depend on the skew-orthogonal polynomial $q^{(+)}_{\nu+2(n+n_{\rm f}-1)}$ and its two integral transforms $\widetilde{q}^{(\rm l,+)}_{\nu+2(n+n_{\rm f}-1)}$ and $\widetilde{q}^{(\rm r,+)}_{\nu+2(n+n_{\rm f}-1)}$.

For $k_{\rm r} = k_{\rm l} = 0$ we find the normalization and in the case $(k_{\rm r}, k_{\rm l}) = (n, n+\nu)$ we have a representation of joint probability density $p_{\rm W}$ as a Pfaffian similar to the one of p_5. Additionally, we can consider the particular cases $(k_{\rm r}, k_{\rm l}) = (n, 0)$ and $(k_{\rm r}, k_{\rm l}) = (0, n+\nu)$ which are the joint probability densities for the eigenvalues $z^{(\rm r)}$ and $z^{(\rm l)}$ separately. These two joint probability densities are the ones for the right handed and the half of the complex modes, namely $z^{(\rm r)}$, and for the left handed and the other half of the complex modes, which is $z^{(\rm l)}$.

Again we recognize a natural splitting of the spectral properties. There are those terms, the sums with Hermite polynomials $p_l^{(+)}$, which describe the broadening of the former zero modes. They are again manifested by the same GUE which we found when

discussing D_5 and are located on the real axis only, notice the Dirac delta functions. Moreover we have the terms for the remaining modes given by the skew-orthogonal polynomials, $q^{(+)}_{\nu+l}$. The corresponding eigenvalues to these modes do not necessarily lie on the real axis. On the contrary most eigenvalues are distributed in the complex plane, see Ref. [19, 24, 34].

There is an interaction between these two kinds of spectra in the integral transform, cf. Eqs. (4.15) and (4.17). This interaction directly follows from the Vandermonde determinant in the joint probability density (2.14). The repulsion obtained by this coupling effects the spectrum located on the real axis as well as the complex one.

As for D_5 all kernels of the result (4.18) can be traced back to two-flavor partition functions. For the kernels $K_1^{(+,n+n_\mathrm{f})}$, Eq. (4.24), $K_3^{(+,n+n_\mathrm{f})}$, Eq. (4.26), and $K_6^{(+,n+n_\mathrm{f})}$, Eq. (4.29), we know already appropriate expressions. In Appendix F we derive the results for the other kernels,

$$\begin{aligned}
\Delta K_2^{(+,n)}(z_1,z_2) &= K_2^{(+,n)}(z_1,z_2) + K_1^{(+,n)}(z_1,z_2) \qquad (4.30)\\
&= g_2^{(+)}(z_2,z_1) + \frac{1}{\pi}\int\limits_{\mathbb{C}} d[\widetilde{z}]\frac{g_2^{(+)}(z_2,\widetilde{z})}{x_1-\widetilde{z}}\\
&\quad\times \operatorname*{Im}_{\varepsilon\to 0}\left\langle \frac{\det(D_\mathrm{W}-\widetilde{z}\mathbb{1}_{2n+\nu})}{\det(D_\mathrm{W}-x_1\mathbb{1}_{2n+\nu}-\imath\varepsilon\gamma_5)}\right\rangle_{n,\nu}\delta(y_1),\\
\Delta K_4^{(+,n)}(z_1,z_2) &= K_4^{(+,n)}(z_1,z_2) - \Delta K_2^{(+,n)}(z_1,z_2) + \Delta K_2^{(+,n)}(z_2,z_1) - K_1^{(+,n)}(z_1,z_2)\\
&= \frac{o_n^{(+)}}{\pi^2}(x_1-x_2)\delta(y_1)\delta(y_2)\operatorname*{Im}_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} \qquad (4.31)\\
&\quad\times\left\langle \frac{1}{\det(D_\mathrm{W}-x_1\mathbb{1}_{2n+\nu+2}-\imath\varepsilon_1\gamma_5)\det(D_\mathrm{W}-x_2\mathbb{1}_{2n+\nu+2}-\imath\varepsilon_2\gamma_5)}\right\rangle_{n+1,\nu},\\
\Delta K_5^{(+,n)}(z_1,z_2) &= K_5^{(+,n)}(z_1,z_2) + K_3^{(+,n)}(z_1,z_2) \qquad (4.32)\\
&= \frac{1}{\pi}\frac{1}{z_1-x_2}\operatorname*{Im}_{\varepsilon\to 0}\left\langle \frac{\det(D_\mathrm{W}-z_1\mathbb{1}_{2n+\nu})}{\det(D_\mathrm{W}-x_2\mathbb{1}_{2n+\nu}-\imath\varepsilon\gamma_5)}\right\rangle_{n,\nu}\delta(y_2).
\end{aligned}$$

The kernel $\Delta K_4^{(+,n+n_\mathrm{f})}$ describes the correlation of the chiral distribution over the real eigenvalues with itself. This can be seen by the γ_5 weight of the imaginary increments in the denominators and the Dirac delta functions, cf. Refs. [22]. The other two kernels (4.30) and (4.32) represent the interaction of the chiral distribution over the real eigenvalues with the remaining spectrum describing the additional real modes and the complex ones.

The quenched one point functions presented in Refs. [22, 19, 24] are given by the kernels $K_3^{(+,n)}(z,z)$ and $K_5^{(+,n)}(z,z)$. The kernel $K_3^{(+,n)}(z,z)$ was denoted by $\rho_r(x)\delta(y)+\rho_c(z)/2$ in Refs. [19, 24] which is the sum of the distribution of the right handed modes and the half of the distribution of the complex eigenvalues. Then the kernel $\Delta K_5^{(+,n+n_\mathrm{f})}(x,x)$ is equal to the chirality distribution over the real eigenvalues $\rho_\chi(x)$, see Refs. [22, 19, 24].

Again the Pfaffian determinant as well as the identification with two-flavor partition

functions of $D_{\rm W}$ carry over to the microscopic limit and, thus, to eigenvalue correlations of the Wilson-Dirac operator in the ϵ-regime. The microscopic limit of the three kernels $K_1^{(+,n)}$, $K_3^{(+,n)}$ and $K_6^{(+,n)}$ are trivial corollaries of Eq. (3.59). The derivation of this limit for the other three kernels will be made elsewhere [68]. Also the discussion of the results will not be done here.

Again we can look what our results mean in the notation of Refs. [20, 21, 22]. In the microscopic limit the following kernels are proportional to the two-flavor partition functions of the chiral Lagrangian

$$\Delta K_4^{(+,\infty)}(z_1,z_2) \propto (\widehat{x}_1-\widehat{x}_2)\operatorname*{Im}_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} Z^\nu_{0/2}(\widehat{m}_6-\widehat{x}_1,\widehat{m}_6-\widehat{x}_2;\widehat{\lambda}_7,\widehat{\lambda}_7;\widehat{a}_8,\widehat{a}_{6/7}=0)\delta(\widehat{y}_1)\delta(\widehat{y}_2), \tag{4.33}$$

$$\Delta K_5^{(+,\infty)}(z_1,z_2) \propto \operatorname*{Im}_{\varepsilon\to 0}\frac{Z^\nu_{1/1}(\widehat{m}_6-\widehat{z}_1,\widehat{m}_6-\widehat{x}_2;\widehat{\lambda}_7,\widehat{\lambda}_7;\widehat{a}_8,\widehat{a}_{6/7}=0)}{\widehat{x}_2-\widehat{z}_1}\delta(\widehat{y}_2), \tag{4.34}$$

$$K_6^{(+,\infty)}(z_1,z_2) \quad \propto (\widehat{z}_1-\widehat{z}_2)Z^\nu_{2/0}(\widehat{m}_6-\widehat{z}_1,\widehat{m}_6-\widehat{z}_2;\widehat{\lambda}_7,\widehat{\lambda}_7;\widehat{a}_8,\widehat{a}_{6/7}=0). \tag{4.35}$$

The other kernels are only integral transforms of these three partition functions. As for D_5 we can create the case $\widehat{a}_{6/7}\neq 0$ by multiplying the expression (4.18) with the partition function of $N_{\rm f}$ fermionic flavors and integrating over Gaussian distributions of $\widehat{m}_6$ and $\widehat{\lambda}_7$. At the end we divide the resulting expression by the partition function with $N_{\rm f}$ fermionic flavor and with $\widehat{a}_{6/7}\neq 0$.

5. Conclusions

We derived the orthogonal and skew-orthogonal polynomials corresponding to the Hermitian as well as the non-Hermitian Wilson random matrix ensemble. The orthogonal polynomials are the Hermite polynomials from order 0 to $\nu-1$ in both cases. They result from the ν-dimensional GUE describing the broadening of the ν generic real modes which are at zero lattice spacing the zero modes. Such a GUE was already discovered in the chirality distribution over the real eigenvalues [22, 19, 24] as well as in the level density of the Hermitian Wilson random matrix ensemble and, thus, the Wilson Dirac operator [20, 21, 32]. Surprisingly this GUE is already the universal result and is a dominant part in the eigenvalue correlations at small lattice spacing since it forms the Dirac delta functions at zero with weight ν in the continuum limit, see Refs. [21, 23].

The remaining spectrum is described by skew-orthogonal polynomials starting from order ν. They describe the remaining spectrum apart from the ν generic real modes. In a unifying way we constructed these polynomials and derived recursion relations which enable us to obtain the odd polynomials by simply acting with a derivative operator on the even ones, cf. Eq. (3.31). This derivative operator can be identified by a creation operator of a harmonic oscillator. Moreover we derived a Christoffel Darboux-like formula (3.37) which is equivalent to the partition function of two fermionic flavors, see Eq. (3.53). The even skew-orthogonal polynomials are equal to one-flavor partition

functions, see Eq. (3.43). With help of this knowledge we were able to derive the Rodrigues formula (3.45) interpolating between the one of the Laguerre polynomials and the one of the Hermite polynomials.

As an application we considered the unquenched k-point correlation functions of the Hermitian and non-Hermitian Wilson random matrix ensemble. We derived a Pfaffian factorization in both cases. The one of the Hermitian matrix was already known before [23] but we traced the entries back to the two-flavor partition functions, see subsection 4.1, which is a better expression for numerical evaluations. The Pfaffian of the non-Hermitian random matrix is a completely new result. We identified its kernels as two-flavor partition functions, too, see subsection 4.2. These partition functions can be readily interpreted as correlations of the complex conjugated pairs, the real eigenvalues corresponding to the right handed modes and the average chirality over the real eigenvalues.

Although the random matrix D_{W} is non-Hermitian we did not need a Hermitization as it was introduced in Ref. [70]. We circumvented this approach by splitting the kernels with bosonic flavors into two kinds of terms. One kind corresponds to the chirality over the real eigenvalues which exhibits no singularities in the bosonic determinants. The other term are integral transforms of partition functions with fermionic flavors instead of bosonic ones. Hence there are no problems of integrability anymore. Especially we have not to double the number of the bosonic dimensions in the superspace when applying the supersymmetry method.

The Pfaffian factorization as well as the identification with two-flavor partition functions carry over to the microscopic limit and, thus, to the spectral properties of the Wilson-Dirac operator in the ϵ-regime [27, 28, 29, 30]. Hence the results shown in Sec. 4 are a good starting point for an analytical study of the Hermitian and non-Hermitian Wilson-Dirac operator. In particular the calculation of the individual eigenvalue distributions will benefit of the structure since a representation as Fredholm Pfaffians are possible, see Ref. [65]. Fredholm determinants and Pfaffians are compact expressions simplifying the perturbative expansion of the gap probability in the k-point correlations function to obtain the individual eigenvalue distributions.

Moreover, the skew-orthogonal polynomials and the Christoffel Darboux-like formula also appearing as kernels of the Pfaffian determinants reduce to a quickly converging sum, see Eq. (3.58), and two-fold integrals over phases, see Eqs. (3.59) and (3.60), respectively.

The Pfaffian of the k-point correlation function will persist in the continuum limit. It is in agreement with Ref. [67] where a non-trivial Pfaffian determinant was derived for $\beta = 2$ random matrix ensembles. A similar but not completely equivalent structure was derived in Refs. [71, 72] for $\beta = 2$ ensembles, too. Hence Pfaffians seem to be more universal than the determinantal structures in the eigenvalues statistics of RMT.

The Pfaffian determinants we found reflect the breaking of the generic pairing of eigenvalues in the continuum limit to no reflection symmetry at all in the Hermitian case and the reflection symmetry at the real axis in the non-Hermitian one. For example

at $a = 0$ and fixed matrix D_5 we can say if λ is an eigenvalue of D_5 then $-\lambda$ is also one. This is not anymore true at finite a.

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Appendix A. De Bruijn-like integration theorems

We generalize the de Bruijn-like integration theorem [73] to an integrand which is a product of one determinant and one Pfaffian, see Appendix A.1, and of two determinants, see Appendix A.2.

Appendix A.1. With a Pfaffian integrand

Let N_1, N_2 and N_3 be three positive integers fulfilling the condition $2N_3, N_2 \geq N_1 > 0$. We consider the following integral

$$I_1 = \int \prod_{j=1}^{N_1} d[z_j] \det \begin{bmatrix} \{A_c(z_b)\}_{\substack{1\leq b\leq N_1 \\ 1\leq c\leq N_2}} \\ \{B_{bc}\}_{\substack{1\leq b\leq N_2-N_1 \\ 1\leq c\leq N_2}} \end{bmatrix} \times \mathrm{Pf} \begin{bmatrix} \{C(z_b,z_c)\}_{1\leq b,c\leq N_1} & \{D_c(z_b)\}_{\substack{1\leq b\leq N_1 \\ 1\leq c\leq 2N_3-N_1}} \\ \{-D_b(z_c)\}_{\substack{1\leq b\leq 2N_3-N_1 \\ 1\leq c\leq N_1}} & \{E_{bc}\}_{1\leq b,c\leq 2N_3-N_1} \end{bmatrix}. \tag{A.1}$$

The matrix B is an arbitrary constant matrix whereas E is an anti-symmetric constant matrix. The matrix valued functions A, C and D are sufficiently integrable and C is anti-symmetric in its entries.

After an expansion of the first determinant in Eq. (A.1) in the entries $A_c(z_b)$ we can integrate over the variables z [58], i.e.

$$I_1 = \frac{1}{(N_2-N_1)!} \sum_{\omega\in\widetilde{S}(N_2)} \operatorname{sign}\omega \det[B_{b\omega(c)}]_{\substack{1\leq b\leq N_2-N_1 \\ N_1+1\leq c\leq N_2}} \tag{A.2}$$

$$\times\mathrm{Pf}\begin{bmatrix} \left\{\int d[z] A_{\omega(b)}(z_1)A_{\omega(c)}(z_2)C(z_1,z_2)\right\}_{1\leq b,c\leq N_1} & \left\{\int d[z] A_{\omega(b)}(z)D_c(z)\right\}_{\substack{1\leq b\leq N_1 \\ 1\leq c\leq 2N_3-N_1}} \\ \left\{-\int d[z] A_{\omega(c)}(z)D_b(z)\right\}_{\substack{1\leq b\leq 2N_3-N_1 \\ 1\leq c\leq N_1}} & \{E_{bc}\}_{1\leq b,c\leq 2N_3-N_1} \end{bmatrix}.$$

The remaining determinant can be combined with the Pfaffian by the sum. Thereby we use the identity

$$\det[B_{b\omega(c)}]_{\substack{1\le b\le N_2-N_1\\ N_1+1\le c\le N_2}} = (-1)^{(N_2-N_1)(N_2-N_1-1)/2} \times \mathrm{Pf}\begin{bmatrix} 0 & \{B_{c\omega(b)}\}_{\substack{N_1+1\le b\le N_2\\ 1\le c\le N_2-N_1}} \\ \{-B_{b\omega(c)}\}_{\substack{1\le b\le N_2-N_1\\ N_1+1\le c\le N_2}} & 0 \end{bmatrix}. \tag{A.3}$$

This yields the result

$$I_1 = (-1)^{(N_2-N_1)(N_1+N_2-1)/2} N_1! \times \mathrm{Pf}\left[\begin{array}{c|c|c} \int d[z]A_b(z_1)A_c(z_2)C(z_1,z_2) & \int d[z]A_b(z)D_c(z) & B_{cb} \\ \hline -\int d[z]A_c(z)D_b(z) & E_{bc} & 0 \\ \hline -B_{bc} & 0 & 0 \end{array}\right]. \tag{A.4}$$

The number of the first set of columns and rows is N_2, the one of the second set is $2N_3 - N_1$ and the one of the third part $N_2 - N_1$. Hence, we take the Pfaffian of a $2(N_2 + N_3 - N_1) \times 2(N_2 + N_3 - N_1)$ anti-symmetric matrix.

Appendix A.2. With a determinantal integrand

Now we study the integral with a determinant instead of a Pfaffian, cf. Eq. (A.1), i.e.

$$I_2 = \int \prod_{j=1}^{N_R} d[z_{jR}] \prod_{j=1}^{N_L} d[z_{jL}] \det\begin{bmatrix} \{A_c(z_{bR})\}_{\substack{1\le b\le N_R\\ 1\le c\le N_1}} \\ \{B_c(z_{bL})\}_{\substack{1\le b\le N_L\\ 1\le c\le N_1}} \\ \{C_{bc}\}_{\substack{1\le b\le N_1-N_R-N_L\\ 1\le c\le N_1}} \end{bmatrix} \times \det\begin{bmatrix} \{D(z_{bR},z_{cL})\}_{\substack{1\le b\le N_R\\ 1\le c\le N_L}} & \{E_c(z_{bR})\}_{\substack{1\le b\le N_R\\ 1\le c\le N_2-N_L}} \\ \{F_b(z_{cL})\}_{\substack{1\le b\le N_2-N_R\\ 1\le c\le N_L}} & \{H_{bc}\}_{\substack{1\le b\le N_2-N_R\\ 1\le c\le N_2-N_L}} \end{bmatrix}. \tag{A.5}$$

The matrices C and H are arbitrary constant matrices and the matrix valued functions A, B, D, E and F are chosen such that the integrals exist. The positive integers N_1, N_2, N_R and N_L have the relations $N_1 \ge N_R + N_L$ and $N_2 \ge N_R, N_L$. Without loss of generality we can assume $N_L \ge N_R \ge 0$.

In the first step we split both matrices F and H into two blocks, i.e.

$$[F_b(z_{cL})]_{\substack{1\le b\le N_2-N_R\\ 1\le c\le N_L}} = \begin{bmatrix} \{F_{b1}(z_{cL})\}_{\substack{1\le b\le N_L-N_R\\ 1\le c\le N_L}} \\ \{F_{b2}(z_{cL})\}_{\substack{1\le b\le N_2-N_L\\ 1\le c\le N_L}} \end{bmatrix}, \tag{A.6}$$

$$[H_{bc}]_{\substack{1\le b\le N_2-N_R\\1\le c\le N_2-N_L}} = \begin{bmatrix} \{H_{bc1}\}_{\substack{1\le b\le N_L-N_R\\1\le c\le N_2-N_L}} \\ \{H_{bc2}\}_{1\le b,c\le N_2-N_L} \end{bmatrix}, \tag{A.7}$$

where we assume that H_2 is invertible. Later on we will relax this restriction since I_2 is a polynomial in the constant matrices C and H. We pull H_2 out of the second determinant and have

$$I_2 = \det H_2 \int \prod_{j=1}^{N_R} d[z_{jR}] \prod_{j=1}^{N_L} d[z_{jL}] \det \begin{bmatrix} \{A_c(z_{bR})\}_{\substack{1\le b\le N_R\\1\le c\le N_1}} \\ \{B_c(z_{bL})\}_{\substack{1\le b\le N_L\\1\le c\le N_1}} \\ \{C_{bc}\}_{\substack{1\le b\le N_1-N_R-N_L\\1\le c\le N_1}} \end{bmatrix} \tag{A.8}$$

$$\times \det \begin{bmatrix} \left\{ D(z_{bR},z_{cL}) - \sum_{1\le i,j\le N_2-N_L} E_i(z_{bR}) \left(H_2^{-1}\right)_{ij} F_{j2}(z_{cL}) \right\}_{\substack{1\le b\le N_R\\1\le c\le N_L}} \\ \left\{ F_{b1}(z_{cL}) - \sum_{1\le i,j\le N_2-N_L} H_{bi1} \left(H_2^{-1}\right)_{ij} F_{j2}(z_{cL}) \right\}_{\substack{1\le b\le N_L-N_R\\1\le c\le N_L}} \end{bmatrix}.$$

After an expansion in both determinants we obtain

$$I_2 = N_L! \det H_2 \sum_{\omega\in\widetilde{S}(N_1)} \operatorname{sign}\omega \prod_{b=1}^{N_R} \int d[z_R]d[z_L] A_{\omega(b)}(z_R) B_{\omega(b+N_R)}(z_L) \tag{A.9}$$

$$\times \left[D(z_R,z_L) - \sum_{1\le i,j\le N_2-N_L} E_i(z_R) \left(H_2^{-1}\right)_{ij} F_{j2}(z_L) \right]$$

$$\times \prod_{b=1}^{N_L-N_R} \int d[z_L] B_{\omega(b+2N_R)}(z_L) \left[F_{b1}(z_L) - \sum_{1\le i,j\le N_2-N_L} H_{bi1} \left(H_2^{-1}\right)_{ij} F_{j2}(z_L) \right]$$

$$\times \prod_{b=1}^{N_1-N_R-N_L} C_{b\omega(b+N_R+N_L)}\,.$$

Notice that the sum over the permutation of the second determinant can be absorbed into the first one which gives $N_L!$.

To shorten the notation we define the following matrices which are integrals over one or two variables

$$O_{bc} = \int d[z_R]d[z_L] \left(A_b(z_R)B_c(z_L) - A_c(z_R)B_b(z_L)\right) D(z_R,z_L)\,, \tag{A.10}$$

$$P_{bc} = \int d[z_R] A_b(z_R) E_c(z_R)\,, \tag{A.11}$$

$$Q_{bc\alpha} = \int d[z_L] F_{b\alpha}(z_L) B_c(z_L) \quad,\ \alpha\in\{1,2\}\,. \tag{A.12}$$

Then the integral (A.9) reads

$$I_2 = N_L! 2^{-N_R} \det H_2 \sum_{\omega\in\widetilde{S}(N_1)} \operatorname{sign}\omega \tag{A.13}$$

$$\times \prod_{b=1}^{N_R}\left[O_{\omega(b)\omega(b+N_R)} - \sum_{1\le i,j\le N_2-N_L}\left(P_{\omega(b)i}Q_{j\omega(b+N_R)2} - P_{\omega(b+N_R)i}Q_{j\omega(b)2}\right)\left(H_2^{-1}\right)_{ij}\right]$$
$$\times \prod_{b=1}^{N_L-N_R}\left[Q_{b\omega(b+2N_R)1} - \sum_{1\le i,j\le N_2-N_L} H_{bi1}\left(H_2^{-1}\right)_{ij} Q_{j\omega(b+2N_R)2}\right] \prod_{b=1}^{N_1-N_R-N_L} C_{b\omega(b+N_R+N_L)}\,.$$

This sum can be represented as a Pfaffian, i.e.

$$I_2 = (-1)^{N_1(N_1-1)/2+N_R(N_R+1)/2} N_L!N_R!\det H_2\ \mathrm{Pf}\left[\begin{array}{cc|c} & & C_{cb} \\ & & 0 \\ \hline -C_{bc} & 0 & 0 \end{array}\right] \tag{A.14}$$

with

$$[R_{bc}] = \begin{pmatrix} O_{bc} & Q_{cb1} \\ -Q_{bc1} & 0 \end{pmatrix} \tag{A.15}$$
$$+ \sum_{1\le i,j\le N_2-N_L} \begin{pmatrix} Q_{ib2} & P_{bi} \\ 0 & -H_{bi1} \end{pmatrix}\begin{pmatrix} 0 & \left(H_2^{-1}\right)_{ji} \\ -\left(H_2^{-1}\right)_{ij} & 0 \end{pmatrix}\begin{pmatrix} Q_{jc2} & 0 \\ P_{cj} & -H_{cj1} \end{pmatrix}.$$

Pushing the determinant of H_2 into the Pfaffian we have the final result

$$I_2 = (-1)^{N_1(N_1-1)/2+N_R(N_R+1)/2+(N_2-N_L)(N_2-N_L+1)/2} N_L!N_R! \tag{A.16}$$
$$\times \mathrm{Pf}\left[\begin{array}{c|c|c|c|c} O_{bc} & Q_{cb1} & C_{cb} & Q_{cb2} & P_{bc} \\ \hline -Q_{bc1} & 0 & 0 & 0 & -H_{bc1} \\ \hline -C_{bc} & 0 & 0 & 0 & 0 \\ \hline -Q_{bc2} & 0 & 0 & 0 & -H_{bc2} \\ \hline -P_{cb} & H_{cb1} & 0 & H_{cb2} & 0 \end{array}\right]$$
$$= (-1)^{N_1(N_1-1)/2+N_R(N_R+1)/2+(N_2-N_L)(N_2-N_L+1)/2} N_L!N_R!$$
$$\times \mathrm{Pf}\left[\begin{array}{c|c|c|c} O_{bc} & Q_{cb} & P_{bc} & C_{cb} \\ \hline -Q_{bc} & 0 & -H_{bc} & 0 \\ \hline -P_{cb} & H_{cb} & 0 & 0 \\ \hline -C_{bc} & 0 & 0 & 0 \end{array}\right]$$

The dimensions of rows and columns are from top to bottom and left to right $(N_1, N_2-N_R, N_2-N_L, N_1-N_R-N_L)$. In Eq. (A.16) we drop the invertibility of the matrix H_2 because I_2 is a polynomial of this matrix.

Appendix B. Derivation of the coefficients in the recursion relation

In Appendix B.1 we show that the recursion relations of the polynomials $q_{\nu+l}^{(\pm)}$ take the form (3.31) and (3.32). The coefficients $\epsilon_l^{(\pm)}$ and $\widetilde{\epsilon}_l^{(\pm)}$ are derived in Appendix B.2.

Appendix B.1. The general form

In the first step we take the scalar product (2.25) of Eq. (3.30) with $p_k^{(\pm)}$, $0 \le k \le \nu-1$. We find

$$\beta_{lk}^{(\pm)} = \frac{1}{h_k}\langle \widetilde{D}^{(\pm)} q_{\nu+l}^{(\pm)} | p_k^{(\pm)}\rangle_{g_1^{(\pm)}} = -\frac{1}{h_k}\langle q_{\nu+l}^{(\pm)} | p_k^{(\pm)\prime}\rangle_{g_1^{(\pm)}} + \frac{\mu_\mathrm{r} \mp \mu_\mathrm{l}}{2h_k}\langle q_{\nu+l}^{(\pm)} | p_k^{(\pm)}\rangle_{g_1^{(\pm)}} = 0, \tag{B.1}$$

Thereby we used the fact that $q^{(\pm)}_{\nu+l}$ and $p^{(\pm)}_k$, $0 \leq k \leq \nu-1$, are orthogonal to each other.

The monic normalization of the polynomials enforces the condition

$$\alpha^{(\pm)}_{l,l+1} = -\frac{n}{a^2}. \tag{B.2}$$

For the other conditions we take the anti-symmetric product of Eq. (3.30) with $q^{(\pm)}_{\nu+l'}$. Let $l = 2i$ and $l' = 2i'$ we have

$$\begin{aligned} \left(\widetilde{D}^{(\pm)} q^{(\pm)}_{\nu+2i} | q^{(\pm)}_{\nu+2i'}\right)_{g_2^{(\pm)}} &= -\left(q^{(\pm)}_{\nu+2i} | \widetilde{D}^{(\pm)} q^{(\pm)}_{\nu+2i'}\right)_{g_2^{(\pm)}} \\ -o^{(\pm)}_{i'} \alpha^{(\pm)}_{2i,2i'+1} \Theta_{i-i'} &= -o^{(\pm)}_{i} \alpha^{(\pm)}_{2i',2i+1} \Theta_{i'-i}, \end{aligned} \tag{B.3}$$

where the integrated Kronecker delta is

$$\Theta_l = \begin{cases} 1, & l \in \mathbb{N}_0, \\ 0, & \text{else.} \end{cases} \tag{B.4}$$

Combining Eqs. (B.2) and (B.3) we find

$$\alpha^{(\pm)}_{2i,2i'+1} = \alpha^{(\pm)}_{2i',2i+1} = -\frac{n}{a^2}\delta_{ii'}. \tag{B.5}$$

With $l = 2i+1$ and $l' = 2i'+1$ we get another relation

$$\begin{aligned} \left(\widetilde{D}^{(\pm)} q^{(\pm)}_{\nu+2i+1} | q^{(\pm)}_{\nu+2i'+1}\right)_{g_2^{(\pm)}} &= -\left(q^{(\pm)}_{\nu+2i+1} | \widetilde{D}^{(\pm)} q^{(\pm)}_{\nu+2i'+1}\right)_{g_2^{(\pm)}} \\ o^{(\pm)}_{i'} \alpha^{(\pm)}_{2i+1,2i'} \Theta_{i-i'+1} &= o^{(\pm)}_{i} \alpha^{(\pm)}_{2i'+1,2i} \Theta_{i'-i+1}, \end{aligned} \tag{B.6}$$

With this we conclude

$$\alpha^{(\pm)}_{2i+1,2i'} = \begin{cases} -\dfrac{n}{a^2}\dfrac{o^{(\pm)}_i}{o^{(\pm)}_{i-1}}, & i' = i-1, \\ \epsilon^{(\pm)}_i, & i' = i, \\ -\dfrac{n}{a^2}, & i' = i+1, \\ 0, & \text{else.} \end{cases} \tag{B.7}$$

The constants $\epsilon^{(\pm)}_i$ cannot be specified by Eq. (B.6).

The last relation which we get by the skew-orthogonality of the polynomials is the one for the choice $l = 2i$ and $l' = 2i'+1$, i.e.

$$\begin{aligned} \left(\widetilde{D}^{(\pm)} q^{(\pm)}_{\nu+2i} | q^{(\pm)}_{\nu+2i'+1}\right)_{g_2^{(\pm)}} &= -\left(q^{(\pm)}_{\nu+2i} | \widetilde{D}^{(\pm)} q^{(\pm)}_{\nu+2i'+1}\right)_{g_2^{(\pm)}} \\ o^{(\pm)}_{i'} \alpha^{(\pm)}_{2i,2i'} \Theta_{i-i'} &= -o^{(\pm)}_{i} \alpha^{(\pm)}_{2i'+1,2i+1} \Theta_{i'-i}, \end{aligned} \tag{B.8}$$

The identity yields

$$\alpha^{(\pm)}_{2i,2i'} = -\alpha^{(\pm)}_{2i+1,2i'+1} = \widetilde{\epsilon}^{(\pm)}_i \delta_{ii'}. \tag{B.9}$$

Again the constants $\widetilde{\epsilon}^{(\pm)}_i$ have to be determined.

Collecting the intermediate results (B.1), (B.5), (B.7) and (B.9) the expansion (3.30) reduces to the results (3.31) and (3.32). The derivation of the constants $\epsilon^{(\pm)}_l$ and $\widetilde{\epsilon}^{(\pm)}_l$ remains.

Appendix B.2. The coefficients $\epsilon_l^{(\pm)}$ and $\widetilde{\epsilon}_l^{(\pm)}$

Considering the anti-symmetric product of Eq. (3.31) with $p^{(\pm)}_{\nu+2l+1}$ we find

$$\begin{aligned}
\left(\widetilde{D}^{(\pm)}q^{(\pm)}_{\nu+2l}|p^{(\pm)}_{\nu+2l+1}\right)_{g_2^{(\pm)}} &= \left(-\frac{n}{a^2}q^{(\pm)}_{\nu+2l+1}+\widetilde{\epsilon}_l^{(\pm)}q^{(\pm)}_{\nu+2l}\middle| p^{(\pm)}_{\nu+2l+1}\right)_{g_2^{(\pm)}}\\
-\left(q^{(\pm)}_{\nu+2l}|\widetilde{D}^{(\pm)}p^{(\pm)}_{\nu+2l+1}\right)_{g_2^{(\pm)}} &= o_l^{(\pm)}\widetilde{\epsilon}_l^{(\pm)}\\
\left(q^{(\pm)}_{\nu+2l}\middle|\frac{n}{a^2}p^{(\pm)}_{\nu+2l+2}-\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2}p^{(\pm)}_{\nu+2l+1}\right)_{g_2^{(\pm)}} &= o_l^{(\pm)}\widetilde{\epsilon}_l^{(\pm)}\\
\frac{n}{a^2}\left(q^{(\pm)}_{\nu+2l}|p^{(\pm)}_{\nu+2l+2}\right)_{g_2^{(\pm)}}-o_l^{(\pm)}\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2} &= o_l^{(\pm)}\widetilde{\epsilon}_l^{(\pm)}.
\end{aligned}\tag{B.10}$$

A similar calculation can be done with the scalar product of Eq. (3.31) with $p^{(\pm)}_{\nu+2l}$

$$\begin{aligned}
\langle\widetilde{D}^{(\pm)}q^{(\pm)}_{\nu+2l}|p^{(\pm)}_{\nu+2l}\rangle_{g_1^{(\pm)}} &= \langle-\frac{n}{a^2}q^{(\pm)}_{\nu+2l+1}+\widetilde{\epsilon}_l^{(\pm)}q^{(\pm)}_{\nu+2l}\middle| p^{(\pm)}_{\nu+2l}\rangle_{g_1^{(\pm)}}\\
-\langle q^{(\pm)}_{\nu+2l}|p^{(\pm)\prime}_{\nu+2l}\rangle_{g_1^{(\pm)}}+\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2}\langle q^{(\pm)}_{\nu+2l}|p^{(\pm)}_{\nu+2l}\rangle_{g_1^{(\pm)}} &= h_{\nu+2l}\widetilde{\epsilon}_l^{(\pm)}\\
-(\nu+2l)\langle q^{(\pm)}_{\nu+2l}|p^{(\pm)}_{\nu+2l-1}\rangle_{g_1^{(\pm)}}+h_{\nu+2l}\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2} &= h_{\nu+2l}\widetilde{\epsilon}_l^{(\pm)}\\
\frac{(\nu+2l)h_{\nu+2l-1}}{o^{(\pm)}_{l-1}}\left(q^{(\pm)}_{\nu+2l-2}|p^{(\pm)}_{\nu+2l}\right)_{g_2^{(\pm)}}+h_{\nu+2l}\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2} &= h_{\nu+2l}\widetilde{\epsilon}_l^{(\pm)}.
\end{aligned}\tag{B.11}$$

In both calculations we employed the definition (3.5), the orthogonality of the polynomials $p_l^{(\pm)}$ and Eqs. (3.19), (3.26), (3.28) and (3.29). The combination of both results yields the recursion relation

$$\frac{\left(q^{(\pm)}_{\nu+2l}|p^{(\pm)}_{\nu+2l+2}\right)_{g_2^{(\pm)}}}{o_l^{(\pm)}}=\frac{a^2}{n}(\mu_{\rm r}\mp\mu_{\rm l})+\frac{\left(q^{(\pm)}_{\nu+2l-2}|p^{(\pm)}_{\nu+2l}\right)_{g_2^{(\pm)}}}{o^{(\pm)}_{l-1}}.\tag{B.12}$$

The starting point of this recursion is $l=0$. Due to the definition (3.5) we know that $q^{(\pm)}_\nu=p^{(\pm)}_\nu$ and $q^{(\pm)}_{\nu+1}=p^{(\pm)}_{\nu+1}$. We conclude

$$\begin{aligned}
\left(q^{(\pm)}_\nu|p^{(\pm)}_{\nu+2}\right)_{g_2^{(\pm)}} &= \left(p^{(\pm)}_\nu|p^{(\pm)}_{\nu+2}\right)_{g_2^{(\pm)}}\\
&= \left(p^{(\pm)}_\nu\middle|\left(-\frac{a^2}{n}\widetilde{D}^{(\pm)}+\frac{a^2(\mu_{\rm r}\mp\mu_{\rm l})}{2n}\right)p^{(\pm)}_{\nu+1}\right)_{g_2^{(\pm)}}\\
&= \left(\left(\frac{a^2}{n}\widetilde{D}^{(\pm)}+\frac{a^2(\mu_{\rm r}\mp\mu_{\rm l})}{2n}\right)p^{(\pm)}_\nu\middle|p^{(\pm)}_{\nu+1}\right)_{g_2^{(\pm)}}\\
&= \left(-p^{(\pm)}_{\nu+1}+\frac{a^2}{n}(\mu_{\rm r}\mp\mu_{\rm l})p^{(\pm)}_\nu|p^{(\pm)}_{\nu+1}\right)_{g_2^{(\pm)}}\\
&= \frac{a^2o^{(\pm)}_0}{n}(\mu_{\rm r}\mp\mu_{\rm l}).
\end{aligned}\tag{B.13}$$

Hence, we can solve the recursion and find

$$\left(q^{(\pm)}_{\nu+2l}|p^{(\pm)}_{\nu+2l+2}\right)_{g_2^{(\pm)}}=\frac{(l+1)a^2o_l^{(\pm)}}{n}(\mu_{\rm r}\mp\mu_{\rm l}),\tag{B.14}$$

$$\widetilde{\epsilon}_l^{(\pm)}=(2l+1)\frac{\mu_{\rm r}\mp\mu_{\rm l}}{2}.\tag{B.15}$$

In a similar way we derive an identity for the constants $\epsilon_l^{(\pm)}$. We take the scalar product of Eq. (3.32) with $p_{\nu+2l}^{(\pm)}$ and obtain

$$\begin{aligned}
\epsilon_l^{(\pm)} &= \frac{1}{h_{\nu+2l}} \left(\langle \widetilde{D}^{(\pm)} q_{\nu+2l+1}^{(\pm)} | p_{\nu+2l}^{(\pm)} \rangle_{g_1^{(\pm)}} + \frac{n}{a^2} \langle q_{\nu+2l+2}^{(\pm)} | p_{\nu+2l}^{(\pm)} \rangle_{g_1^{(\pm)}} \right) \qquad \text{(B.16)} \\
&= \frac{1}{h_{\nu+2l}} \left(-\langle q_{\nu+2l+1}^{(\pm)} | p_{\nu+2l}^{(\pm)\prime} \rangle_{g_1^{(\pm)}} + \frac{\mu_\mathrm{r} \mp \mu_\mathrm{l}}{2} \langle q_{\nu+2l+1}^{(\pm)} | p_{\nu+2l}^{(\pm)} \rangle_{g_1^{(\pm)}} \right. \\
&+ \left. \frac{n}{(\nu+2l+1)a^2} \langle q_{\nu+2l+2}^{(\pm)} | p_{\nu+2l}^{(\pm)\prime} \rangle_{g_1^{(\pm)}} \right) \\
&= \frac{1}{h_{\nu+2l}} \left(-(\nu+2l) \langle q_{\nu+2l+1}^{(\pm)} | p_{\nu+2l-1}^{(\pm)} \rangle_{g_1^{(\pm)}} - \frac{n}{(\nu+2l+1)a^2} \langle \widetilde{D}^{(\pm)} q_{\nu+2l+2}^{(\pm)} | p_{\nu+2l}^{(\pm)} \rangle_{g_1^{(\pm)}} \right. \\
&- \left. \frac{n}{(\nu+2l+1)a^2} \frac{\mu_\mathrm{r} \mp \mu_\mathrm{l}}{2} \langle q_{\nu+2l+2}^{(\pm)} | p_{\nu+2l+1}^{(\pm)} \rangle_{g_1^{(\pm)}} \right) \\
&= \frac{n}{a^2} \left(\frac{\langle q_{\nu+2l+3}^{(\pm)} | p_{\nu+2l+1}^{(\pm)} \rangle_{g_1^{(\pm)}}}{h_{\nu+2l+1}} - \frac{\langle q_{\nu+2l+1}^{(\pm)} | p_{\nu+2l-1}^{(\pm)} \rangle_{g_1^{(\pm)}}}{h_{\nu+2l-1}} \right) \\
&- \frac{(l+1)(\mu_\mathrm{r} \mp \mu_\mathrm{l})}{h_{\nu+2l+1}} \langle q_{\nu+2l+2}^{(\pm)} | p_{\nu+2l+1}^{(\pm)} \rangle_{g_1^{(\pm)}} \\
&= \frac{n}{a^2} \left(\frac{\langle q_{\nu+2l+3}^{(\pm)} | p_{\nu+2l+1}^{(\pm)} \rangle_{g_1^{(\pm)}}}{h_{\nu+2l+1}} - \frac{\langle q_{\nu+2l+1}^{(\pm)} | p_{\nu+2l-1}^{(\pm)} \rangle_{g_1^{(\pm)}}}{h_{\nu+2l-1}} \right) \\
&- \frac{(l+1)(\mu_\mathrm{r} \mp \mu_\mathrm{l})}{o_l^{(\pm)}} \left(q_{\nu+2l}^{(\pm)} | p_{\nu+2l+2}^{(\pm)} \right)_{g_2^{(\pm)}} \\
&= \frac{n}{a^2} \left(\frac{\langle q_{\nu+2l+3}^{(\pm)} | p_{\nu+2l+1}^{(\pm)} \rangle_{g_1^{(\pm)}}}{h_{\nu+2l+1}} - \frac{\langle q_{\nu+2l+1}^{(\pm)} | p_{\nu+2l-1}^{(\pm)} \rangle_{g_1^{(\pm)}}}{h_{\nu+2l-1}} \right) - \frac{(l+1)^2(\mu_\mathrm{r} \mp \mu_\mathrm{l})^2 a^2}{n}
\end{aligned}$$

For a further simplification we need more information, i.e. we have to perform the integral $\langle q_{\nu+2l+1}^{(\pm)} | p_{\nu+2l-1}^{(\pm)} \rangle_{g_1^{(\pm)}}$ for all $l \in \mathbb{N}_0$.

Appendix C. Derivation of the Christoffel Darboux-like formula

Let z_1 and z_2 be restricted to the real axis, i.e. $z_{1/2} = x_{1/2}$. The action of the sum of the two differential operators $\widetilde{D}^{(\pm)}$ with respect to x_1 and x_2 on $\Sigma_{n-1}^{(\pm)}$ is

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{n}{a^2}(x_1+x_2) + (\mu_\mathrm{r} \pm \mu_\mathrm{l}) \right) \Sigma_{n-1}^{(\pm)}(x_1, x_2) \\
&= \sum_{l=0}^{n-1} \frac{1}{o_l^{(\pm)}} \left(\widetilde{\epsilon}_l^{(\pm)} \det \begin{bmatrix} q_{\nu+2l}^{(\pm)}(x_1) & q_{\nu+2l+1}^{(\pm)}(x_1) \\ q_{\nu+2l}^{(\pm)}(x_2) & q_{\nu+2l+1}^{(\pm)}(x_2) \end{bmatrix} + \frac{n}{a^2} \det \begin{bmatrix} q_{\nu+2l+2}^{(\pm)}(x_1) & q_{\nu+2l}^{(\pm)}(x_1) \\ q_{\nu+2l+2}^{(\pm)}(x_2) & q_{\nu+2l}^{(\pm)}(x_2) \end{bmatrix} \right. \\
&+ \left. \widetilde{\epsilon}_l^{(\pm)} \det \begin{bmatrix} q_{\nu+2l+1}^{(\pm)}(x_1) & q_{\nu+2l}^{(\pm)}(x_1) \\ q_{\nu+2l+1}^{(\pm)}(x_2) & q_{\nu+2l}^{(\pm)}(x_2) \end{bmatrix} + \frac{n o_l^{(\pm)}}{a^2 o_{l-1}^{(\pm)}} \det \begin{bmatrix} q_{\nu+2l-2}^{(\pm)}(x_1) & q_{\nu+2l}^{(\pm)}(x_1) \\ q_{\nu+2l-2}^{(\pm)}(x_2) & q_{\nu+2l}^{(\pm)}(x_2) \end{bmatrix} \right) \\
&= -\frac{n}{a^2 o_{n-1}^{(\pm)}} \det \begin{bmatrix} q_{\nu+2n-2}^{(\pm)}(x_1) & q_{\nu+2n}^{(\pm)}(x_1) \\ q_{\nu+2n-2}^{(\pm)}(x_2) & q_{\nu+2n}^{(\pm)}(x_2) \end{bmatrix}. \qquad \text{(C.1)}
\end{aligned}$$

Let $X = (x_1 + x_2)/2$ and $\Delta x = (x_1 - x_2)/2$. Then we rewrite the differential equation to

$$\frac{\partial}{\partial X} \exp\left[-\frac{n}{a^2}X^2 + (\mu_{\rm r} \pm \mu_{\rm l})X\right] \Sigma_{n-1}^{(\pm)}(X + \Delta x, X - \Delta x) \tag{C.2}$$
$$= -\frac{n}{a^2 o_{n-1}^{(\pm)}} \exp\left[-\frac{n}{a^2}X^2 + (\mu_{\rm r} \pm \mu_{\rm l})X\right] \det \begin{bmatrix} q_{\nu+2n-2}^{(\pm)}(X + \Delta x) & q_{\nu+2n}^{(\pm)}(X + \Delta x) \\ q_{\nu+2n-2}^{(\pm)}(X - \Delta x) & q_{\nu+2n}^{(\pm)}(X - \Delta x) \end{bmatrix}.$$

In the next step we integrate this equation from X to ∞ and take into account that the upper boundary vanishes due to the Gaussian. This yields Eq. (3.37) for real entries. The restriction to real z_1 and z_2 can be relaxed since the integrand is absolutely integrable.

Appendix D. Derivation of Eq. (3.44)

We consider Eq. (3.43). The characteristic polynomial in $Z_1^{(l,\nu,\pm)}$ can be raised into the exponent by a Gaussian integral over a complex vector of Grassmann (anti-commuting) variables,

$$\xi = \begin{bmatrix} \xi_{\rm r} \\ \xi_{\rm l} \end{bmatrix} = \begin{bmatrix} \xi_{1{\rm r}} \\ \vdots \\ \xi_{l{\rm r}} \\ \xi_{1{\rm l}} \\ \vdots \\ \xi_{l+\nu,{\rm l}} \end{bmatrix}, \ \xi^\dagger = \begin{bmatrix} \xi_{\rm r}^\dagger, & \xi_{\rm l}^\dagger \end{bmatrix} = \begin{bmatrix} \xi_{1{\rm r}}^*, & \cdots, \xi_{l{\rm r}}^*, \xi_{1{\rm l}}^*, & \cdots, \xi_{l+\nu,{\rm l}}^* \end{bmatrix}. \tag{D.1}$$

The integration is defined by

$$\int \xi_i d\xi_j = \int \xi_i^* d\xi_j^* = \frac{1}{\sqrt{2\pi}} \text{ and } \int d\xi_j = \int d\xi_j^* = 0. \tag{D.2}$$

Moreover we employ the conjugation of the second kind, i.e.

$$(\xi_i^*)^* = -\xi_i \text{ and } (\xi_i \xi_j)^* = \xi_i^* \xi_j^*. \tag{D.3}$$

Good introductions in the standard techniques of supersymmetry can be found in Refs. [49, 74].

We find

$$\begin{aligned} q_{\nu+2l}^{(\pm)}(z) \propto & \int d[H]d[\xi] \exp\left[-\frac{n}{2a^2}(\operatorname{tr} A^2 + \operatorname{tr} B^2) - n \operatorname{tr} WW^\dagger + \operatorname{tr} A(\mu_{\rm r} + \xi_{\rm r}\xi_{\rm r}^\dagger)\right] \\ & \times \exp\left[\operatorname{tr} B(\mu_{\rm l} + \xi_{\rm l}\xi_{\rm l}^\dagger) + \operatorname{tr} W\xi_{\rm l}\xi_{\rm r}^\dagger - \operatorname{tr} W^\dagger \xi_{\rm r}\xi_{\rm l}^\dagger + z(\xi_{\rm r}^\dagger\xi_{\rm r} \pm \xi_{\rm l}^\dagger\xi_{\rm l})\right] \\ \propto & \int d[\xi] \exp\left[\frac{a^2}{2n}(\operatorname{tr}(\mu_{\rm r} + \xi_{\rm r}\xi_{\rm r}^\dagger)^2 + \operatorname{tr}(\mu_{\rm l} + \xi_{\rm l}\xi_{\rm l}^\dagger)^2) - \frac{1}{n}\operatorname{tr}\xi_{\rm r}\xi_{\rm l}^\dagger\xi_{\rm l}\xi_{\rm r}^\dagger\right] \\ & \times \exp\left[z(\xi_{\rm r}^\dagger\xi_{\rm r} \pm \xi_{\rm l}^\dagger\xi_{\rm l})\right] \\ \propto & \int d[\xi] \exp\left[-\frac{a^2}{2n}((\xi_{\rm r}^\dagger\xi_{\rm r})^2 + (\xi_{\rm l}^\dagger\xi_{\rm l})^2) + \frac{1}{n}\xi_{\rm r}^\dagger\xi_{\rm r}\xi_{\rm l}^\dagger\xi_{\rm l}\right] \\ & \times \exp\left[-\left(\frac{a^2\mu_{\rm r}}{n} - z\right)\xi_{\rm r}^\dagger\xi_{\rm r} - \left(\frac{a^2\mu_{\rm l}}{n} \mp z\right)\xi_{\rm l}^\dagger\xi_{\rm l}\right]. \end{aligned} \tag{D.4}$$

With help of the superbosonization formula [53, 54] we express the integral over ξ by a two-fold integral over two phases (3.44).

Appendix E. Derivation of Eq. (3.53)

Equations (2.27) and (2.45) with the parameters $k = k_{\mathrm{r}} = k_{\mathrm{l}} = 0$ and $N_{\mathrm{f}} = 2$ read

$$Z_2^{(n,\nu,\pm)}(-z_1,-z_2) \propto \frac{1}{z_1-z_2} \tag{E.1}$$

$$\times \operatorname{Pf}\left[\begin{array}{c|c|c|c|c} 0 & 0 & h_i\delta_{ij} & p_i^{(\pm)}(z_1) & p_i^{(\pm)}(z_2) \\ \hline 0 & \begin{array}{cc} 0 & o_i^{(\pm)}\delta_{ij} \\ -o_i^{(\pm)}\delta_{ij} & 0 \end{array} & 0 & \begin{array}{c} q_{\nu+2i}^{(\pm)}(z_1) \\ q_{\nu+2i+1}^{(\pm)}(z_1) \end{array} & \begin{array}{c} q_{\nu+2i}^{(\pm)}(z_2) \\ q_{\nu+2i+1}^{(\pm)}(z_2) \end{array} \\ \hline -h_i\delta_{ij} & 0 & 0 & 0 & 0 \\ \hline -p_j^{(\pm)}(z_1) & -q_{\nu+2j}^{(\pm)}(z_1), \quad -q_{\nu+2j+1}^{(\pm)}(z_1) & 0 & 0 & 0 \\ \hline -p_j^{(\pm)}(z_2) & -q_{\nu+2j}^{(\pm)}(z_2), \quad -q_{\nu+2j+1}^{(\pm)}(z_2) & 0 & 0 & 0 \end{array}\right]$$

combined with the derived knowledge in subsection 3.1. This Pfaffian can be expanded in the normalization constants h_j. Then the polynomials $p_l^{(\pm)}$, $0 \le l < \nu - 1$, drop out. Furthermore, we make use of identity (4.3) and get

$$\begin{aligned} & Z_2^{(n,\nu,\pm)}(-z_1,-z_2) \propto \frac{1}{z_1-z_2} \\ & \times \operatorname{Pf}\left[\sum_{j=0}^{n} \frac{1}{o_j^{(\pm)}} \begin{pmatrix} q_{\nu+2j}^{(\pm)}(z_1) & q_{\nu+2j+1}^{(\pm)}(z_1) \\ q_{\nu+2j}^{(\pm)}(z_2) & q_{\nu+2j+1}^{(\pm)}(z_2) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{\nu+2j}^{(\pm)}(z_1) & q_{\nu+2j}^{(\pm)}(z_2) \\ q_{\nu+2j+1}^{(\pm)}(z_1) & q_{\nu+2j+1}^{(\pm)}(z_2) \end{pmatrix}\right] \\ & \propto \frac{1}{z_1-z_2} \sum_{j=0}^{n} \frac{1}{o_j^{(\pm)}} \left(q_{\nu+2j}^{(\pm)}(z_1) q_{\nu+2j+1}^{(\pm)}(z_2) - q_{\nu+2j+1}^{(\pm)}(z_1) q_{\nu+2j}^{(\pm)}(z_2) \right). \end{aligned} \tag{E.2}$$

This result is proportional to the sum $\Sigma_n^{(\pm)}$, cf. (3.35).

Appendix F. Simplification of the kernels

In Appendix F.1 and Appendix F.2 we simplify the kernels of D_5. Derivations of the kernels of D_{W} are given in Appendix F.3, Appendix F.4 and Appendix F.5.

Appendix F.1. The kernel $K_1^{(-,n)}$

With help of Eqs. (2.27-2.30) it can be readily shown that

$$K_1^{(-,n)}(x_1,x_2) = \frac{1}{(2n+\nu)!} \prod_{j=0}^{\nu-1} \frac{1}{h_j} \prod_{j=0}^{n-1} -\frac{1}{o_j^{(-)}} \int\limits_{\mathbb{R}^{2n+\nu}} d[\widetilde{x}] \Delta_{2n+\nu}(\widetilde{x}) \tag{F.1}$$

$$\times \operatorname{Pf}\left[\begin{array}{c|cc|c} g_2^{(-)}(\widetilde{x}_i,\widetilde{x}_j) & g_2^{(-)}(\widetilde{x}_i,x_1) & g_2^{(-)}(\widetilde{x}_i,x_2) & \widetilde{x}_i^{j-1} g_1^{(-)}(\widetilde{x}_i) \\ \hline \begin{array}{c} g_2^{(-)}(x_1,\widetilde{x}_j) \\ g_2^{(-)}(x_2,\widetilde{x}_j) \end{array} & \begin{array}{c} 0 \\ g_2^{(-)}(x_2,x_1) \end{array} & \begin{array}{c} g_2^{(-)}(x_1,x_2) \\ 0 \end{array} & \begin{array}{c} x_1^{j-1} g_1^{(-)}(x_1) \\ x_2^{j-1} g_1^{(-)}(x_2) \end{array} \\ \hline -\widetilde{x}_j^{i-1} g_1^{(-)}(\widetilde{x}_j) & -x_1^{i-1} g_1^{(-)}(x_1) & -x_2^{i-1} g_1^{(-)}(x_2) & 0 \end{array}\right].$$

The indices i and j run from 1 to $2n+\nu$ in the first row and column and 1 to ν in the last ones.

In the next step we extend the Vandermonde determinant by two Dirac delta functions such that the integration is over $2n+\nu+2$ variables,

$$K_1^{(-,n)}(x_1,x_2) = \frac{(-1)^{(2n+\nu)(2n+\nu-1)/2}}{(2n+\nu+2)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}-\frac{1}{o_j^{(-)}}\int\limits_{\mathbb{R}^{2n+\nu+2}} d[\widetilde{x}] \tag{F.2}$$
$$\times\ \det\begin{bmatrix} \widetilde{x}_j^{i-1} \\ \delta(\widetilde{x}_j - x_1) \\ \delta(\widetilde{x}_j - x_2)\end{bmatrix}\mathrm{Pf}\left[\begin{array}{c|c} g_2^{(-)}(\widetilde{x}_i,\widetilde{x}_j) & \widetilde{x}_i^{j-1}g_1^{(-)}(\widetilde{x}_i) \\ \hline -\widetilde{x}_j^{i-1}g_1^{(-)}(\widetilde{x}_j) & 0\end{array}\right].$$

The two Dirac delta functions can be expressed by the imaginary parts of the Cauchy transforms in two variables. Using the identity [58]

$$\det\begin{bmatrix} \widetilde{x}_j^{i-1} \\ \dfrac{1}{\widetilde{x}_j - x_1 - \imath\varepsilon_1} \\ \dfrac{1}{\widetilde{x}_j - x_2 - \imath\varepsilon_2}\end{bmatrix} = (-1)^{(2n+\nu+2)(2n+\nu+1)/2}\frac{(x_1+\imath\varepsilon_1 - x_2 - \imath\varepsilon_2)\Delta_{2n+\nu+2}(\widetilde{x})}{\prod\limits_{j=1}^{2n+\nu+2}(\widetilde{x}_j - x_1 - \imath\varepsilon_1)(\widetilde{x}_j - x_2 - \imath\varepsilon_2)} \tag{F.3}$$

we find the expression

$$K_1^{(-,n)}(x_1,x_2) = \frac{1}{(2n+\nu+2)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}-\frac{1}{o_j^{(-)}}\frac{x_2-x_1}{\pi^2}\operatorname*{Im}_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\int\limits_{\mathbb{R}^{2n+\nu+2}} d[\widetilde{x}] \tag{F.4}$$
$$\times\frac{\Delta_{2n+\nu+2}(\widetilde{x})}{\prod\limits_{j=1}^{2n+\nu+2}(\widetilde{x}_j - x_1 - \imath\varepsilon_1)(\widetilde{x}_j - x_2 - \imath\varepsilon_2)}\mathrm{Pf}\left[\begin{array}{c|c} g_2^{(-)}(\widetilde{x}_i,\widetilde{x}_j) & \widetilde{x}_i^{j-1}g_1^{(-)}(\widetilde{x}_i) \\ \hline -\widetilde{x}_j^{i-1}g_1^{(-)}(\widetilde{x}_j) & 0\end{array}\right].$$

This result is the partition function of D_5 with two bosonic flavors, see Eq. (4.8).

Appendix F.2. The kernel $K_2^{(-,n)}$

Again we start from an identity between the kernel and an integral weighted by the joint probability density p_5, see Eq. (2.6), i.e.

$$K_2^{(-,n)}(x_1,x_2) = \frac{1}{(2n+\nu-1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}-\frac{1}{o_j^{(-)}}\int\limits_{\mathbb{R}^{2n+\nu-1}} d[\widetilde{x}]\Delta_{2n+\nu}(\widetilde{x},x_1) \tag{F.5}$$
$$\times\ \mathrm{Pf}\left[\begin{array}{c|c|c} g_2^{(-)}(\widetilde{x}_i,\widetilde{x}_j) & g_2^{(-)}(\widetilde{x}_i,x_2) & \widetilde{x}_i^{j-1}g_1^{(-)}(\widetilde{x}_i) \\ \hline g_2^{(-)}(x_2,\widetilde{x}_j) & 0 & x_2^{j-1}g_1^{(-)}(x_2) \\ \hline -\widetilde{x}_j^{i-1}g_1^{(-)}(\widetilde{x}_j) & -x_2^{i-1}g_1^{(-)}(x_2) & 0\end{array}\right].$$

Notice that we integrate this time over $2n+\nu-1$ variables. Hence the range of the indices i and j is from 1 to $2n+\nu-1$ in the first row and column and from 1 to ν in the last ones.

The integral is extended to $2n+\nu$ variables by introducing a Dirac delta function,

$$K_1^{(-,n)}(x_1,x_2) = \frac{(-1)^{(2n+\nu-1)(2n+\nu-2)/2}}{(2n+\nu+2)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}-\frac{1}{o_j^{(-)}}\int\limits_{\mathbb{R}^{2n+\nu}} d[\widetilde{x}]\frac{\prod_{j=1}^{2n+\nu}(\widetilde{x}_j-x_1)}{x_2-x_1} \tag{F.6}$$
$$\times\ \det\begin{bmatrix}\widetilde{x}_j^{i-1}\\ \delta(\widetilde{x}_j-x_2)\end{bmatrix}\mathrm{Pf}\left[\begin{array}{c|c} g_2^{(-)}(\widetilde{x}_i,\widetilde{x}_j) & \widetilde{x}_i^{j-1}g_1^{(-)}(\widetilde{x}_i)\\ \hline -\widetilde{x}_j^{i-1}g_1^{(-)}(\widetilde{x}_j) & 0\end{array}\right].$$

We employ again the Cauchy integral as a representation of the Dirac delta function and an equation similar to Eq. (F.3). This yields the result (4.9) which is the partition function of D_5 with one fermionic flavor and one bosonic one.

Appendix F.3. The kernel $K_2^{(+,n)}$

Also for D_{W} the kernels have a representation as an integral over the eigenvalues weighted by the joint probability density p_{W} (2.14),

$$K_2^{(+,n)}(z_1,z_2) = \frac{(-1)^{n(n+1)/2+\nu(\nu+1)/2}}{n!(n+\nu)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]\Delta_{2n+\nu}(\widetilde{z}) \tag{F.7}$$
$$\times\ \det\left[\begin{array}{c|c} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) & g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},z_1)\\ \hline g_2^{(+)}(z_2,\widetilde{z}_j^{(\mathrm{l})}) & g_2^{(+)}(z_2,z_1)\\ \hline -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) & -x_1^{i-1}g_1^{(+)}(x_1)\delta(y_1)\end{array}\right].$$

The index j in the first column takes the values 1 to $n+\nu$ while i goes from 1 to n in the first row and from 1 to ν in the last one.

We expand the determinant in the row with the variable z_2 and have

$$K_2^{(+,n)}(z_1,z_2) = g_2^{(+)}(z_2,z_1) \tag{F.8}$$
$$+\ \frac{(-1)^{n(n+1)/2+(\nu+2)(\nu+1)/2}}{n!(n+\nu-1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]d[\widehat{z}]g_2^{(+)}(z_2,\widehat{z})$$
$$\times\ \Delta_{2n+\nu}(\widetilde{z},\widehat{z})\det\left[\begin{array}{c|c} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) & g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},z_1)\\ \hline -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) & -x_1^{i-1}g_1^{(+)}(x_1)\delta(y_1)\end{array}\right].$$

The integration of the second term is extended by a Dirac delta function. However this distribution can only be symmetrized with respect to the $\widetilde{z}^{(\mathrm{l})}$ integration in contrast to the calculation in Appendix F.1 and Appendix F.2. We add and subtract a Dirac delta function for the integration over $\widetilde{z}^{(\mathrm{r})}$. Collecting these steps we find

$$K_2^{(+,n)}(z_1,z_2) = g_2^{(+)}(z_2,z_1) \tag{F.9}$$
$$+\ \frac{(-1)^{n(n+1)/2+\nu(\nu+1)/2}}{(n-1)!(n+\nu)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]d[\widehat{z}]g_2^{(+)}(z_2,\widehat{z})$$
$$\times\ \Delta_{2n+\nu}(\widetilde{z},\widehat{z})\det\begin{bmatrix} g_2^{(+)}(z_1,\widetilde{z}_j^{(\mathrm{l})})\\ g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})})\\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\end{bmatrix}$$

$$+\frac{(-1)^{n(n-1)/2+\nu}}{n!(n+\nu)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu+1}}d[\widetilde{z}]d[\widehat{z}]\frac{g_2^{(+)}(z_2,\widehat{z})}{z_1-\widehat{z}}\det(\widetilde{z}-\widehat{z}\mathbb{1}_{2n+\nu})$$
$$\times\ \det\left[\begin{array}{c|c}(\widetilde{z}_i^{(\mathrm{r})})^{j-1} & -\delta^{(2)}(\widetilde{z}_i^{(\mathrm{r})}-z_1)\\ \hline (\widetilde{z}_i^{(\mathrm{l})})^{j-1} & \delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_1)\end{array}\right]\det\left[\begin{array}{c} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})})\\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\end{array}\right]$$

with $\delta^{(2)}(z)=\delta(x)\delta(y)$.

The minus sign in front of $\delta^{(2)}(\widetilde{z}_i^{(\mathrm{r})}-z_1)$ is needful to construct the chirality distribution over the real eigenvalues. Thereby we need the following relation [22] between the real eigenvalues of $D_{\mathrm{W}}+m\mathbb{1}_{2n+\nu}$, $\lambda_i^{(\mathrm{W})}+m$, and the eigenvalues of $D_5+m\gamma_5$, $\lambda_i^{(5)}(m)$,

$$\left.\frac{\partial\lambda_i^{(5)}}{\partial m}\right|_{\lambda_i^{(5)}=\lambda_i^{(\mathrm{W})}+m=0}=\langle\psi_i|\gamma_5|\psi_i\rangle, \tag{F.10}$$

$$\lambda_i^{(5)}(m)=\langle\psi_i|\gamma_5|\psi_i\rangle(\lambda_i^{(\mathrm{W})}+m)+o(\lambda_i^{(\mathrm{W})}+m), \tag{F.11}$$

where ψ_i is the eigenvector to the eigenvalue $\lambda_i^{(\mathrm{W})}$ of D_{W}. The right hand side of Eq. (F.10) is the chirality of the corresponding eigenvector. Since the eigenvectors of the complex eigenvalues have vanishing chirality, Eq. (F.10) is only applicable to the real modes of D_{W}. The following short calculation will show the connection between the chiral distribution over the real eigenvalues and the third term in Eq. (F.9),

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\lim_{\varepsilon\to0}\frac{\delta(y)}{\det(D_{\mathrm{W}}-x\mathbb{1}_{2n+\nu}-\imath\varepsilon\gamma_5)}&=\frac{1}{\pi}\mathrm{Im}\lim_{\varepsilon\to0}\frac{\delta(y)}{\prod_{j=1}^{2n+\nu}(\lambda_j^{(5)}(-x)-\imath\varepsilon)}\\ &=\frac{\delta(y)}{\prod_{j=1}^{2n+\nu}\lambda_j^{(5)}(-x)}\sum_{j=1}^{2n+\nu}\lambda_j^{(5)}(-x)\delta(\lambda_j^{(5)}(-x))\\ &=\sum_{\lambda_j^{(\mathrm{W})}\text{ is real}}\frac{\lambda_j^{(\mathrm{W})}-x}{\prod_{i=1}^{2n+\nu}(\lambda_i^{(\mathrm{W})}-x)}\delta(\lambda_j^{(\mathrm{W})}-x)\delta(y)\mathrm{sign}\langle\psi_j|\gamma_5|\psi_j\rangle\\ &=\sum_{\widetilde{z}_j^{(\mathrm{r})}\text{ is real}}\frac{\widetilde{z}_j^{(\mathrm{r})}-z}{\det(D_{\mathrm{W}}-z\mathbb{1}_{2n+\nu})}\delta^{(2)}(\widetilde{z}_j^{(\mathrm{r})}-z)\\ &\quad-\sum_{\widetilde{z}_j^{(\mathrm{l})}\text{ is real}}\frac{\widetilde{z}_j^{(\mathrm{l})}-z}{\det(D_{\mathrm{W}}-z\mathbb{1}_{2n+\nu})}\delta^{(2)}(\widetilde{z}_j^{(\mathrm{l})}-z)\end{aligned} \tag{F.12}$$

Hereby we have to understand the whole calculation, in particular the limit of the imaginary increment $\imath\varepsilon$, in a weak sense. The complex conjugated pairs of the integration variables in the third term of Eq. (F.9) do not contribute. We recognize this by expanding the second determinant in the two-point weights and the first determinant

in the Dirac delta distribution such that we consider the integral

$$I = \int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]\,\det(\widetilde{z}-\widehat{z}\mathbb{1}_{2n+\nu})\prod_{j=1}^{n} g_2^{(+)}(\widetilde{z}_j^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})})\prod_{j=n+1}^{n+\nu} g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\Delta_\nu(\widetilde{z}_{>n}^{(\mathrm{l})}) \tag{F.13}$$
$$\times\ \det\left[\begin{array}{c|c} (\widetilde{z}_1^{(\mathrm{r})})^{j-1} & -\delta^{(2)}(\widetilde{z}_1^{(\mathrm{r})}-z_1) \\ \hline (\widetilde{z}_i^{(\mathrm{r})})^{j-1} & 0 \\ \hline (\widetilde{z}_1^{(\mathrm{l})})^{j-1} & \delta^{(2)}(\widetilde{z}_1^{(\mathrm{l})}-z_1) \\ \hline (\widetilde{z}_i^{(\mathrm{l})})^{j-1} & 0 \end{array}\right]$$

If $\widetilde{z}_1^{(\mathrm{l})} = \widetilde{z}_1^{(\mathrm{r})\,*} = z$ the integrand is anti-symmetric under the complex conjugation of z, i.e. $z \leftrightarrow z^*$. The determinant is symmetric under $z \leftrightarrow z^*$ while $g_2^{(+)}(z,z^*) \sim g_{\mathrm{c}}(z) = -g_{\mathrm{c}}(z^*)$, see Eq. (2.17). Thus the integral over the imaginary part of z vanishes. The same discussion can be made for all complex conjugated pairs.

Expanding the determinant in the Dirac delta function and using the calculation (F.12) we find

$$K_2^{(+,n)}(z_1,z_2) = g_2^{(+)}(z_2,z_1) - \frac{(-1)^{n(n-1)/2+\nu(\nu-1)/2}}{(n-1)!(n+\nu-1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}} \tag{F.14}$$
$$\times \int\limits_{\mathbb{C}^2} d[\widehat{z}]\,g_2^{(+)}(z_1,\widehat{z}_1)g_2^{(+)}(z_2,\widehat{z}_2)(\widehat{z}_1-\widehat{z}_2)\int\limits_{\mathbb{C}^{2n+\nu-2}} d[\widetilde{z}]\Delta_{2n+\nu-2}(\widetilde{z})$$
$$\times\ \det(\widetilde{z}-\widehat{z}_1\mathbb{1}_{2n+\nu-2})\det(\widetilde{z}-\widehat{z}_2\mathbb{1}_{2n+\nu-2})\det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\end{bmatrix}$$
$$+\ \frac{(-1)^{n(n+1)/2+\nu(\nu-1)/2}}{n!(n+\nu)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\frac{\delta(y_1)}{\pi}\int\limits_{\mathbb{C}} d[\widehat{z}]\frac{g_2^{(+)}(z_2,\widehat{z})}{x_1-\widehat{z}}\operatorname*{Im}_{\varepsilon\to 0}\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]$$
$$\times\ \Delta_{2n+\nu}(\widetilde{z})\frac{\det(D_{\mathrm{W}}-\widehat{z}\mathbb{1}_{2n+\nu})}{\det(D_{\mathrm{W}}-x_1\mathbb{1}_{2n+\nu}-\imath\varepsilon\gamma_5)}\det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\end{bmatrix}$$

The second term is an integral transform of the partition of D_{W} with two fermionic flavors and the last term is an integral over the partition function with one bosonic and one fermionic flavor. Notice that the integral over $\widehat{z}$ does not commute with the limit $\varepsilon \to 0$ because of the singularity at x_1. This singularity cancels with a term after we take the limit. Hence, the expression (F.14) is equal to the result (4.30).

Appendix F.4. The kernel $K_4^{(+,n)}$

The starting point for this kernel is the identity

$$K_4^{(+,n)}(z_1,z_2) = \frac{(-1)^{n(n-1)/2+\nu(\nu-1)/2}}{(n+1)!(n+\nu-1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]\Delta_{2n+\nu}(\widetilde{z}) \tag{F.15}$$
$$\times\det\left[\begin{array}{c|cc} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) & g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},z_1) & g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},z_2) \\ \hline -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) & -x_1^{i-1}g_1^{(+)}(x_1)\delta(y_1) & -x_2^{i-1}g_1^{(+)}(x_2)\delta(y_2)\end{array}\right].$$

The index j runs from 1 to $n+\nu-1$ and the index i takes the values 1 to $n+1$ in the upper row and from 1 to ν in the lower one. We introduce two Dirac delta functions and, thus, extend the integral by two additional $\widetilde{z}_j^{(\mathrm{l})}$ variables,

$$K_4^{(+,n)}(z_1,z_2) = \frac{(-1)^{n(n+1)/2}}{(n+1)!(n+\nu+1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu+2}} d[\widetilde{z}] \tag{F.16}$$

$$\times\ \det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) \end{bmatrix}\det\left[\begin{array}{c|cc} (\widetilde{z}_i^{(\mathrm{r})})^{j-1} & 0 & 0 \\ \hline (\widetilde{z}_i^{(\mathrm{l})})^{j-1} & \delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_1) & \delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_2)\end{array}\right].$$

We extend the determinant by Dirac delta functions of $\widetilde{z}_i^{(\mathrm{r})}$ similar to the calculation in Appendix F.3,

$$K_4^{(+,n)}(z_1,z_2) = \frac{(-1)^{n(n+1)/2}}{(n+1)!(n+\nu+1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu+2}} d[\widetilde{z}] \tag{F.17}$$

$$\times\ \det\left[\begin{array}{c|cc} (\widetilde{z}_i^{(\mathrm{r})})^{j-1} & -\delta^{(2)}(\widetilde{z}_i^{(\mathrm{r})}-z_1) & -\delta^{(2)}(\widetilde{z}_i^{(\mathrm{r})}-z_2) \\ \hline (\widetilde{z}_i^{(\mathrm{l})})^{j-1} & \delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_1) & \delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_2)\end{array}\right]$$

$$\times\ \det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) \end{bmatrix} - \frac{(-1)^{n(n+1)/2+\nu}}{n!(n+\nu+1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}$$

$$\times\int\limits_{\mathbb{C}^{2n+\nu+1}} d[\widetilde{z}]\det\begin{bmatrix} g_2^{(+)}(z_2,\widetilde{z}_j^{(\mathrm{l})}) \\ g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) \end{bmatrix}\det\left[\begin{array}{c|c} (\widetilde{z}_i^{(\mathrm{r})})^{j-1} & -\delta^{(2)}(\widetilde{z}_i^{(\mathrm{r})}-z_1) \\ \hline (\widetilde{z}_i^{(\mathrm{l})})^{j-1} & \delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_1)\end{array}\right]$$

$$+\frac{(-1)^{n(n+1)/2+\nu}}{n!(n+\nu+1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}$$

$$\times\int\limits_{\mathbb{C}^{2n+\nu+1}} d[\widetilde{z}]\det\begin{bmatrix} g_2^{(+)}(z_1,\widetilde{z}_j^{(\mathrm{l})}) \\ g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) \end{bmatrix}\det\left[\begin{array}{c|c} (\widetilde{z}_i^{(\mathrm{r})})^{j-1} & -\delta^{(2)}(\widetilde{z}_i^{(\mathrm{r})}-z_2) \\ \hline (\widetilde{z}_i^{(\mathrm{l})})^{j-1} & \delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_2)\end{array}\right]$$

$$+\frac{(-1)^{n(n-1)/2+\nu(\nu-1)/2}}{(n-1)!(n+\nu+1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]\Delta_{2n+\nu}(\widetilde{z})\det\begin{bmatrix} g_2^{(+)}(z_1,\widetilde{z}_j^{(\mathrm{l})}) \\ g_2^{(+)}(z_2,\widetilde{z}_j^{(\mathrm{l})}) \\ g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) \end{bmatrix}.$$

In the final step we expand the last three terms in $g_2^{(+)}(z_j,\widetilde{z}_j^{(\mathrm{l})})$. The Dirac delta functions can be rewritten as limits of Cauchy transforms, see Eq. (F.12),

$$K_4^{(+,n)}(z_1,z_2) = \frac{(-1)^{n(n-1)/2+\nu(\nu-1)/2}}{(n+1)!(n+\nu+1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\frac{x_2-x_1}{\pi^2}\delta(y_1)\delta(y_2)\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} \tag{F.18}$$

$$\times\int\limits_{\mathbb{C}^{2n+\nu+2}} d[\widetilde{z}]\Delta_{2n+\nu+2}(\widetilde{z})\frac{\det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) \end{bmatrix}}{\det(D_{\mathrm{W}}-x_1\mathbb{1}_{2n+\nu+2}-\imath\varepsilon_1\gamma_5)\det(D_{\mathrm{W}}-x_2\mathbb{1}_{2n+\nu+2}-\imath\varepsilon_2\gamma_5)}$$

$$
\begin{aligned}
&- g_2(z_2,z_1) - \frac{(-1)^{n(n+1)/2+\nu(\nu-1)/2}}{n!(n+\nu)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\frac{\delta(y_1)}{\pi}\int\limits_{\mathbb{C}} d[\widehat{z}]\frac{g_2^{(+)}(z_2,\widehat{z})}{x_1-\widehat{z}}\operatorname*{Im}_{\varepsilon\to 0}\\
&\times\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]\Delta_{2n+\nu}(\widetilde{z})\frac{\det(D_{\mathrm{W}}-\widehat{z}\mathbb{1}_{2n+\nu})}{\det(D_{\mathrm{W}}-x_1\mathbb{1}_{2n+\nu}-\imath\varepsilon\gamma_5)}\det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\end{bmatrix}\\
&+ g_2(z_1,z_2) + \frac{(-1)^{n(n+1)/2+\nu(\nu-1)/2}}{n!(n+\nu)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\frac{\delta(y_2)}{\pi}\int\limits_{\mathbb{C}} d[\widehat{z}]\frac{g_2^{(+)}(z_1,\widehat{z})}{x_2-\widehat{z}}\operatorname*{Im}_{\varepsilon\to 0}\\
&\times\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]\Delta_{2n+\nu}(\widetilde{z})\frac{\det(D_{\mathrm{W}}-\widehat{z}\mathbb{1}_{2n+\nu})}{\det(D_{\mathrm{W}}-x_1\mathbb{1}_{2n+\nu}-\imath\varepsilon\gamma_5)}\det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\end{bmatrix}\\
&+\int\limits_{\mathbb{C}^2} d[\widehat{z}]g_2^{(+)}(z_1,\widehat{z}_1)g_2^{(+)}(z_2,\widehat{z}_2)(\widehat{z}_1-\widehat{z}_2)\Sigma_{n-1}^{(+)}(\widehat{z}_1,\widehat{z}_2).
\end{aligned}
$$

Only the first term is new in comparison to the kernel $K_2^{(+,n)}$. It is the partition function of D_{W} with two bosonic flavors which agrees with the result (4.31).

Appendix F.5. The kernel $K_5^{(+,n)}$

Also for this kernel we start with

$$
\begin{aligned}
K_5^{(+,n)}(z_1,z_2) &= \frac{(-1)^{n(n+1)/2+\nu(\nu-1)/2}}{n!(n+\nu-1)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu-1}} d[\widetilde{z}]\Delta_{2n+\nu}(\widetilde{z},z_1) \qquad \text{(F.19)}\\
&\times\ \det\left[\begin{array}{c|c} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) & g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},z_2) \\ \hline -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})}) & -x_2^{i-1}g_1^{(+)}(x_2)\delta(y_2)\end{array}\right],
\end{aligned}
$$

where the indices of the determinant are $j\in\{1,\dots,n+\nu-1\}$, $i\in\{1,\dots,n\}$ in the first row and $i\in\{1,\dots,\nu\}$ in the last one. The extension with a Dirac delta function yields

$$
\begin{aligned}
K_5^{(+,n)}(z_1,z_2) &= \frac{(-1)^{n(n-1)/2+\nu}}{n!(n+\nu)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}]\frac{\det(\widetilde{z}-z_1\mathbb{1}_{2n+\nu})}{z_1-z_2} \qquad \text{(F.20)}\\
&\times\ \det\left[\begin{array}{c|c} (\widetilde{z}_i^{(\mathrm{r})})^{j-1} & 0 \\ \hline (\widetilde{z}_i^{(\mathrm{l})})^{j-1} & \delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_2)\end{array}\right]\det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\end{bmatrix}.
\end{aligned}
$$

We proceed in the same way as in Appendix F.3 by extending the first determinant by $-\delta^{(2)}(\widetilde{z}_i^{(\mathrm{l})}-z_2)$ and expanding the resulting correction in $g_2^{(+)}(z_2,\widetilde{z}_i^{(\mathrm{l})})$. Then we find

$$
\begin{aligned}
K_5^{(+,n)}(z_1,z_2) &= \frac{(-1)^{n(n+1)/2+\nu(\nu-1)/2}}{n!(n+\nu)!}\prod_{j=0}^{\nu-1}\frac{1}{h_j}\prod_{j=0}^{n-1}\frac{1}{o_j^{(+)}}\frac{1}{\pi}\frac{\delta(y_2)}{z_1-x_2}\operatorname*{Im}_{\varepsilon\to 0}\int\limits_{\mathbb{C}^{2n+\nu}} d[\widetilde{z}] \qquad \text{(F.21)}\\
&\times\ \Delta_{2n+\nu}(\widetilde{z})\frac{\det(D_{\mathrm{W}}-z_1\mathbb{1}_{2n+\nu})}{\det(D_{\mathrm{W}}-x_2\mathbb{1}_{2n+\nu}-\imath\varepsilon\gamma_5)}\det\begin{bmatrix} g_2^{(+)}(\widetilde{z}_i^{(\mathrm{r})},\widetilde{z}_j^{(\mathrm{l})}) \\ -(\widetilde{x}_j^{(\mathrm{l})})^{i-1}g_1^{(+)}(\widetilde{x}_j^{(\mathrm{l})})\delta(\widetilde{y}_j^{(\mathrm{l})})\end{bmatrix}\\
&+\int\limits_{\mathbb{C}} d[\widehat{z}]g_2^{(+)}(z_2,\widehat{z})(\widehat{z}-z_1)\Sigma_{n-1}^{(+)}(\widehat{z},z_1).
\end{aligned}
$$

The first term is the partition function of D_{W} with one bosonic and one fermionic flavor and the second term is the kernel $K_3^{(+,n)}$. Therefore Eq. (F.21) is the result (4.32).

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Spectral Properties of the Wilson Dirac Operator and random matrix theory

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Random Matrix Theory has been successfully applied to lattice Quantum Chromodynamics. In particular, a great deal of progress has been made on the understanding, numerically as well as analytically, of the spectral properties of the Wilson Dirac operator. In this paper, we study the infra-red spectrum of the Wilson Dirac operator via Random Matrix Theory including the three leading order a^2 correction terms that appear in the corresponding chiral Lagrangian. A derivation of the joint probability density of the eigenvalues is presented. This result is used to calculate the density of the complex eigenvalues, the density of the real eigenvalues and the distribution of the chiralities over the real eigenvalues. A detailed discussion of these quantities shows how each low energy constant affects the spectrum. Especially we consider the limit of small and large (which is almost the mean field limit) lattice spacing. Comparisons with Monte Carlo simulations of the Random Matrix Theory show a perfect agreement with the analytical predictions. Furthermore we present some quantities which can be easily used for comparison of lattice data and the analytical results.

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I. INTRODUCTION

The drastically increasing computational power as well as algorithmic improvements over the last decades provide us with deep insights in non-perturbative effects of Quantum Chromodynamics (QCD). However, the artefacts of the discretization, i.e. a finite lattice spacing, are not yet completely under control. In particular, in the past few years a large numerical [1–7] and analytical [8–14] effort was undertaken to determine the low energy constants of the terms in the chiral Lagrangian that describe the discretization errors. It is well known that new phase structures arise such as the Aoki phase [15] and the Sharpe-Singleton scenario [16]. A direct analytical understanding of lattice QCD seems to be out of reach. Fortunately, as was already realized two decades ago, the low lying spectrum of the continuum QCD Dirac operator can be described in terms of Random Matrix Theories (RMTs) [17, 18].

Recently, RMTs were formulated to describe discretization effects for staggered [19] as well as Wilson [9, 10] fermions. Although these RMTs are more complicated than the chiral Random Matrix Theory formulated in [17, 18], in the case of Wilson fermions a complete analytical solution of the RMT has been achieved [9–11, 13, 20–23]. Since the Wilson RMT shares the global symmetries of the Wilson Dirac operator it will be equivalent to the corresponding (partially quenched) chiral Lagrangian in the microscopic domain (also known as the ϵ-domain) [24–29].

Quite recently, there has been a breakthrough in deriving eigenvalue statistics of the infra-red spectrum of the Hermitian [20] as well as the non-Hermitian [21–23] Wilson Dirac operator. These results explain [13] why the Sharpe-Singleton scenario is only observed for the case of dynamical fermions [1, 5, 30–36] and not in the quenched theory [37, 38] while the Aoki phase has been seen in both cases. First comparisons of the analytical predictions with lattice data show a promising agreement [4, 6, 7]. Good fits of the low energy constants are expected for the distributions of individual eigenvalues [9, 10, 39].

Up to now, mostly the effects of W_8 [10, 20, 23], and quite recently also of W_6 [12, 13, 40], on the Dirac spectrum were studied in detail. In this article, we will discuss the effect of all three low energy constants. Thereby we start from the Wilson RMT for the non-Hermitian Wilson Dirac operator proposed in Refs. [9]. In Sec. II we recall this Random Matrix Theory and its properties. Furthermore we derive the joint probability density of the eigenvalues which so far was only stated without proof in Refs. [13, 22]. We also discuss the approach to the continuum limit in terms of the Dirac spectrum.

In Sec. III, we derive the level densities of $D_{\rm W}$ starting from the joint probability density. Note that due to its γ_5-Hermiticity $D_{\rm W}$ has complex eigenvalues as well as exactly real eigenvalues. Moreover, the real modes split into those corresponding to eigenvectors with positive and negative chirality. In Sec. IV, we discuss the spectrum of the quenched non-Hermitian Wilson Dirac operator in the microscopic limit in detail. In particular the asymptotics at small and large lattice spacing is studied. The latter limit is equal to a mean field limit for some quantities which can be trivially read off.

In Sec. V we summarize our results. In particular we present easily measurable quantities which can be used for fitting the three low energy constants $W_{6/7/8}$ and the chiral condensate Σ. Detailed derivations are given in several appendices. The joint probability density is derived in Appendix A. Some useful integral identities are given in Appendix B and in Appendix C we perform the microscopic limit of the graded partition function that enters in the distribution of the chiralities over the real eigenvalues of $D_{\rm W}$. Finally, some asymptotic results are derived in Appendix D.

II. WILSON RANDOM MATRIX THEORY AND ITS JOINT PROBABILITY DENSITY

In Sec. II A we introduce the Random Matrix Theory for the infra-red spectrum of the Wilson Dirac operator and recall its most important properties. Its joint probability density is given in Sec. II B, and the continuum limit is derived in Sec. II C.

A. The random matrix ensemble

We consider the random matrix ensemble [9, 10]

$$D_{\rm W} = \begin{pmatrix} A & W \\ -W^\dagger & B \end{pmatrix} \tag{1}$$

distributed by the probability density

$$P(D_{\rm W}) = \left(\frac{n}{2\pi a^2}\right)^{[n^2+(n+\nu)^2]/2} \left(-\frac{n}{2\pi}\right)^{n(n+\nu)} \exp\left[-\frac{a^2}{2}\left(\mu_{\rm r}^2 + \frac{n+\nu}{n}\mu_{\rm l}^2\right)\right]$$
$$\times \exp\left[-\frac{n}{2a^2}(\mathrm{tr}\, A^2 + \mathrm{tr}\, B^2) - n\mathrm{tr}\, WW^\dagger + \mu_{\rm r}\,\mathrm{tr}\, A + \mu_{\rm l}\,\mathrm{tr}\, B\right]. \quad (2)$$

The Hermitian matrices A and B break chiral symmetry and their dimensions are $n \times n$ and $(n+\nu) \times (n+\nu)$, respectively, where ν is the index of the Dirac operator. Both $\mu_{\rm r}$ and $\mu_{\rm l}$ are one dimensional real variables. The chiral RMT describing continuum QCD [17] is given by the ensemble (1) with A and B replaced by zero. The $N_{\rm f}$ flavor RMT partition function is defined by

$$Z^\nu_{N_{\rm f}}(m) = \int D[D_{\rm W}]P(D_{\rm W})\mathrm{det}^{N_{\rm f}}(D_{\rm W} + m). \quad (3)$$

Without loss of generality we can assume $\nu \geq 0$ since the results are symmetric under $\nu \to -\nu$ together with $\mu_{\rm r} \leftrightarrow \mu_{\rm l}$.

The Gaussian integrals over the two variables $\mu_{\rm r}$ and $\mu_{\rm l}$ yield the two low energy constants W_6 and W_7 [9, 10]. The reason is that the integrated probability density

$$P(D_{\rm W}, W_{6/7} \neq 0) = \int_{-\infty}^{\infty} P(D_{\rm W}) \exp\left[-\frac{a^2(\mu_{\rm r}+\mu_{\rm l})^2}{16V|W_6|} - \frac{a^2(\mu_{\rm r}-\mu_{\rm l})^2}{16V|W_7|}\right] \frac{a^2 d\mu_{\rm r}\, d\mu_{\rm l}}{8\pi V\sqrt{W_6 W_7}}$$

generates the terms $(\mathrm{tr}\, A + \mathrm{tr}\, B)^2$ and $(\mathrm{tr}\, A - \mathrm{tr}\, B)^2$ which correspond to the squares of traces in the chiral Lagrangian [24–27]. In the microscopic domain the corresponding partition function for $N_{\rm f}$ fermionic flavors is then given by

$$Z^\nu_{N_{\rm f}}(\widetilde{m}) = \int\limits_{\mathrm{U}(N_{\rm f})} d\mu(U) \exp\left[\frac{\Sigma V}{2}\mathrm{tr}\,\widetilde{m}(U + U^{-1}) - \widetilde{a}^2 V W_6 \mathrm{tr}^{\,2}(U + U^{-1})\right]$$
$$\times \exp\left[-\widetilde{a}^2 V W_7 \mathrm{tr}^{\,2}(U - U^{-1}) - \widetilde{a}^2 V W_8 \mathrm{tr}\,(U^2 + U^{-2})\right] \det{}^\nu U \quad (4)$$

with the physical quark masses $\widetilde{m} = \mathrm{diag}\,(\widetilde{m}_1, \ldots, \widetilde{m}_{N_{\rm f}})$, the space-time volume V, the physical lattice spacing $\widetilde{a}$ and the chiral condensate Σ. The low energy constant W_8 is generated by the term $\mathrm{tr}\, A^2 + \mathrm{tr}\, B^2$ in Eq. (2) and is *a priori* positive. We include the lattice spacing a in the standard deviation of A and B, cf. Eq. (2), out of convenience for deriving the joint probability density. We employ the sign convention of Refs. [9, 10] for the low energy constants.

The microscopic limit ($n \to \infty$) is performed in Sec. III. In this limit the rescaled lattice spacing $\widehat{a}_8^2 = na^2/2 = \widetilde{a}^2 V W_8$, the rescaled parameters $\widehat{m}_6 = a^2(\mu_{\rm r} + \mu_{\rm l})$ and $\widehat{\lambda}_7 = a^2(\mu_{\rm r} - \mu_{\rm l})$, and the rescaled eigenvalues $\widehat{Z} = 2nZ = \mathrm{diag}\,(2nz_1, \ldots, 2nz_{2n+\nu})$ of $D_{\rm W}$ are kept fixed for $n \to \infty$. The mass $\widehat{m}_6$ and axial mass $\widehat{\lambda}_7$ are distributed with respect to Gaussians with variance $8\widehat{a}_6^2 = -8\widetilde{a}^2 V W_6$ and $8\widehat{a}_7^2 = -8\widetilde{a}^2 V W_7$, respectively. Note the minus sign in front of $W_{6/7}$. As was shown in Ref. [13] the opposite sign is inconsistent with the symmetries of the Wilson Dirac operator. The notation is slightly different from what is used in the literature to get rid of the imaginary unit in $\widehat{a}_6$ and $\widehat{a}_7$.

The joint probability density $p(Z)$ of the eigenvalues $Z = \mathrm{diag}\,(z_1, \ldots, z_{2n+\nu})$ of $D_{\rm W}$ can be defined by

$$I[f] = \int\limits_{\mathbb{C}^{(2n+\nu)\times(2n+\nu)}} f(D_{\rm W})P(D_{\rm W})d[D_{\rm W}] = \int\limits_{\mathbb{C}^{(2n+\nu)}} f(Z)p(Z)d[Z], \quad (5)$$

where f is an arbitrary $\mathrm{U}(n, n+\nu)$ invariant function. The random matrix $D_{\rm W}$ is $\gamma_5 = \mathrm{diag}\,(\mathbb{1}_n, -\mathbb{1}_{n+\nu})$ Hermitian, i.e.

$$D_{\rm W}^\dagger = \gamma_5 D_{\rm W} \gamma_5. \quad (6)$$

Hence, the eigenvalues z come in complex conjugate pairs or are exactly real. The matrix $D_{\rm W}$ has ν generic real modes and $2(n-l)$ additional real eigenvalues ($0 \leq l \leq n$). The index l decreases by one when a complex conjugate pair enters the real axis.

B. The joint probability density of $D_{\rm W}$

Let D_l be $D_{\rm W}$ if it can be quasi-diagonalized by a non-compact unitary rotation $U \in \mathrm{U}(n, n+\nu)$, i.e. $U\gamma_5 U^\dagger = \gamma_5$, to

$$D_l = U Z_l U^{-1} = U \left(\begin{array}{cc|cc} x_1 & 0 & 0 & 0 \\ 0 & x_2 & y_2 & 0 \\ \hline 0 & -y_2 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{array}\right) U^{-1}, \quad (7)$$

where the real diagonal matrices $x_1 = \mathrm{diag}\,(x_1^{(1)}, \ldots, x_{n-l}^{(1)})$, $x_2 = \mathrm{diag}\,(x_1^{(2)}, \ldots, x_l^{(2)})$, $y_2 = \mathrm{diag}\,(y_1^{(2)}, \ldots, y_l^{(2)})$ and $x_3 = \mathrm{diag}\,(x_1^{(3)}, \ldots, x_{n+\nu-l}^{(3)})$ have the dimension $n-l$, l, l and $n+\nu-l$, respectively. The matrices x_1 and x_3 comprise all real eigenvalues of D_l corresponding to the right-handed and left-handed modes, respectively. We refer to an eigenvector ψ of D_{W} as right-handed if the chirality is positive definite, i.e.

$$\langle\psi|\gamma_5|\psi\rangle > 0, \tag{8}$$

and as left-handed if the chirality is negative definite. The eigenvectors corresponding to complex eigenvalues have vanishing chirality. The complex conjugate pairs are $(z_2 = x_2 + \imath y_2, z_2^* = x_2 - \imath y_2)$. Note that it is not possible to diagonalize D_{W} with a $\mathrm{U}(n, n+\nu)$ transformation with complex conjugate eigenvalues. Moreover we emphasize that almost all γ_5-Hermitian matrices can be brought to the form (7) excluding a set of measure zero.

The quasi-diagonalization $D_l = UZ_lU^{-1}$ determines U up to a $\mathrm{U}^{\,2n+\nu-l}(1) \times \mathrm{O}^{\,l}(1,1)$ transformation while the set of eigenvalues Z_l can be permuted in $l!(n-l)!(n+\nu-l)!2^l$ different ways. The factor 2^l is due to the complex conjugation of each single complex pair. The Jacobian of the transformation to eigenvalues and the coset $\mathbb{G}_l = \mathrm{U}\,(n, n+\nu)/[\mathrm{U}^{\,2n+\nu-l}(1) \times \mathrm{O}^{\,l}(1,1)]$ is given by

$$|\Delta_{2n+\nu}(Z_l)|^2, \tag{9}$$

where the Vandermonde determinant is defined as

$$\Delta_{2n+\nu}(Z) = \prod_{1\le i<j\le 2n+\nu} (z_i - z_j) = (-1)^{n+\nu(\nu-1)/2} \det\left[z_i^{j-1}\right]_{1\le i,j\le 2n+\nu}. \tag{10}$$

The functional $I[f]$ in Eq. (5) is a sum over $n+1$ integrations on disjoint sets, i.e.

$$I[f] = \sum_{l=0}^{n} \frac{1}{2^l(n-l)!l!(n+\nu-l)!} \int\limits_{\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l} f(Z_l) \left[\int\limits_{\mathbb{G}_l} P(UZ_lU^{-1}) d\mu_{\mathbb{G}_l}(U)\right] |\Delta_{2n+\nu}(Z_l)|^2 d[Z_l], \tag{11}$$

where we have normalized the terms with respect to the number of possible permutations of the eigenvalues in Z_l. Thus we have for the joint probability density over all sectors of eigenvalues

$$p(Z)d[Z] = \sum_{l=0}^{n} p_l(Z_l)d[Z_l] \sum_{l=0}^{n} \frac{|\Delta_{2n+\nu}(Z_l)|^2 d[Z_l]}{2^l(n-l)!l!(n+\nu-l)!} \int\limits_{\mathbb{G}_l} P(UZ_lU^{-1}) d\mu_{\mathbb{G}_l}(U). \tag{12}$$

Here $p_l(Z_l)$ is the joint probability density for a fixed number of complex conjugate eigenvalue pairs, namely l. The integration over U is non-trivial and will be worked out in detail in Appendix A.

In a more mathematical language the normalization factor in Eq. (12) can be understood as follows. If the permutation group of N elements is denoted by $\mathbf{S}(N)$ while the group describing the reflection $y \to -y$ is $\mathbb{Z}_2$, the factor $2^l(n-l)!l!(n+\nu-l)!$ is the volume of the finite subgroup $\mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbb{Z}_2^l$ of $\mathrm{U}\,(n, n+\nu)$ which correctly normalizes each summand. Originally we had to divide $\mathrm{U}\,(n, n+\nu)$ by the set $\mathrm{U}^{\,2n+\nu-l}(1) \times \mathrm{O}^{\,l}(1,1) \times \mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbb{Z}_2^l$ because it is the maximal subgroup whose image of the adjoint mapping commutes with Z_l. The reasoning is as follows. Let $\Sigma[Z_l] = \{UZ_lU^{-1} | U \in \mathrm{U}\,(n, n+\nu)\}$ be the orbit of Z_l and $\Sigma_{\mathrm{c}}[Z_l] = \{\widehat{Z}_l \in \Sigma[Z_l] | [\widehat{Z}_l, Z_l]_- = \widehat{Z}_l Z_l - Z_l \widehat{Z}_l = 0\}$ a subset of this orbit. Then all orderings in each of the three sets of eigenvalues x_1, (z_2, z_2^*) and x_3 as well as the reflections $y_j^{(2)} \to -y_j^{(2)}$ are in $\Sigma_{\mathrm{c}}[Z_l]$. This subset $\Sigma_{\mathrm{c}}[Z_l] \subset \Sigma[Z_l]$ can be represented by the finite group $\mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbb{Z}_2^l$. This group is called the Weyl group in group theory. The Lie group $\mathrm{U}^{\,2n+\nu-l}(1) \times \mathrm{O}^{\,l}(1,1)$ acts on $\Sigma_{\mathrm{c}}[Z_l]$ as the identity since it commutes with Z_l. The group $\mathrm{U}^{\,2n+\nu-l}(1)$ represents $2n+\nu-l$ complex phases along the diagonal commuting with the set which consists of Z_l with a fixed l. Each non-compact orthogonal group $\mathrm{O}\,(1,1)$ reflects the invariance of a single complex conjugate eigenvalue pair under a hyperbolic transformation which is equal to a Lorentz-transformation in a 1+1 dimensional space-time.

There are two ways to deal with the invariance under $\mathrm{U}^{\,2n+\nu-l}(1) \times \mathrm{O}^{\,l}(1,1) \times \mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbb{Z}_2^l$ in an integral such that we correctly weigh all points. We have either to divide $\mathrm{U}\,(n, n+\nu)$ by the whole subgroup or we integrate over a larger coset and reweight the measure by the volume of the subgroups not excluded. The ordering enforced by $\mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbb{Z}_2^l$ is difficult to handle in calculations. Therefore, we have decided for a reweighting of the integration measure by $1/[(n-l)!(n+\nu-l)!l!2^l]$. However the Lie group $\mathrm{U}^{\,2n+\nu-l}(1) \times \mathrm{O}^{\,l}(1,1)$, in particular the hyperbolic subgroups, has to be excluded since its volume is infinite.

In this section as well as in Appendix A, we use the non-normalized Haar-measures induced by the pseudo metric

$$\mathrm{tr}\, dD_{\mathrm{W}}^2 = \mathrm{tr}\, dA^2 + \mathrm{tr}\, dB^2 - 2\mathrm{tr}\, dW\, dW^\dagger. \tag{13}$$

Therefore the measures for $D_{\rm W}$ and Z_l are

$$d[D_{\rm W}] = \prod_{j=1}^{n} dA_{jj} \prod_{1\le i<j\le n} 2\, d\,{\rm Re}\, A_{ij} d\,{\rm Im}\, A_{ij} \prod_{j=1}^{n+\nu} dB_{jj} \tag{14}$$
$$\times \prod_{1\le i<j\le n+\nu} 2\, d\,{\rm Re}\, B_{ij} d\,{\rm Im}\, B_{ij} \prod_{\substack{1\le i\le n\\ 1\le j\le n+\nu}} (-2)\, d\,{\rm Re}\, W_{ij} d\,{\rm Im}\, W_{ij},$$

$$d[Z_l] = \prod_{j=1}^{n-l} dx_j^{(1)} \prod_{j=1}^{l} 2\imath\, dx_j^{(2)} dy_j^{(2)} \prod_{j=1}^{n+\nu-l} dx_j^{(3)}. \tag{15}$$

The Haar measure $d\mu_{\mathbb{G}_l}$ for the coset $\mathbb{G}_l$ is also induced by $d[D_{\rm W}]$ and results from the pseudo metric, i.e.

$$\mathrm{tr}\, dD_{\rm W}^2 = \mathrm{tr}\, dZ_l^2 + \mathrm{tr}\, [U^{-1}dU, Z_l]_-^2. \tag{16}$$

The reason for this unconventional definition is the non-normalizability of the measure $d\mu_{\mathbb{G}_l}$ because $\mathbb{G}_l$ is non-compact for $l > 0$. Hence the normalization resulting from definition (16) seems to be the most natural one, and it helps in keeping track of the normalizations.

In Appendix A we solve the coset integral (12). The first step is to linearize the quadratic terms in UZ_lU^{-1} by introducing auxiliary Gaussian integrals over additional matrices which is along the idea presented in Ref. [20]. In this way we split the integrand in a part invariant under $\mathrm{U}(n, n+\nu)$ and a non-invariant part resulting from an external source. The group integrals appearing in this calculations are reminiscent of the Itsykson-Zuber integral. However they are over non-compact groups and, thus, much more involved than in Ref. [20]. Because of the $\mathrm{U}(n)\times\mathrm{U}(n+\nu)$ invariance of the probability density of $D_{\rm W}$, the joint eigenvalue distribution is a symmetric function of n eigenvalues which we label by "r" and $n+\nu$ eigenvalues labelled by "l". The γ_5-Hermiticity imposes reality constraints on the eigenvalues resulting in Dirac delta-functions in the joint probability density. Similarly to the usual Itzykson-Zuber integral, the symmetric function of the eigenvalues turns out to be particularly simple (see Appendix A)

$$p(Z)d[Z] = c(1+a^2)^{-n(n+\nu-1/2)} a^{-n-\nu^2} \exp\left[-\frac{a^4}{4(1+a^2)}(\mu_{\rm r}-\mu_{\rm l})^2\right] \tag{17}$$
$$\times \Delta_{2n+\nu}(Z) \det \begin{bmatrix} \left\{ g_2(z_i^{(\rm r)}, z_j^{(\rm l)}) dx_i^{(\rm r)} dy_i^{(\rm r)} dx_j^{(\rm l)} dy_j^{(\rm l)} \right\}_{\substack{1\le i\le n\\ 1\le j\le n+\nu}} \\ \left\{ \left(x_j^{(\rm l)}\right)^{i-1} g_1(x_j^{(\rm l)}) \delta(y_j^{(\rm l)}) dx_j^{(\rm l)} dy_j^{(\rm l)} \right\}_{\substack{1\le i\le \nu\\ 1\le j\le n+\nu}} \end{bmatrix}.$$

The last ν rows become zero in the continuum limit resulting in ν exact zero modes (see subsection II C). At finite a they can be interpreted as broadened "zero modes". The functions in the determinant are given by

$$g_2(z_1, z_2) = g_r(x_1, x_2)\delta(y_1)\delta(y_2) + g_{\rm c}(z_1)\delta(x_1-x_2)\delta(y_1+y_2), \tag{18}$$

$$g_{\rm r}(x_1, x_2) = \exp\left[-\frac{n}{4a^2}\left(x_1+x_2-\frac{a^2(\mu_{\rm r}+\mu_{\rm l})}{n}\right)^2 + \frac{n}{4}(x_1-x_2)^2\right] \tag{19}$$
$$\times \left[\mathrm{sign}\,(x_1-x_2) - \mathrm{erf}\left[\sqrt{\frac{n(1+a^2)}{4a^2}}(x_1-x_2) - \sqrt{\frac{a^2}{4n(1+a^2)}}(\mu_{\rm r}-\mu_{\rm l})\right]\right],$$

$$g_{\rm c}(z) = -2\imath\, \mathrm{sign}\,(y) \exp\left[-\frac{n}{a^2}\left(x-\frac{a^2(\mu_{\rm r}+\mu_{\rm l})}{2n}\right)^2 - ny^2\right], \tag{20}$$

$$g_1(x) = \exp\left[-\frac{n}{2a^2}\left(x-\frac{a^2\mu_{\rm l}}{n}\right)^2\right]. \tag{21}$$

We employ the error function "erf" and the function "sign" which yields the sign of the argument. The constant is equal to

$$\frac{1}{c} = (-1)^{\nu(\nu-1)/2+n(n-1)/2} \left(\frac{16\pi}{n}\right)^{n/2} (2\pi)^{\nu/2} n^{-\nu^2/2-n(n+\nu)} \prod_{j=0}^{n} j! \prod_{j=0}^{n+\nu} j!, \tag{22}$$

and is essentially the volume of the coset $[\mathrm{U}(n)\times\mathrm{U}(n+\nu)]/[\mathbf{S}(n)\times\mathbf{S}(n+\nu)]$.

The two-point weight g_2 consists of two parts. The first term, g_{r}, represents a pair of real modes where one eigenvalue corresponds to a right-handed eigenvector and the other one to a left-handed one. The second term, g_{c}, enforces that a complex eigenvalue comes with its complex conjugate, only. The function g_1 is purely Gaussian. As we will see in the next subsection, in the small a limit this will result in a distribution of the former zero modes that is broadened to the Gaussian Unitary Ensemble (GUE) [4, 6, 7, 10, 20, 23].

For N_{f} dynamical quarks with quark mass m_f the joint probability density is simply given by [13]

$$p^{(N_{\mathrm{f}})}(z)=\prod_{f=1}^{N_{\mathrm{f}}}\prod_{k=1}^{2n+\nu}(z_k+m_f)p(Z). \tag{23}$$

The expansion in g_{c} yields the joint probability density for a fixed number of complex conjugate pairs,

$$\begin{aligned}p_l(Z_l)d[Z_l]=&\frac{(-1)^{(n-l)l}c(1+a^2)^{-n(n+\nu-1/2)}a^{-n-\nu^2}n!(n+\nu)!}{(n-l)!l!(n+\nu-l)!}\exp\left[-\frac{a^4}{4(1+a^2)}(\mu_{\mathrm{r}}-\mu_{\mathrm{l}})^2\right]\\ &\times\Delta_{2n+\nu}(Z)\det\begin{bmatrix}\{g_r(x_i^{(1)},x_j^{(3)})dx_i^{(1)}dx_j^{(3)}\}_{\substack{1\le i\le n-l\\1\le j\le n+\nu-l}}\\ \{(x_j^{(3)})^{i-1}g_1(x_j^{(3)})dx_j^{(3)}\}_{\substack{1\le i\le \nu\\1\le j\le n+\nu-l}}\end{bmatrix}\prod_{j=1}^{l}g_c(z_j^{(2)})dx_j^{(2)}dy_j^{(2)}.\end{aligned} \tag{24}$$

The factorials in the prefactor are the combinatorial factor which results from the expansion of the determinant in co-factors with l columns and l rows less. Note that they correspond to the coset of finite groups, $[\mathbf{S}(n)\times\mathbf{S}(n+\nu)]/[\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l)]$, which naturally occurs when diagonalizing D_{W} in a fixed sector, see the discussion after Eq. (12).

C. The continuum limit

In this section, we take the continuum limit of the joint probability density p, i.e. $a\to 0$ at fixed z, μ_{r} and μ_{l}. In this limit the probability density (2) of D_{W} trivially becomes the one of chiral RMT which is equivalent to continuum QCD in the ϵ-regime [17]. We expect that this is also the case for the joint probability density.

The small a limit of the two point weight (18) is given by

$$g_2(z_1,z_2)\overset{a\ll 1}{=}-2\imath\,\mathrm{sign}\,(y_1)\sqrt{\frac{a^2\pi}{n}}\exp\left[-ny_1^2\right]\delta(x_1)\delta(x_2)\delta(y_1+y_2). \tag{25}$$

The function g_{r} vanishes due to the error function which cancels with the sign function. The expansion of the determinant (17) yields $(n+\nu)!/\nu!$ terms which are all the same. Thus, we have

$$\begin{aligned}\lim_{a\to 0}p(Z)d[Z]=&\,c(-1)^{\nu(\nu-1)/2}\frac{(n+\nu)!}{\nu!}\left(-2\imath\sqrt{\frac{\pi}{n}}\right)^n\\ &\times\lim_{a\to 0}a^{-\nu^2}\Delta_{2n+\nu}(\imath y,-\imath y,x)\Delta_\nu(x)\prod_{j=1}^{n}\mathrm{sign}\,(y_j)\exp\left[-ny_j^2\right]dy_j\prod_{j=1}^{\nu}\exp\left[-\frac{n}{2a^2}x_j^2\right]dx_j.\end{aligned} \tag{26}$$

Thereby we have already evaluated the Dirac delta-functions. The real part of the complex eigenvalues $z_j^{(\mathrm{r}/\mathrm{l})}$, $1\le j\le n$, and the imaginary part of $z_j^{(\mathrm{l})}$, $n+1\le j\le n+\nu$, vanish and they become the variables $\pm\imath y_j$, $1\le j\le n$, and x_j, $1\le j\le\nu$, respectively. Note, that the random variables x scale with a while y is of order 1. Therefore the distribution of the two sets of eigenvalues factorizes into a product that can be identified as the joint probability density of a $\nu\times\nu$ dimensional GUE on the scale of a and the chiral Unitary Ensemble on the scale 1,

$$\begin{aligned}\lim_{a\to 0}p(Z)d[Z]=&\frac{1}{(2\pi)^{\nu/2}}\left(\frac{n}{a^2}\right)^{\nu^2/2}\prod_{j=0}^{\nu}\frac{1}{j!}\Delta_\nu^2(x)\prod_{j=1}^{\nu}\exp\left[-\frac{n}{2a^2}x_j^2\right]dx_j\\ &\times\frac{n^{n^2+\nu n}}{n!}\prod_{j=0}^{n-1}\frac{1}{(j+\nu)!j!}\Delta_n^2(y^2)\prod_{j=1}^{n}2\Theta(y_j)y_j^{2\nu+1}\exp\left[-ny_j^2\right]dy_j,\end{aligned} \tag{27}$$

where Θ is the Heaviside distribution.

III. FROM THE JOINT PROBABILITY DENSITY TO THE LEVEL DENSITIES

The level density is obtained by integrating the joint probability density (17) over all eigenvalues of $D_{\rm W}$ except one. We can choose to exclude an eigenvalue of $z^{(\rm r)}$ or one of the $z^{(\rm l)}$'s. When we exclude $z_1^{(\rm r)}$ we have to expand the determinant (17) with respect to the first row. All resulting terms are the same and consist of a term for which $z_1^{(\rm r)}$ is complex and a term for which $z_1^{(\rm r)}$ is real. We thus have [23]

$$\int p(Z) \prod_{z_j \neq z_1^{(\rm r)}} d[z_j] = \rho_{\rm r}\,(x_1^{(\rm r)})\delta(y_1^{(\rm r)}) + \frac{1}{2}\rho_{\rm c}(z_1^{(\rm r)}). \tag{28}$$

When excluding $z_1^{(\rm l)}$ and expanding the determinant (17) with respect to the first column we notice that the first n terms are the same while the remaining ν terms have to be treated separately. Again the spectral density is the sum of the density of the real modes, which are left-handed in this case, and the density of the complex modes [23]

$$\int p(Z) \prod_{z_j \neq z_1^{(\rm l)}} d[z_j] = \rho_{\rm l}\,(x_1^{(\rm l)})\delta(y_1^{(\rm l)}) + \frac{1}{2}\rho_{\rm c}(z_1^{(\rm l)}). \tag{29}$$

The level densities $\rho_{\rm r}$ and $\rho_{\rm l}$ are the densities of the real right- and left-handed modes, respectively. Interestingly the level density of the complex modes appears symmetrically in both equations. The reason is the vanishing chirality of eigenvectors corresponding to the complex eigenvalues.

Let us consider the case when excluding $z_1^{(\rm r)}$. The Vandermonde determinant without a factor $(z_1^{(\rm r)}-z_1^{(\rm l)})\prod_{k=2}^{n}(z_1^{(\rm r)}-z_k^{(\rm r)})(z_1^{(\rm l)}-z_k^{(\rm r)})\prod_{j=2}^{n+\nu}(z_1^{(\rm r)}-z_j^{(\rm l)})(z_1^{(\rm l)}-z_j^{(\rm l)})$ and the cofactor from expanding the first row of the determinant is equal to the joint probability density with one pair $(z^{(\rm r)}, z^{(\rm l)})$ less. The $z_1^{(\rm l)}$-integral over this distribution together with the factor $\prod_{k=2}^{n}(z_1^{(\rm r)}-z_k^{(\rm r)})(z_1^{(\rm l)}-z_k^{(\rm r)})\prod_{j=2}^{n+\nu}(z_1^{(\rm r)}-z_j^{(\rm l)})(z_1^{(\rm l)}-z_j^{(\rm l)})$ can be identified as the partition function with two additional flavors. We thus find

$$\rho_{\rm r}\,(x) \propto \int_{-\infty}^{\infty} g_{\rm r}(x,x')(x-x')Z_{N_{\rm f}+2}^{n-1,\nu}(x,x',m_k)dx', \tag{30}$$

$$\rho_{\rm c}(z) \propto g_{\rm c}(z)(z-z^*)Z_{N_{\rm f}+2}^{n-1,\nu}(z,z^*,m_k). \tag{31}$$

The fermionic partition function is given by

$$Z_{N_{\rm f}+2}^{n-1,\nu}(z_1,z_2,m_k) = \int \det(D_{\rm W} - z_1\mathbb{1}_{2n+\nu-2})\det(D_{\rm W} - z_2\mathbb{1}_{2n+\nu-2})\prod_{k=1}^{N_{\rm f}}(D_{\rm W}+m_k\mathbb{1}_{2n+\nu-2})P(D_{\rm W})d[D_{\rm W}], \tag{32}$$

where $D_{\rm W}$ is given as in Eq. (1) only that n is replaced by $n-1$. In the microscopic limit this is simply a unitary matrix integral which can be easily numerically evaluated. Note that the integral over the variables $\mu_{\rm r\,/l}$ which introduces the low energy constants $W_{6/7}$ can already be performed at this step.

Considering the exclusion of $z_1^{(\rm l)}$ we have to expand the determinant in the joint probability density with respect to the first column resulting in a much more complicated expression

$$\begin{aligned}\rho_{\rm l}\,(x) \propto n \int_{\mathbb{C}} d[\widetilde{z}](x-\widetilde{z})g_2(x,\widetilde{z})Z_{N_{\rm f}=2}^{n-1,\nu}(x,\widetilde{z}) + \alpha\delta(y)\sum_{p=1}^{\nu}(-1)^{\nu-p} \\ \times\begin{pmatrix} n+\nu-1 \\ \nu-p \end{pmatrix} x^{p-1}g_1(x)\int_{\mathbb{R}^{\nu-p}}\prod_{j=1}^{\nu-p}dx_j x_j^p g_1(x_j)\Delta_{\nu-p}(x_1,\cdots,x_{\nu-p}) \\ \times\Delta_{\nu-p+1}(x,x_1,\cdots,x_{\nu-p})Z_{N_{\rm f}=\nu-p+1}^{n,\,p}(x,x_1,\cdots,x_{\nu-p})\end{aligned} \tag{33}$$

with a certain constant α which we will specify in the microscopic limit. The global proportionality constant is up to a factor n the same as the one in Eqs. (30) and (31). Again $g_2(z,z_{\rm r})$ is the sum of a term comprising the density of the complex eigenvalues and a term giving the real eigenvalue density. For the complex eigenvalue density we find the same expression as obtained by integration over $z_1^{(\rm l)}$.

For $\nu = 1$, the density of the real eigenvalues simplifies to

$$\rho_{\rm l}\,(x)|_{\nu=1} \propto n\int_{-\infty}^{\infty} dx_{\rm r}\,(x_{\rm r}-x)g_r(x_{\rm r},x)Z_{N_{\rm f}=2}^{n-1,1}(x_{\rm r},x) + \alpha g_1(x)Z_{N_{\rm f}=1}^{n,\,0}(x) \tag{34}$$

since there is no integration in the term proportional to α, cf. Eq. (33). The distribution of chirality over the real modes is the difference

$$\rho_\chi(x) = \rho_{\rm l}(x) - \rho_{\rm r}(x), \tag{35}$$

resulting in

$$\rho_\chi|_{\nu=1} \sim \alpha g_1(x) Z^{n,0}_{N_{\rm f}=1}(x) + n \int\limits_{-\infty}^{\infty} dx'(x'-x)(g_r(x',x)+g_r(x,x'))Z^{n-1,1}_{N_{\rm f}=2}(x',x), \tag{36}$$

where we used that the two-flavor partition function is symmetric in x and x'. For $\mu_r = \mu_l$, the last two terms cancel resulting in a very simple expression for $\rho_\chi(x)$. Note that the integral over the second term always vanishes such that it does not contribute to the normalization of the distribution of chirality over the real modes

$$\int dx \rho_\chi(x) = \nu \tag{37}$$

which is 1 for $\nu = 1$. For $\nu = 2$ we find

$$\begin{aligned}\rho_\chi(x)|_{\nu=2} \sim{}& \alpha x g_1(x) Z^{n,1}_{N_{\rm f}=1}(x) - \alpha(n+1) g_1(x) \int\limits_{-\infty}^{\infty} dx' x'(x-x') g_1(x') Z^{n,0}_{N_{\rm f}=2}(x,x') \\ &+ n \int\limits_{-\infty}^{\infty} dx'(x'-x)(g_r(x',x)+g_r(x,x'))Z^{n-1,2}_{N_{\rm f}=2}(x',x).\end{aligned} \tag{38}$$

In the microscopic limit the two-flavor partition functions can be replaced by a unitary matrix integral which still can be easily numerically evaluated including the integrals over $\widehat{m}_6$ and $\widehat{\lambda}_7$.

For large values of ν the expression of the distribution of chirality over the real modes obtained from expanding the determinant gets increasingly complicated. However, there is an alternative expression in terms of a supersymmetric partition function [23, 42],

$$\rho_\chi(x) \propto \lim_{\varepsilon\to 0} {\rm Im} \left.\frac{\partial}{\partial J}\right|_{J=0} \int \frac{\det(D_{\rm W} - (x+J)\mathbb{1}_{2n+\nu})}{\det(D_{\rm W} - x\mathbb{1}_{2n+\nu} - \imath\varepsilon\gamma_5)} P(D_{\rm W}) d[D_{\rm W}]. \tag{39}$$

In the ensuing sections we will use this expression to calculate the microscopic limit of the distribution of chirality over the real modes.

A. Microscopic Limit of the Eigenvalue Densities

The goal of this section is to derive the microscopic limit of $\rho_{\rm r}$, ρ_χ, and $\rho_{\rm c}$ including those terms involving non-zero values of W_6 and W_7 in the chiral Lagrangian. We only give results for the quenched case. It is straightforward to include dynamical quarks but this will be worked out in a forthcoming publication. The result for the distribution of chirality over the real modes with dynamical quarks for $W_6 = W_7 = 0$ was already given in [42], and an explicit expression for the density of the complex eigenvalues in the presence of dynamical quarks and non-zero values W_6, W_7 and W_8 was derived in [13].

The microscopic limit of the spectral densities is obtained from the microscopic limit of the partition functions and the functions appearing in the joint probability density. We remind the reader that the microscopic parameters which are kept fixed for $V \to \infty$, are defined by

$$\begin{aligned}&\widehat{a}_6^2 = -\widetilde{a}^2 V W_6, \qquad \widehat{a}_7^2 = -\widetilde{a}^2 V W_7, \qquad \widehat{a}_8^2 = na^2/2 = \widetilde{a}^2 V W_8, \\ &\widehat{m}_6 = a^2(\mu_r + \mu_l), \qquad \widehat{\lambda}_7 = a^2(\mu_r - \mu_l), \qquad \widehat{x} = 2nx.\end{aligned} \tag{40}$$

The microscopic limit of the probability density of $\widehat{m}_6$ and $\widehat{\lambda}_7$ is given by

$$p(\widehat{m}_6, \widehat{\lambda}_7) = \frac{1}{16\pi \widehat{a}_6 \widehat{a}_7} \exp\left[-\frac{\widehat{m}_6^2}{16\widehat{a}_6^2} - \frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}\right], \tag{41}$$

and the functions that appear in the joint probability density simplify to

$$\widehat{g}_{\rm r}\left(\widehat{x},\widehat{x}',\widehat{m}_6,\widehat{\lambda}_7\right)=\exp\left[-\frac{(\widehat{x}+\widehat{x}'-2\widehat{m}_6)^2}{32\widehat{a}_8^2}\right]\left[\mathrm{sign}\left(\widehat{x}-\widehat{x}'\right)-\mathrm{erf}\left[\frac{(\widehat{x}-\widehat{x}')/2-\widehat{\lambda}_7}{\sqrt{8\widehat{a}_8}}\right]\right], \tag{42}$$

$$\widehat{g}_{\rm c}(\widehat{z})=-2\imath\,\mathrm{sign}\left(\widehat{y}\right)\exp\left[-\frac{(\widehat{x}-\widehat{m}_6)^2}{8\widehat{a}_8^2}\right], \tag{43}$$

$$\widehat{g}_1(\widehat{x})=\exp\left[-\frac{(\widehat{x}-\widehat{m}_6+\widehat{\lambda}_7)^2}{16\widehat{a}_8^2}\right]. \tag{44}$$

The microscopic limit of the spectral densities obtained in Eqs. (30), (31) and (39) is given by

$$\begin{aligned}\rho_{\rm r}\left(\widehat{x}\right)=&\frac{1}{32\sqrt{2\pi}\widehat{a}_8}\int_{\mathbb{R}^3}d\widehat{m}_6d\widehat{\lambda}_7d\widehat{x}'p(\widehat{m}_6,\widehat{\lambda}_7)(\widehat{x}-\widehat{x}')\widehat{g}_{\rm r}\left(\widehat{x},\widehat{x}',\widehat{m}_6,\widehat{\lambda}_7\right)\\&\times Z^{\nu}_{2/0}(\widehat{x}+\widehat{m}_6,\widehat{x}'+\widehat{m}_6,\widehat{\lambda}_7,\widehat{a}_8),\end{aligned} \tag{45}$$

$$\begin{aligned}\rho_{\rm c}(\widehat{z})=&\frac{\imath\widehat{y}}{32\sqrt{2\pi}\widehat{a}_8}\int_{\mathbb{R}^2}d\widehat{m}_6d\widehat{\lambda}_7p(\widehat{m}_6,\widehat{\lambda}_7)\widehat{g}_c(\widehat{z},\widehat{z}^*,\widehat{m}_6)\\&\times Z^{\nu}_{2/0}(\widehat{z}+\widehat{m}_6,\widehat{z}^*+\widehat{m}_6,\widehat{\lambda}_7,\widehat{a}_8),\end{aligned} \tag{46}$$

$$\rho_\chi(\widehat{x})=\frac{1}{\pi}\lim_{\varepsilon\to0}\mathrm{Im}\int d\widehat{m}_6d\widehat{\lambda}_7p(\widehat{m}_6,\widehat{\lambda}_7)G_{1/1}(\widehat{x}+\widehat{m}_6,\widehat{\lambda}_7+\imath\varepsilon,\widehat{a}_8). \tag{47}$$

The resolvent $G_{1/1}$ follows from the graded partition function

$$\begin{aligned}G_{1/1}(\widehat{x}+\widehat{m}_6,\widehat{\lambda}_7+\imath\varepsilon,\widehat{a}_8)=&\left.\frac{d}{d\widehat{x}'}Z^{\nu}_{1/1}(\widehat{x}+\widehat{m}_6,\widehat{x}'+\widehat{m}_6,\widehat{\lambda}_7+\imath\varepsilon,\widehat{a}_8)\right|_{\widehat{x}'=\widehat{x}}\\=&\lim_{n\to\infty}\frac{1}{2n}\int\mathrm{tr}\,\frac{1}{D_{\rm W}-2n\widehat{x}\mathbb{1}_{2n+\nu}-\imath\varepsilon\gamma_5}P(D_{\rm W})d[D_{\rm W}].\end{aligned} \tag{48}$$

The two-flavor partition function is up to a constant defined by

$$Z^{\nu}_{2/0}(z_1+m_6,z_2+m_6,\lambda_7,a)\propto\int\det(z_1\mathbb{1}_{2n+\nu-2}-D_{\rm W})\det(z_2\mathbb{1}_{2n+\nu-2}-D_{\rm W})P(D_{\rm W})d[D_{\rm W}]. \tag{49}$$

The microscopic limit of the two-flavor partition function follows from the chiral Lagrangian (4). In the diagonal representation of the unitary 2×2 matrix, it can be simplified by means of an Itzykson-Zuber integral and is given by

$$\begin{aligned}Z^{\nu}_{2/0}(\widehat{z}_1,\widehat{z}_2,\widehat{\lambda}_7,\widehat{a}_8)=&\frac{1}{2\pi^2}\int d\varphi_1d\varphi_2\sin^2((\varphi_1-\varphi_2)/2)e^{\imath\nu(\varphi_1+\varphi_2)}\exp\left[\imath\widehat{\lambda}_7(\sin\varphi_1+\sin\varphi_2)-4\widehat{a}_8^2(\cos^2\varphi_1+\cos^2\varphi_2)\right]\\&\times\frac{\exp\left[\widehat{z}_1\cos\varphi_1+\widehat{z}_2\cos\varphi_2\right]-\exp\left[\widehat{z}_2\cos\varphi_1+\widehat{z}_1\cos\varphi_2\right]}{(\cos\varphi_1-\cos\varphi_2)(\widehat{z}_1-\widehat{z}_2)}.\end{aligned} \tag{50}$$

The normalization is chosen such that we find the well known result [41],

$$Z^{\nu}_{2/0}(\widehat{z}_1,\widehat{z}_2,\widehat{\lambda}_7=0,\widehat{a}_8=0)=\frac{\widehat{z}_1I_{\nu+1}(\widehat{z}_1)I_\nu(\widehat{z}_2)-\widehat{z}_2I_{\nu+1}(\widehat{z}_2)I_\nu(\widehat{z}_1)}{\widehat{z}_1^2-\widehat{z}_2^2}, \tag{51}$$

at vanishing lattice spacing, where I_ν is the modified Bessel function of the first kind.

The microscopic limit of the graded partition function follows from the chiral Lagrangian [9] which can be written as an integral over a $(1/1)\times(1/1)$ supermatrix [42]

$$U=\begin{bmatrix}e^{\vartheta}&\eta^*\\\eta&e^{\imath\varphi}\end{bmatrix},\ \vartheta\in\mathbb{R},\ \varphi\in[0,2\pi], \tag{52}$$

with η and η^* two independent Grassmann variables, see Refs. [50–54] for the supersymmetry method in random matrix theory. Let the normalization of the integration over the Grassmann variables be

$$\int\eta^*\eta d\eta d\eta^*=\frac{1}{2\pi}. \tag{53}$$

Then the graded partition function is

$$Z^{\nu}_{1/1}(\widehat{z}_1,\widehat{z}_2,\widehat{\lambda}_7\pm\imath\varepsilon,\widehat{a}_8)=\int\frac{\imath de^{\imath\varphi}}{2\pi}de^{\vartheta}d\eta d\eta^{*}\mathrm{Sdet}^{\,\nu}U\exp[-\widehat{a}_8^2\mathrm{Str}\,(U^2+U^{-2})] \\ \times\exp\left[\pm\frac{\imath}{2}\mathrm{Str}\,\widehat{Z}(U-U^{-1})-\left(\varepsilon\pm\frac{\imath\widehat{\lambda}_7}{2}\right)\mathrm{Str}\,(U+U^{-1})\right], \tag{54}$$

where $\widehat{Z}=\mathrm{diag}(\widehat{z}_1,\widehat{z}_2)$ and the normalization adjusted by the continuum limit [55, 56]

$$Z^{\nu}_{1/1}(\widehat{z}_1,\widehat{z}_2,\widehat{\lambda}_7=0,\widehat{a}_8=0)=\widehat{z}_1K_{\nu+1}(\widehat{z}_1)I_{\nu}(\widehat{z}_2)-\widehat{z}_2I_{\nu+1}(\widehat{z}_2)K_{\nu}(\widehat{z}_1). \tag{55}$$

The function K_ν is the modified Bessel function of the second kind.

There are various ways to calculate the integral (54). One possibility is a brute force evaluation of the Grassmann integrals as in [9, 40]. Then the Gaussian integrals over $\widehat{m}_6$ and $\widehat{\lambda}_7$ can be performed analytically leaving us with a non-singular two-dimensional integral. A second possibility would be to rewrite the integrals as in [42]. Then we end up with a two dimensional singular integral (see Appendix C) which can be evaluated numerically with some effort. The third way to evaluate the integral, is a variation of the method in [42] and results in a one dimensional integral and a sum over Bessel functions that can be easily numerically evaluated (see section IV C).

IV. THE EIGENVALUE DENSITIES AND THEIR PROPERTIES

To illustrate the effect of non-zero $\widehat{a}_6$ and $\widehat{a}_7$ we first discuss the case $\widehat{a}_8=0$. For the general case, with $\widehat{a}_8$ also non-zero, we will discuss the density of the additional real eigenvalues, the density of the complex eigenvalues, and finally the distribution of chirality over the real eigenvalues of D_{W}.

A. Spectrum of D_{W} for $\widehat{a}_8=0$

.

The low-energy constants $\widehat{a}_6$ and $\widehat{a}_7$ are introduced through the addition of the Gaussian stochastic variables $\widehat{m}_6+\widehat{\lambda}_7\gamma_5$ to $D_{\mathrm{W}}|_{a=0}$ resulting in the massive Dirac operator [60]

$$D=D_{\mathrm{W}}|_{a=0}+(\widehat{m}+\widehat{m}_6)\mathbb{1}+\widehat{\lambda}_7\gamma_5. \tag{56}$$

For $\widehat{a}_8=0$ the Dirac operator $D_{\mathrm{W}}|_{a=0}$ is anti-Hermitian, and the eigenvalues of $D_{\mathrm{W}}(\widehat{\lambda}_7,\widehat{m}_6)=D-\widehat{m}$ are given by

$$\widehat{z}_{\pm}=\widehat{m}_6\pm\imath\sqrt{\lambda_{\mathrm{W}}^2-\widehat{\lambda}_7^2}, \tag{57}$$

where $\imath\lambda_{\mathrm{W}}$ is an eigenvalue of $D_{\mathrm{W}}|_{a=0}$. The density of the eigenvalues of D is obtained after integrating over the Gaussian distribution of $\widehat{m}_6$ and $\widehat{\lambda}_7$.

As can be seen from Eq. (57), in case $\widehat{a}_6=\widehat{a}_8=0$ and $\widehat{a}_7\neq0$, the eigenvalues of D are either purely imaginary or purely real depending on whether $\widehat{\lambda}_7$ is smaller or larger than λ_{W}, respectively. Paired imaginary eigenvalues penetrate the real axis only through the origin when varying $\widehat{\lambda}_7$, see Fig. 1. Introducing a non-zero W_6, broadens the spectrum by a Gaussian parallel to the real axis but nothing crucial happens because $\widehat{m}_6$ is just an additive constant to the eigenvalues, cf. Fig. 1.

In the continuum the low lying spectral density of the quenched Dirac operator is given by [17]

$$\rho_{\mathrm{cont.}}(\widehat{z})=\delta(\widehat{x})\left[\nu\delta(\widehat{y})+\frac{|\widehat{y}|}{2}(J_\nu^2(\widehat{y})-J_{\nu-1}(\widehat{y})J_{\nu+1}(\widehat{y}))\right]=\delta(\widehat{x})\left[\nu\delta(\widehat{y})+\rho_{\mathrm{NZ}}(\widehat{y})\right]. \tag{58}$$

The function J_ν is the Bessel function of the first kind. The level density ρ_{NZ} describes the density of the generic non-zero eigenvalues, only.

For non-zero $W_{6/7}$ the distribution of the zero modes represented by the Dirac delta-functions in Eq. (58) is broadened by a Gaussian, i.e.

$$\rho_\chi(\widehat{z},\widehat{a}_8=0)=\frac{\nu}{\sqrt{16\pi(\widehat{a}_6^2+\widehat{a}_7^2)}}\exp\left[-\frac{\widehat{x}^2}{16(\widehat{a}_6^2+\widehat{a}_7^2)}\right]. \tag{59}$$

Complex modes have vanishing chirality and do not contribute to the distribution of chirality over the real modes. Additional pairs of real modes also do not contribute to ρ_χ. The reason is the symmetric integration of $\widehat{\lambda}_7$ over the real axis. The eigenvalues remain the same under the change $\widehat{\lambda}_7 \to -\widehat{\lambda}_7$, see Eq. (57). However the corresponding eigenvectors interchange the sign of the chirality which can be seen by the symmetry relation

$$D_{\rm W}\,(\widehat{\lambda}_7,\widehat{m}_6) = -\gamma_5 D_{\rm W}\,(-\widehat{\lambda}_7,-\widehat{m}_6)\gamma_5. \tag{60}$$

Thus the normalized eigenfunctions ($\langle\psi_\pm|\psi_\pm\rangle = 1$) corresponding to the eigenvalues $\widehat{z}_\pm$, i.e.

$$D_{\rm W}\,(\widehat{\lambda}_7,\widehat{m}_6)|\psi_\pm\rangle = \widehat{z}_\pm|\psi_\pm\rangle, \tag{61}$$

also fulfills the identity

$$D_{\rm W}\,(-\widehat{\lambda}_7,-\widehat{m}_6)\gamma_5|\psi_\pm\rangle = -\widehat{z}_\pm\gamma_5|\psi_\pm\rangle. \tag{62}$$

Since the quark mass $\widehat{m}_6$ enters with unity we have also

$$D_{\rm W}\,(-\widehat{\lambda}_7,\widehat{m}_6)\gamma_5|\psi_\pm\rangle = \widehat{z}_\mp\gamma_5|\psi_\pm\rangle. \tag{63}$$

The wave-functions $\gamma_5|\psi_\pm\rangle$ share the same chirality with $|\psi_\pm\rangle$. Moreover $|\psi_+\rangle$ and $|\psi_-\rangle$ have opposite chirality because the pair of eigenvalues $\widehat{z}_\pm$ is assumed to be real and their difference $|\widehat{z}_+ - \widehat{z}_-|$ non-zero. This can be seen by the eigenvalue equations

$$\begin{aligned} D_{\rm W}\,(\widehat{\lambda}_7,\widehat{m}_6=0)|\psi_\pm\rangle &= \pm\sqrt{\widehat{\lambda}_7^2-\lambda_{\rm W}^2}\,|\psi_\pm\rangle, \\ \langle\psi_\pm|D_{\rm W}\,(-\widehat{\lambda}_7,\widehat{m}_6=0) &= \langle\gamma_5 D_{\rm W}\,(-\widehat{\lambda}_7,\widehat{m}_6=0)\gamma_5\psi_\pm| = \mp\sqrt{\widehat{\lambda}_7^2-\lambda_{\rm W}^2}\,\langle\psi_\pm|. \end{aligned} \tag{64}$$

In the second equation we used the γ_5-Hermiticity of $D_{\rm W}$. We multiply the first equation with $\langle\psi_\pm|$ and the second with $|\psi_\pm\rangle$ and employ the normalization of the eigenmodes such that we find

$$\begin{aligned} \langle\psi_\pm|D_{\rm W}\,(\widehat{\lambda}_7,\widehat{m}_6=0)|\psi_\pm\rangle &= \pm\sqrt{\widehat{\lambda}_7^2-\lambda_{\rm W}^2}\,, \\ \langle\psi_\pm|D_{\rm W}\,(-\widehat{\lambda}_7,\widehat{m}_6=0)|\psi_\pm\rangle &= \mp\sqrt{\widehat{\lambda}_7^2-\lambda_{\rm W}^2}\,. \end{aligned} \tag{65}$$

We subtract the second line from the first and use the identity $D_{\rm W}\,(\widehat{\lambda}_7,\widehat{m}_6=0) - D_{\rm W}\,(-\widehat{\lambda}_7,\widehat{m}_6=0) = 2\widehat{\lambda}_7\gamma_5$, i.e.

$$\widehat{\lambda}_7\langle\psi_\pm|\gamma_5|\psi_\pm\rangle = \pm\sqrt{\widehat{\lambda}_7^2-\lambda_{\rm W}^2}\,, \tag{66}$$

which indeed shows the opposite chirality of $|\psi_+\rangle$ and $|\psi_-\rangle$. Thus $|\psi_+\rangle$ and $\gamma_5|\psi_-\rangle$ have opposite sign of chirality but their corresponding eigenvalues are the same. Therefore the average of their chiralities at a specific eigenvalue vanishes.

The density of the complex eigenvalues can be obtained by integrating over those $\lambda_{\rm W}$ fulfilling the condition $|\lambda_{\rm W}| > |\widehat{\lambda}_7|$. After averaging over $\widehat{m}_6$ and $\widehat{\lambda}_7$ we find

$$\begin{aligned} \rho_{\rm c}(\widehat{z}=\widehat{x}+i\widehat{y},\widehat{a}_8=0) &= \frac{\exp\left[-\widehat{x}^2/(16\widehat{a}_6^2)\right]}{16\pi|\widehat{a}_6\widehat{a}_7|}\int\limits_{\mathbb{R}^2}\rho_{\rm NZ}(\lambda_{\rm W})\exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}\right]\delta\left(\sqrt{\lambda_{\rm W}^2-\widehat{\lambda}_7^2}-|\widehat{y}|\right)\Theta(|\lambda_{\rm W}|-|\widehat{\lambda}_7|)d\lambda_{\rm W}\,d\widehat{\lambda}_7 \\ &= \frac{\exp\left[-\widehat{x}^2/(16\widehat{a}_6^2)\right]}{4\pi|\widehat{a}_6\widehat{a}_7|}\int\limits_{|\widehat{y}|}^{\infty}\frac{|\widehat{y}|\rho_{\rm NZ}(\lambda_{\rm W})d\lambda_{\rm W}}{\sqrt{\lambda_{\rm W}^2-\widehat{y}^2}}\exp\left[\frac{\lambda_{\rm W}^2-\widehat{y}^2}{16\widehat{a}_7^2}\right]. \end{aligned} \tag{67}$$

The original continuum result is smoothened by a distribution with a Gaussian tail. The oscillations in the microscopic spectral density dampen due to a non-zero W_7 similar to the effect of a non-zero value W_8, cf. Ref. [23]. We also expect a loss of the height of the first eigenvalue distributions around the origin. Pairs of eigenvalues are moving from the imaginary axis into the real axis and thus lowering their probability density on the imaginary axis. The density $\rho_{\rm c}$ for non-zero $\widehat{a}_8$ will be discussed in full detail in Sec. IV B 2.

The density of the additional real modes can be obtained by integrating the continuum distribution, $\rho_{\rm NZ}$ over $|\lambda_{\rm W}| < |\lambda_7|$ analogous to the complex case. We find

$$\begin{aligned} \rho_{\rm r}(\widehat{x},W_8=0) &= \frac{1}{16\pi|\widehat{a}_6\widehat{a}_7|}\int\limits_{\mathbb{R}^3}\rho_{\rm NZ}(\lambda_{\rm W})\exp\left[-\frac{\widehat{m}_6^2}{16\widehat{a}_6^2}-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}\right]\delta\left(\sqrt{\widehat{\lambda}_7^2-\lambda_{\rm W}^2}-|\widehat{m}_6-\widehat{x}|\right)\Theta(|\widehat{\lambda}_7|-|\lambda_{\rm W}|)d\lambda_{\rm W}\,d\widehat{\lambda}_7 d\widehat{m}_6 \\ &= \int\limits_{\mathbb{R}^2}\frac{|\widehat{m}_6|d\widehat{m}_6 d\lambda_{\rm W}}{8\pi|\widehat{a}_6\widehat{a}_7|\sqrt{\lambda_{\rm W}^2+\widehat{m}_6^2}}\rho_{\rm NZ}(\lambda_{\rm W})\exp\left[-\frac{\lambda_{\rm W}^2+\widehat{m}_6^2}{16\widehat{a}_7^2}-\frac{(\widehat{m}_6+\widehat{x})^2}{16\widehat{a}_6^2}\right]. \end{aligned} \tag{68}$$

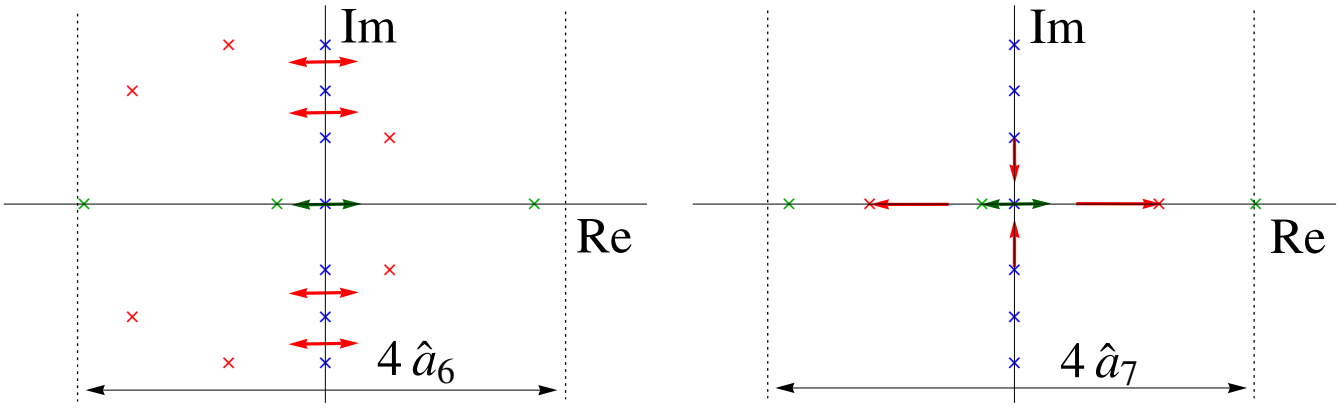

FIG. 1. Schematic plots of the effects of W_6 (left plot) and of W_7 (right plot). The low energy constant W_6 broadens the spectrum parallel to the real axis according to a Gaussian with width $4\widehat{a}_6 = 4\sqrt{-VW_6}\widetilde{a}^2$, but does not change the continuum spectrum in a significant way. When W_7 is switched on and $W_6 = 0$ the purely imaginary eigenvalues invade the real axis through the origin and only the real (green crosses along the real axis) are broadened by a Gaussian with width $4\widehat{a}_7 = 4\sqrt{-VW_7}\widetilde{a}^2$.

The number of additional real modes given by the integral of $\rho_{\rm r}(\widehat{x})$ over $\widehat{x}$ only depends on $\widehat{a}_7$, as it should be since $\widehat{m}_6$ is just an additive constant to the eigenvalues. Moreover $\rho_{\rm r}$ will inherit the oscillatory behavior of $\rho_{\rm NZ}$ although most of it will be damped by the Gaussian cut-off. The mixture of this effect with the effect of a non-zero W_8 is highly non-trivial, but we expect that, at small lattice spacings, we can separate both contributions. For a sufficiently small value of $\widehat{a}_6$ the behavior of $\rho_{\rm r}(\widehat{x})$ for $\widehat{x} \to 0$ is given by $\rho_{\rm r}(\widehat{x}) = \widetilde{c}|\widehat{x}| + \ldots$ with $\widetilde{c} > 0$ for vanishing W_8 and, thus, $\rho_{\rm r}(\widehat{x}) = c_0 + c_1\widehat{x}^2 + \ldots$ with $c_0, c_1 > 0$ for non-zero W_8. Hence, we will see a soft repulsion of the additional real eigenvalues from the origin which still allows real eigenvalues to be zero.

The discussion of the real modes for non-zero $\widehat{a}_8$ as well is given in Sec. IV B 1.

B. Eigenvalue densities for non-zero values of W_6, W_7 and W_8

In this subsection all three low-energy constants are non-zero. As in the previous subsection, we will consider the density of the real eigenvalues of $D_{\rm W}$, the density of the complex eigenvalues of $D_{\rm W}$, and the distribution of the chiralities over the real eigenvalues of $D_{\rm W}$. The expressions for these distributions were already given in section III, but in this section we further simplify them and calculate the asymptotic expressions for large and small values of $\widehat{a}$.

1. *Density of the additional real modes*

The quenched eigenvalue density of the additional real modes is given by Eq. (45). The Gaussian average over the variables $\widehat{m}_6$ and $\widehat{\lambda}_7$ can be worked out analytically. The result is given by (see Appendix B for integrals that were used to obtain this result)

$$\rho_{\rm r}(\widehat{x}) = \frac{1}{16\pi^2} \int\limits_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1 - \varphi_2}{2}\right] e^{\imath\nu(\varphi_1+\varphi_2)} \frac{\widetilde{k}(\widehat{x},\varphi_1,\varphi_2) - \widetilde{k}(\widehat{x},\varphi_2,\varphi_1)}{\cos\varphi_2 - \cos\varphi_1} \tag{69}$$

with

$$\widetilde{k}(\widehat{x},\varphi_1,\varphi_2) = \exp\left[4\widehat{a}_6^2(\cos\varphi_1 - \cos\varphi_2)^2 - 4\widehat{a}_7^2(\sin\varphi_1 + \sin\varphi_2)^2 + 4\widehat{a}_8^2\left(\cos\varphi_1 - \frac{\widehat{x}}{8\widehat{a}_8^2}\right)^2 - 4\widehat{a}_8^2\left(\cos\varphi_2 - \frac{\widehat{x}}{8\widehat{a}_8^2}\right)^2\right]$$
$$\times\left[\mathrm{erf}\left[\frac{\widehat{x} - 8(\widehat{a}_6^2 + \widehat{a}_8^2)\cos\varphi_1 + 8\widehat{a}_6^2\cos\varphi_2}{\sqrt{8(\widehat{a}_8^2 + 2\widehat{a}_6^2)}}\right] + \mathrm{erf}\left[\frac{8(\widehat{a}_6^2 + \widehat{a}_8^2)\cos\varphi_1 - 8\widehat{a}_6^2\cos\varphi_2 - 8\imath\widehat{a}_7^2\sin\varphi_1 - 8\imath\widehat{a}_7^2\sin\varphi_2 - \widehat{x}}{\sqrt{16(\widehat{a}_8^2 + \widehat{a}_6^2 + \widehat{a}_7^2)}}\right]\right]. \tag{70}$$

The effect of each low energy constant on $\rho_{\rm r}$ is shown in Fig. 2.

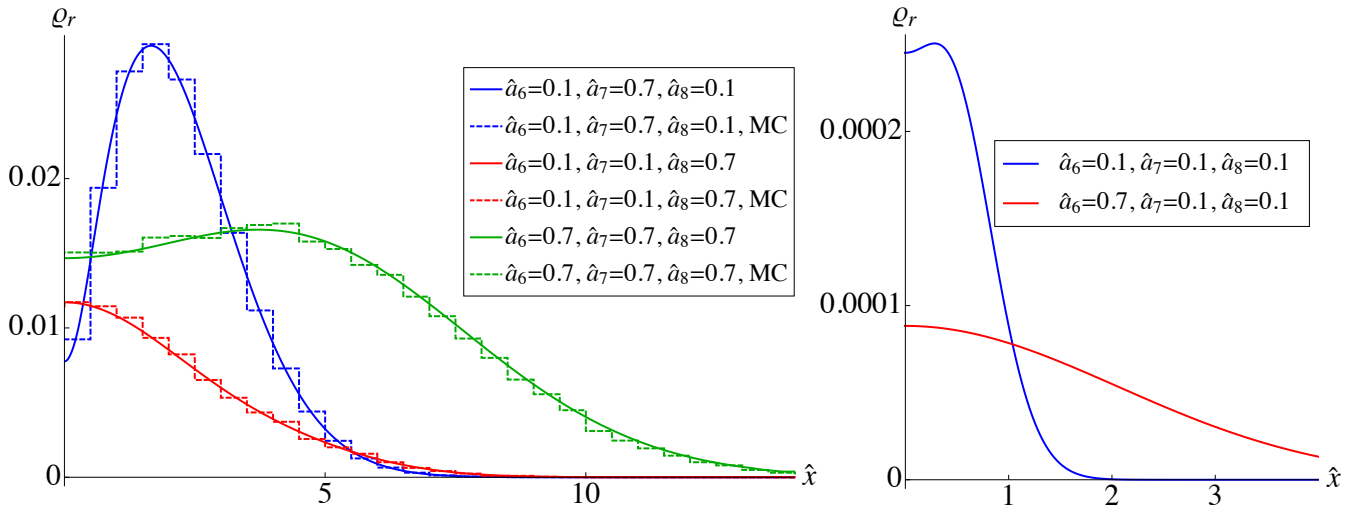

FIG. 2. The density of additional real modes is shown for various parameters $\widehat{a}_{6/7/8}$. The analytical results (solid curves) agree with the Monte Carlo simulations of the Random Matrix Theory (histogram [MC] with bin size 0.5 and with different ensemble and matrix sizes such that statistics are about 1-5%) for $\nu = 1$. We plot only the positive real axis since $\rho_{\rm r}$ is symmetric. Notice that the height of the two curves for $\widehat{a}_7 = \widehat{a}_8 = 0.1$ (right plot) are two orders smaller than the height of the other curves (left plot) and because of bad statistics we have not performed simulations for this case. Notice the soft repulsion of the additional real modes from the origin at large $\widehat{a}_7 = \sqrt{-VW_7}\widetilde{a}$ as discussed in the introductory section. The parameter $\widehat{a}_6 = \sqrt{-VW_6}\widetilde{a}$ smoothens the distribution.

At small lattice spacing, $\widehat{a} \ll 1$, the density $\rho_{\rm r}$ has support on the scale of $\widehat{a}$. In particular it is given by derivatives of a specific function, i.e.

$$\rho_{\rm r}(\widehat{x}) \overset{\widehat{a}\ll 1}{=} \frac{1}{4}\left(\frac{1}{(\nu!)^2}\frac{\partial^{2\nu}}{\partial t_1^\nu \partial t_2^\nu} - \frac{1}{(\nu-1)!(\nu+1)!}\frac{\partial^{2\nu}}{\partial t_1^{\nu-1}\partial t_2^{\nu+1}}\right)\Big|_{t_1=t_2=0} \frac{\widehat{k}(\widehat{x},t_1,t_2)-\widehat{k}(\widehat{x},t_2,t_1)}{t_2-t_1}, \tag{71}$$

where

$$\begin{aligned}\widehat{k}(\widehat{x},t_1,t_2) = \exp&\left[\widehat{a}_6^2(t_1-t_2)^2+\widehat{a}_7^2(t_1+t_2)^2+\widehat{a}_8^2\left(t_1-\frac{\widehat{x}}{4\widehat{a}_8^2}\right)^2-\widehat{a}_8^2\left(t_2-\frac{\widehat{x}}{4\widehat{a}_8^2}\right)^2\right]\\ &\times\left[\mathrm{erf}\left[\frac{\widehat{x}-4(\widehat{a}_6^2+\widehat{a}_8^2)t_1+4\widehat{a}_6^2t_2}{\sqrt{8(\widehat{a}_8^2+2\widehat{a}_6^2)}}\right]+\mathrm{erf}\left[\frac{4(\widehat{a}_6^2+\widehat{a}_7^2+\widehat{a}_8^2)t_1-4(\widehat{a}_6^2-\widehat{a}_7^2)t_2-\widehat{x}}{\sqrt{16(\widehat{a}_8^2+\widehat{a}_6^2+\widehat{a}_7^2)}}\right]\right].\end{aligned} \tag{72}$$

The error functions guarantee a Gaussian tail on the scale of $\widehat{a}$. Furthermore, the height of the density is of order $\widehat{a}^{2\nu+1}$. Hence, additional real modes are strongly suppressed for $\nu > 0$ and the important contributions only result from $\nu = 0$. This behavior becomes clearer for the expression of the average number of the additional real modes. This quantity directly follows from the result (70),

$$\begin{aligned}N_{\rm add} &= 2\int_{-\infty}^{\infty}\rho_{\rm r}(\widehat{x})d\widehat{x} \\ &= \int_0^{2\pi}\frac{d\Phi}{4\pi}\cos[2\nu\Phi]\frac{1-\exp\left[-(4\widehat{a}_8^2+8\widehat{a}_7^2)\sin^2\Phi\right]I_0\left[(4\widehat{a}_8^2-8\widehat{a}_7^2)\sin^2\Phi\right]}{\sin^2\Phi} \\ &= \sum_{n=\nu+1}^{\infty}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^{\nu-1+n}\frac{(2n-2)!\left(\widehat{a}_8^2-2\widehat{a}_7^2\right)^{2j}\left(\widehat{a}_8^2+2\widehat{a}_7^2\right)^{n-2j}}{2^{2j-1}\Gamma(n-\nu)\Gamma(n+\nu)\Gamma(n-2j+1)(j!)^2},\end{aligned} \tag{73}$$

where the symbol $\lfloor n/2\rfloor$ denotes the largest integer smaller than or equal to $n/2$.

The average number of the real modes does not depend on the low energy constant $W_6 = -\widehat{a}_6^2/(\widetilde{a}^2V)$ because this constant induces overall fluctuations of the Dirac spectrum parallel to the $\widehat{x}$-axis.

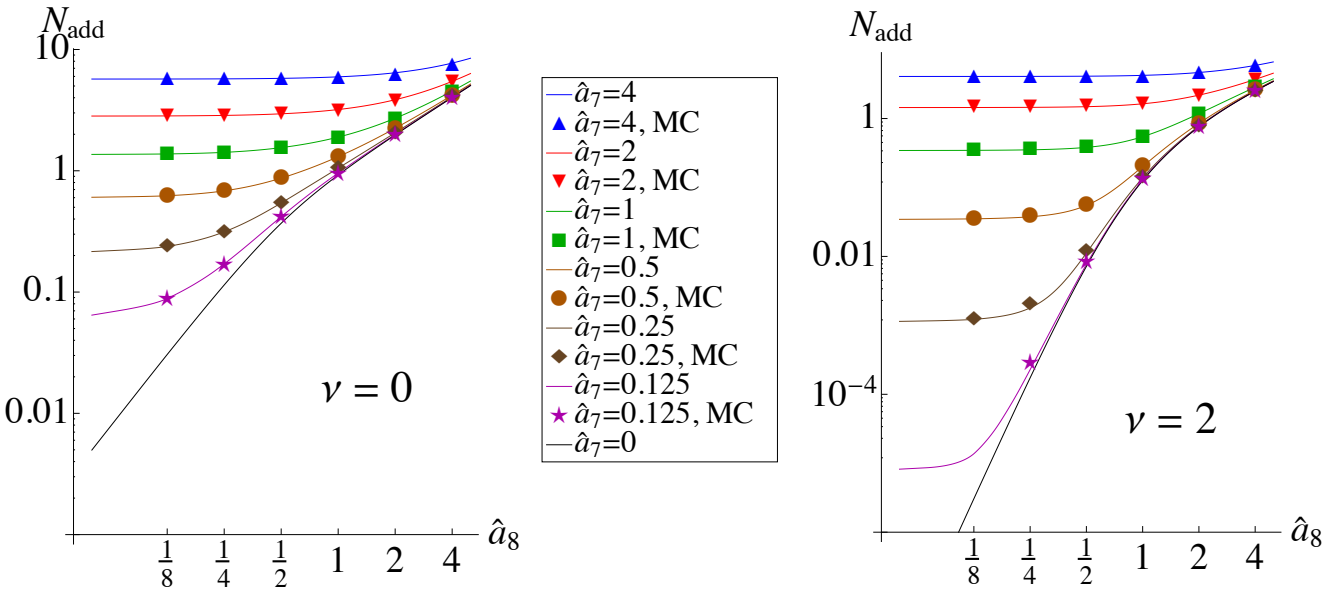

FIG. 3. Log-log plots of $N_{\rm add}$ as a function of $\widehat{a}_8 = \sqrt{VW_8\widetilde{a}^2}$ for $\nu = 0$ (left plot) and $\nu = 2$ (right plot). The analytical results (solid curves) are compared to Monte Carlo simulations of RMT (symbols; ensemble and matrix size varies such that the statical error is about 1-5%). Notice that W_6 has no effect on $N_{\rm add}$. The saturation around zero is due to a non-zero value of $\widehat{a}_7 = \sqrt{-VW_7\widetilde{a}^2}$. For $\widehat{a}_7 = 0$ (lowest curves) the average number of additional real modes behaves like $\widehat{a}_8^{2\nu+2}$, see Ref. [23].

The asymptotics of $N_{\rm add}$ at small and large lattice spacing is given by

$$N_{\rm add} = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor \nu/2 \rfloor} \frac{\left(\widehat{a}_8^2 - 2\widehat{a}_7^2\right)^{2j} \left(\widehat{a}_8^2 + 2\widehat{a}_7^2\right)^{\nu-2j+1}}{2^{2j-1}\Gamma(\nu-2j+2)(j!)^2} \propto \widehat{a}^{2\nu+2}, & \widehat{a} \ll 1, \\ \displaystyle\sqrt{\frac{64\widehat{a}_7^2}{\pi^3}} E\left(\sqrt{1 - \frac{\widehat{a}_8^2}{2\widehat{a}_7^2}}\right) \propto \widehat{a}, & \widehat{a} \gg 1, \end{cases} \tag{74}$$

see Appendix D 1 for a derivation. The function E is the elliptic integral of the second kind, i.e

$$E(x) = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2\varphi}\, d\varphi. \tag{75}$$

In Ref. [23] this result was derived for $\widehat{a}_6 = \widehat{a}_7 = 0$. Notice that for large lattice spacings the number of additional real modes increases linearly with $\widehat{a}$ and is independent of ν.

The average number of additional real modes can be used to fix the low energy constants from lattice simulations. For $\nu = 0$, a sufficient number of eigenvalues [61] can be generated to keep the statistical error small. For $\nu = 0$ and $\nu = 1$ the average number of additional real modes is given by

$$N_{\rm add}^{\nu=0} \overset{\widehat{a}\ll 1}{=} 2(\widehat{a}_8^2 + 2\widehat{a}_7^2) = 2V\widetilde{a}^2(W_8 - 2W_7), \tag{76}$$

$$N_{\rm add}^{\nu=1} \overset{\widehat{a}\ll 1}{=} (\widehat{a}_8^2 + 2\widehat{a}_7^2)^2 + \frac{1}{2}(\widehat{a}_8^2 - 2\widehat{a}_7^2)^2 = V^2\widetilde{a}^4 \left[(W_8 - 2W_7)^2 + \frac{1}{2}(W_8 + 2W_7)^2\right]. \tag{77}$$

These simple relations can be used to fit lattice data at small lattice spacing. In Fig. 3 we illustrate the behavior of $N_{\rm add}$ by a log-log plot.

The density $\rho_{\rm r}$ takes a much simpler form at large lattice spacing. Then, the integrals can be evaluated by a saddle

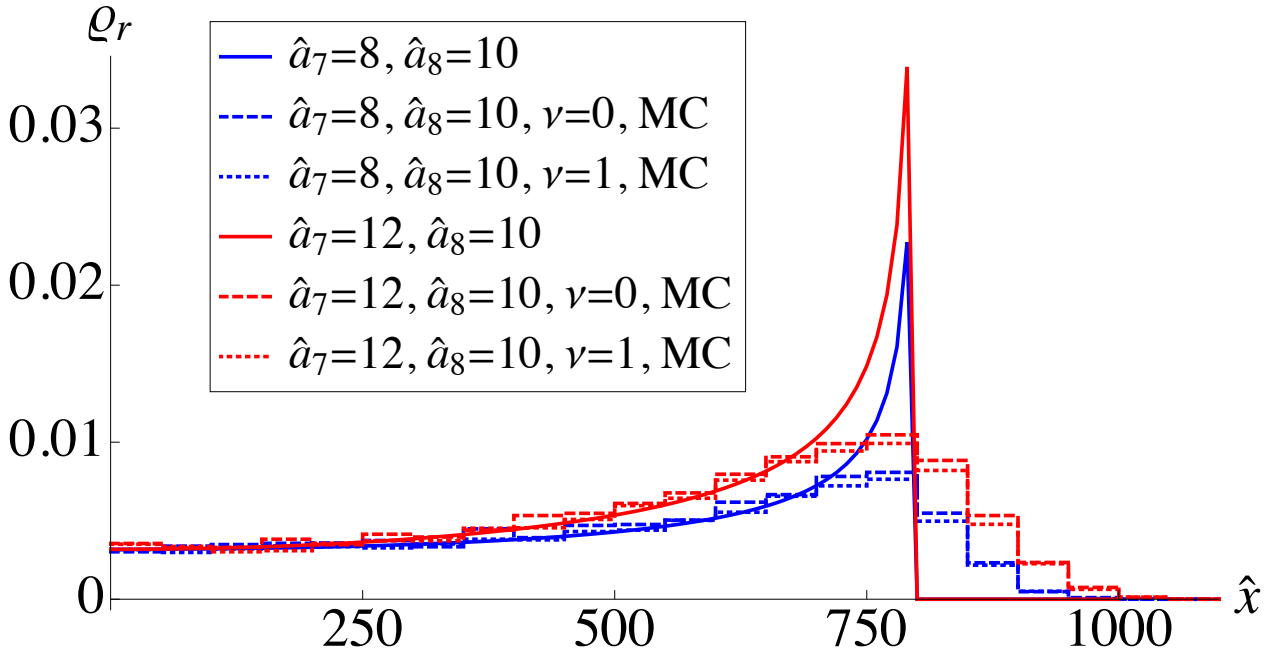

FIG. 4. At large lattice spacing the density of additional real modes develops square root singularities at the boundaries. The analytical results at $\widehat{a} \to \infty$ (solid curves) are compared to Monte Carlo simulations at non-zero, but large lattice spacing (histogram [MC], with bin size 50, $\widehat{a}_6 = \sqrt{-VW_6\widetilde{a}^2} = 0.01$ and $n = 2000$ for an ensemble of 1000 matrices). Due to the finite matrix size and the finite lattice spacing, $\rho_{\rm r}$ has a tail which drops off much faster than the size of the support. The low energy constant $\widehat{a}_8 = \sqrt{VW_8\widetilde{a}^2}$ is chosen equal to 10. Therefore the boundary is at $\widehat{x} = 800$ which is confirmed by the Monte Carlo simulations. The dependence on W_6 and ν is completely lost.

point approximation resulting in the expression (see Appendix D 2)

$$\rho_{\rm r}(\widehat{x}) \overset{\widehat{a}\gg 1}{=} \begin{cases} \dfrac{1}{8\pi^2\widehat{a}_7\widehat{a}_6}\displaystyle\int_0^\infty d\widetilde{x}\cosh\left(\frac{\widetilde{x}\widehat{x}}{8\widehat{a}_6^2}\right)K_0\left(\frac{\widetilde{x}^2}{32\widehat{a}_7^2}\right)\widetilde{x}\exp\left[-\frac{\widetilde{x}^2}{32\widehat{a}_7^2}-\frac{\widetilde{x}^2+\widehat{x}^2}{16\widehat{a}_6^2}\right], & \widehat{a}_8 = 0, \\ \dfrac{\Theta(8\widehat{a}_8^2-|\widehat{x}|)}{2(2\pi)^{3/2}\widehat{a}_8^2}\sqrt{\widehat{a}_8^2+2\widehat{a}_7^2\dfrac{\widehat{x}^2}{(8\widehat{a}_8^2)^2-\widehat{x}^2}}, & \widehat{a}_8 \neq 0. \end{cases} \tag{78}$$

Notice that we have square root singularities at the two edges of the support if both $\widehat{a}_7 \neq 0$ and $\widehat{a}_8 \neq 0$, cf. Fig. 4. So the effect of the low energy constant W_7 is different than what we would have expected naively.

2. *Density of the complex eigenvalues*

The expression for the density of the complex eigenvalues given in Eq. (46) can be simplified by performing the integral of $\widehat{m}_6$ and $\widehat{\lambda}_7$ resulting in

$$\rho_{\rm c}(\widehat{z}) = \frac{|\widehat{y}|}{2(2\pi)^{5/2}\sqrt{\widehat{a}_8^2+2\widehat{a}_6^2}}\int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1-\varphi_2}{2}\right]\cos[\nu(\varphi_1+\varphi_2)]\mathrm{sinc}\left[\widehat{y}(\cos\varphi_1-\cos\varphi_2)\right] \tag{79}$$

$$\times\exp\left[-4\widehat{a}_8^2\left(\left(\cos\varphi_1-\frac{\widehat{x}}{8\widehat{a}_8^2}\right)^2+\left(\cos\varphi_2-\frac{\widehat{x}}{8\widehat{a}_8^2}\right)^2\right)+\frac{4\widehat{a}_6^2\widehat{a}_8^2}{\widehat{a}_8^2+2\widehat{a}_6^2}\left(\cos\varphi_1+\cos\varphi_2-\frac{\widehat{x}}{4\widehat{a}_8^2}\right)^2-4\widehat{a}_7^2(\sin\varphi_1+\sin\varphi_2)^2\right].$$

The function $\mathrm{sinc}(x) = \sin x/x$ is the *sinus cardinalis*. This result reduces to the expressions obtained in Ref. [23] for $\widehat{a}_6 = \widehat{a}_7 = 0$.

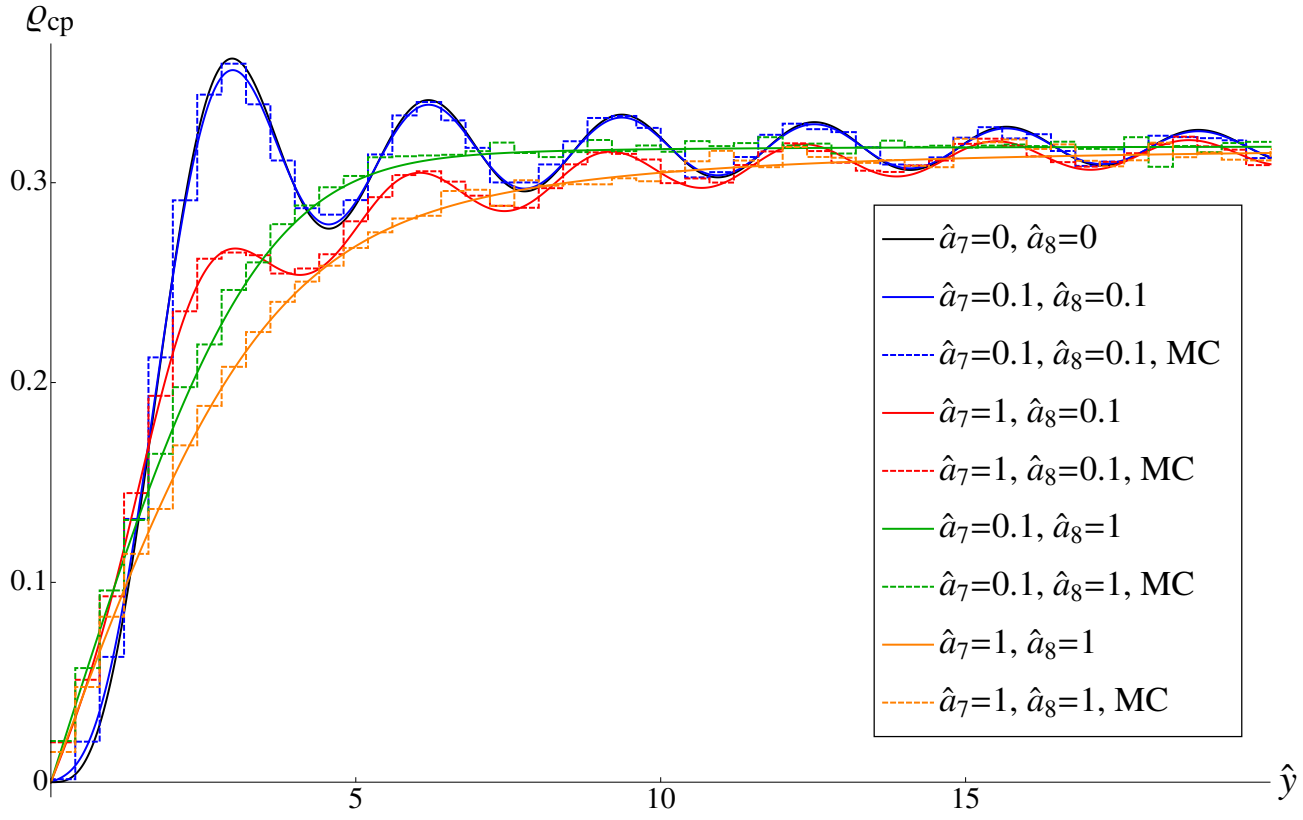

FIG. 5. Comparison of the analytical result (solid curves) and Monte Carlo simulations of the Random Matrix Theory (histogram [MC] with bin size equal to 0.4 and varying ensemble size and matrix size such that the statistical error is about 1-5%) for the density of the complex eigenvalues projected onto the imaginary axis. The index of the Wilson Dirac operator is $\nu = 1$ for all curves. Notice that $\widehat{a}_6 = \sqrt{-VW_6}\widetilde{a}$ does not affect this density. The comparison of $\widehat{a}_7 = \widehat{a}_8 = 0.1$ with the continuum result (black, thick curve) shows that $\rho_{\rm cp}$ is still a good quantity to extract the chiral condensate Σ at small lattice spacing.

To compare to numerical simulations it is useful to consider the projection of the complex modes onto the imaginary axis. The result for the projected eigenvalue density can be simplified to

$$\begin{aligned}\rho_{\rm cp}(\widehat{y}) &= \int_{-\infty}^{\infty} \rho_{\rm c}(\widehat{x}+i\widehat{y})d\widehat{x} \\ &= \frac{|\widehat{y}|}{(2\pi)^2}\int_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1-\varphi_2}{2}\right] \mathrm{sinc}\left[\widehat{y}(\cos\varphi_1-\cos\varphi_2)\right]\cos[\nu(\varphi_1+\varphi_2)] \\ &\quad\times \exp\left[-2\widehat{a}_8^2(\cos\varphi_1-\cos\varphi_2)^2 - 4\widehat{a}_7^2(\sin\varphi_1+\sin\varphi_2)^2\right].\end{aligned} \tag{80}$$

Again this function is independent of W_6 as was the case for $N_{\rm add}$. The reason is that the Gaussian broadening with respect to the mass $\widehat{m}_6$ is absorbed by the integral over the real axis. At small lattice spacing $\rho_{\rm cp}$ approaches the continuum result $\rho_{\rm NZ}$ given in Eq. (58) (see Fig. 5). Therefore it is still a good quantity to determine the chiral condensate Σ from lattice simulations. In Fig. 5, we compare the projected spectral density (solid curves) with numerical results from an ensemble of random matrices (histograms). The spectral density at a couple of lattice spacings away from the origin can be used to determine the chiral condensate according to the Banks-Casher formula.

At small lattice spacing, $\rho_{\rm c}$ factorizes into a Gaussian distribution of the real part of the eigenvalues and of the

level density of the continuum limit,

$$\rho_{\rm c}(\widehat{z}) \overset{\widehat{a}\ll 1}{=} \frac{|\widehat{y}|}{2(2\pi)^{5/2}\sqrt{\widehat{a}_8^2+2\widehat{a}_6^2}}\exp\left[-\frac{\widehat{x}^2}{8(\widehat{a}_8^2+2\widehat{a}_6^2)}\right]\int\limits_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1-\varphi_2}{2}\right]\cos[\nu(\varphi_1+\varphi_2)]{\rm sinc}\left[\widehat{y}(\cos\varphi_1-\cos\varphi_2)\right]$$
$$= \frac{1}{\sqrt{8\pi(\widehat{a}_8^2+2\widehat{a}_6^2)}}\exp\left[-\frac{\widehat{x}^2}{8(\widehat{a}_8^2+2\widehat{a}_6^2)}\right]\rho_{\rm NZ}(\widehat{y}). \tag{81}$$

Therefore the support of $\rho_{\rm c}$ along the real axis is on the scale $\widehat{a}$ while it is of order 1 along the imaginary axis. It also follows from perturbation theory in the non-Hermitian part of the Dirac operator that the first order correction to the continuum result is a Gaussian broadening perpendicular to the imaginary axis. The width of the Gaussian can be used to determine the combination $\widehat{a}_8^2+2\widehat{a}_6^2 = V\widetilde{a}^2(W_8-2W_6)$ from fitting the results to lattice simulations. Since most of the eigenvalues of $D_{\rm W}$ occur in complex conjugate pairs at small lattice spacing, it is expected to have a relatively small statistical error in this limit. A further reduction of the statistical error can be achieved by integrating the spectral density over $\widehat{y}$ up to the Thouless energy (see Ref. [59] for a definition of the Thouless energy in QCD).

The behavior drastically changes in the limit of large lattice spacing. Then the density reads (see Appendix D 3)

$$\rho_{\rm c}(\widehat{z}) \overset{\widehat{a}\gg 1}{=} \begin{cases} \dfrac{\Theta(8\widehat{a}_8^2-|\widehat{x}|)}{16\pi\widehat{a}_8^2}{\rm erf}\left[\dfrac{|\widehat{y}|}{\sqrt{8\widehat{a}_8^2}}\sqrt{\dfrac{(8\widehat{a}_8^2)^2-\widehat{x}^2}{(8\widehat{a}_8^2)^2-(1-2\widehat{a}_7^2/\widehat{a}_8^2)\widehat{x}^2}}\right], & \widehat{a}_8>0, \\ \dfrac{|\widehat{y}|}{16\pi^2|\widehat{a}_6\widehat{a}_7|}\exp\left[-\dfrac{\widehat{x}^2}{16\widehat{a}_6^2}+\dfrac{\widehat{y}^2}{32\widehat{a}_7^2}\right]K_0\left(\dfrac{\widehat{y}^2}{32\widehat{a}_7^2}\right), & \widehat{a}_8=0. \end{cases} \tag{82}$$

There is no dependence on ν, and in the case of $\widehat{a}_8>0$, the result does not depend on $\widehat{a}_6$ as well and becomes a strip of width $16\widehat{a}_8^2$ along the imaginary axis. To have any structure, the imaginary part of the eigenvalues has to be of order $\widehat{a}$. In the mean field limit, where $|\widehat{y}|/\sqrt{8\widehat{a}_8^2}\gg 1$, $\rho_{\rm c}$ is equal to $1/(16\pi\widehat{a}_8^2)$ on a strip of width $16\widehat{a}_8^2$. Hence, the low energy constants $W_{6/7}$, do not alter the mean field limit of $\rho_{\rm c}$, cf. Ref. [23]. This was already observed in Ref. [13].

The effect of $\widehat{a}_6$ is an overall Gaussian fluctuation perpendicular to the strip of the eigenvalues, and for $\widehat{a}_8=0$, when there is no strip, only the Gaussian fluctuations remain. The second case of Eq. (82) can also be obtained from Eq. (67) since for large $\widehat{y}$, $\rho_{\rm NZ}$ is equal to $1/\pi$.

C. The distribution of chirality over the real eigenvalues

The distribution of chirality over the real eigenvalues given in Eq. (47) is an expression in terms of the graded partition function $Z^\nu_{1/1}$ and the partition function of two fermionic flavors, $Z^\nu_{2/0}$, which is evaluated in Appendix C. Including the integrals over $\widehat{m}_6$ and $\widehat{\lambda}_7$ we obtain from Eq. (C7)

$$\rho_\chi(\widehat{x}) = \frac{(-1)^\nu}{(16\pi)^{3/2}\widehat{a}_8^2|\widehat{a}_7|}\int\limits_{-\infty}^{\infty} d\widehat{\lambda}_7 \int\limits_{\mathbb{R}^2}\frac{ds_1 ds_2}{s_1-\imath s_2}(\imath s_2+\widehat{\lambda}_7)^\nu(s_1-\widehat{\lambda}_7)^\nu \tag{83}$$
$$\times\exp\left[-\frac{1}{16\widehat{a}_8^2}\left((s_1-\widehat{x})^2+(s_2+\imath\widehat{x})^2\right)+\frac{\widehat{a}_6^2}{16\widehat{a}_8^4}(s_1-\imath s_2)^2-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}\right]$$
$$\times\left[\frac{\delta^{(\nu-1)}(s_1+\widehat{\lambda}_7)}{(\nu-1)!(s_1-\widehat{\lambda}_7)^\nu}\left(\frac{s_1^2-\widehat{\lambda}_7^2}{s_2^2+\widehat{\lambda}_7^2}\right)^{\nu/2}Z^\nu_{1/1}\left(\sqrt{s_1^2-\widehat{\lambda}_7^2},\imath\sqrt{s_2^2+\widehat{\lambda}_7^2};\widehat{a}=0\right)\right.$$
$$\left.-{\rm sign}(\widehat{\lambda}_7)\Theta(|\widehat{\lambda}_7|-|s_1|)\left(s_1^2+s_2^2\right)\frac{Z^\nu_{2/0}\left(\sqrt{s_1^2-\widehat{\lambda}_7^2},\imath\sqrt{s_2^2+\widehat{\lambda}_7^2};\widehat{a}=0\right)}{[(s_1^2-\widehat{\lambda}_7^2)(s_2^2+\widehat{\lambda}_7^2)]^{\nu/2}}\right].$$

We recognize the two terms that were obtained in Eqs. (36) and (38) from the expansion in the first column of the determinant in the joint probability density.

Equation (83) is a complicated expression which is quite hard to numerically evaluate. However, it is possible to derive an alternative expression in terms of an integral over the supersymmetric coset manifold $U\in {\rm Gl}(1/1)/{\rm U}(1/1)$.

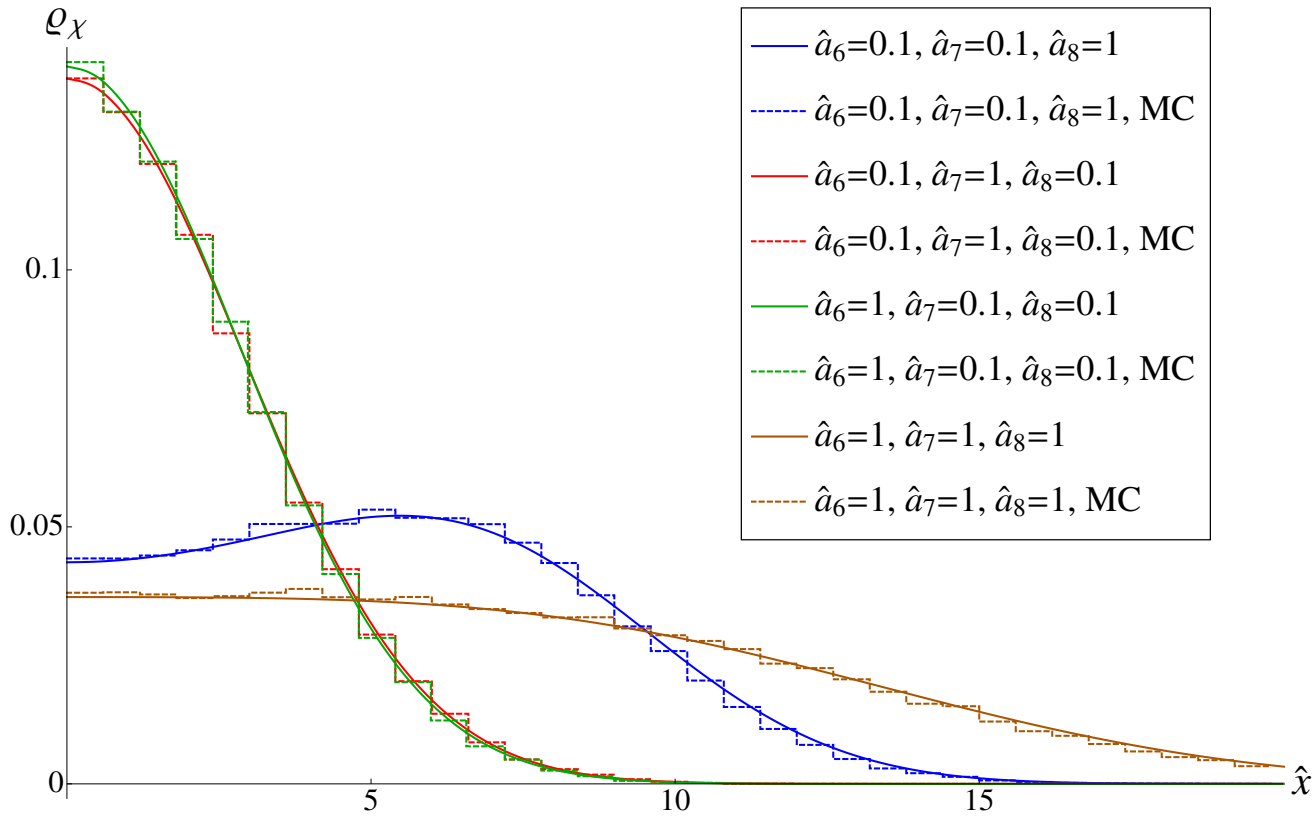

FIG. 6. The analytical result (solid curves) for ρ_χ is compared to Monte Carlo simulations of RMT (histogram [MC] with bin size 0.6 and varying ensemble and matrix size such that the statistical error is about 1-5%) for $\nu = 1$. We plotted only the positive real axis since the distribution is symmetric around the origin. At small $\widehat{a}_8 = \sqrt{VW_8\widetilde{a}^2}$ the distributions for $(\widehat{a}_6, \widehat{a}_7) = (\sqrt{-VW_6\widetilde{a}^2}, \sqrt{-VW_7\widetilde{a}^2}) = (1, 0.1), (0.1, 1)$ are almost the same Gaussian as the analytical result predicts. At large $\widehat{a}_8$ the maximum reflects the predicted square root singularity which starts to build up. We have not included the case $\widehat{a}_{6/7/8} = 0.1$ since it exceeds the other curves by a factor of 10 to 100.

We start from the equality

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2} - \frac{\imath\widehat{\lambda}_7}{2}\mathrm{Str}(U+U^{-1})\right] d\widehat{\lambda}_7 \qquad (84)\\
&= 4\sqrt{\pi}\widehat{a}_7 \exp\left[-\widehat{a}_7^2 \mathrm{Str}^{\,2}(U+U^{-1})\right]\\
&= \exp\left[4\widehat{a}_7^2(\mathrm{Sdet}U + \mathrm{Sdet}U^{-1} - 2)\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2} - \frac{\imath\widehat{\lambda}_7}{2}\mathrm{Str}(U-U^{-1})\right] d\widehat{\lambda}_7\\
&= \sum_{j=-\infty}^{\infty} I_j(8\widehat{a}_7^2)\mathrm{Sdet}^j U e^{-8\widehat{a}_7^2} \int_{-\infty}^{\infty} \exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2} - \frac{\imath\widehat{\lambda}_7}{2}\mathrm{Str}(U-U^{-1})\right] d\widehat{\lambda}_7,
\end{aligned}$$

based on an identity for the Gl(1/1)/U (1/1) graded unitary matrices,

$$\mathrm{Str}^2(U+U^{-1}) = 8 - 4(\mathrm{Sdet}U + \mathrm{Sdet}U^{-1}) + \mathrm{Str}^2(U-U^{-1}), \qquad (85)$$

and the expansion of the generating function for the modified Bessel functions of the first kind, I_j,

$$\exp\left[x\left(t+\frac{1}{t}\right)\right] = \sum_{j=-\infty}^{\infty} I_j(2x)t^j. \qquad (86)$$

This allows us to absorb $\widehat{m}_6$ and $\widehat{\lambda}_7$ by a shift of the eigenvalues of the auxiliary supermatrix σ introduced to linearize the terms quadratic in U. The integral over U can now be identified as a graded 1/1 partition function at $\widehat{a} = 0$ and

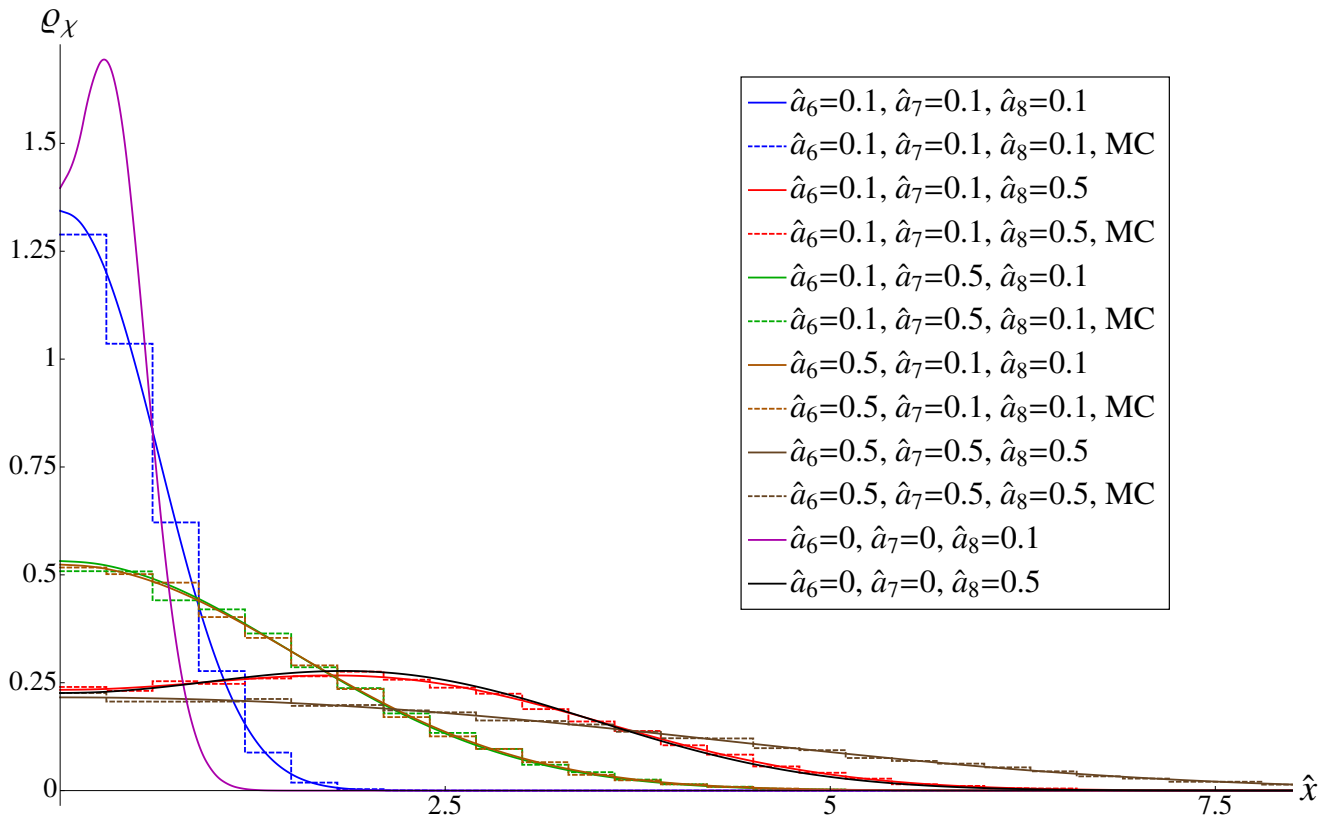

FIG. 7. We compare the analytical result of ρ_χ (solid curves) with Monte Carlo simulations of RMT (histogram [MC] with bin size 0.6 and with varying ensemble and matrix size such that the statistical error is about 1-5%) for $\nu = 2$. Again we only plotted the positive real-axis because ρ_χ is symmetric in the quenched theory. Two curves with $W_{6/7} = 0$ and $W_8 = 0.1, 0.5$ (highest, purple and thick, black curve) are added to emphasize that the two peaks (ρ_χ has to be reflected at the origin) can be strongly suppressed by non-zero $W_{6/7}$ although they are only of the same order as W_8. Recall that the two peaks are relics of a 2×2 GUE which is formed by W_8.

we obtain the result

$$\rho_\chi(\widehat{x}) = \frac{\exp(-8\widehat{a}_7^2)}{16\pi\widehat{a}_8^2} \sum_{j=1}^{\infty} \left(I_{j-\nu}(8\widehat{a}_7^2) - I_{j+\nu}(8\widehat{a}_7^2)\right) \int_{\mathbb{R}^2} \exp\left[-\frac{1}{16\widehat{a}_8^2}\left((s_1-\widehat{x})^2 + (s_2+\imath\widehat{x})^2\right) + \frac{\widehat{a}_6^2+\widehat{a}_7^2}{16\widehat{a}_8^4}(s_1-\imath s_2)^2\right]$$
$$\times \frac{(-|s_1|)^j \delta^{(j-1)}(s_1)}{(j-1)!} Z^j_{1/1}\left(|s_1|, \imath s_2; \widehat{a}=0\right) \frac{ds_1 ds_2}{s_1 - \imath s_2}. \tag{87}$$

Notice that the $j = 0$ term does not contribute to the distribution of chirality over the real modes because of the symmetry of the modified Bessel function $I_\nu = I_{-\nu}$. The derivatives of Dirac delta-function originate from the $\mathrm{Im}[1/(s_1 - \imath\epsilon)^j]$-term.

The representation (87) is effectively a one-dimensional integral due to the Dirac delta-function. Please notice that Eq. (87) reduces to Eq. (59) for $\widehat{a}_8 = 0$. Two plots, Fig. 6 ($\nu = 1$) and Fig. 7, ($\nu = 2$) illustrate the effect of each low-energy constant $\widehat{a}_{6/7/8}$ on the distribution ρ_χ.

For $\widehat{a}_7 = 0$ and $\nu = 1$ one can derive a more compact result in a straightforward way starting from the expression (36). In this case the two-point weight for two real eigenvalues $g_{\mathrm{r}}(x_1, x_2)$ is anti-symmetric in its two arguments, see Eq. (19). Then the integral in Eq. (36) involving $Z^1_{2/0}$ is absent. Employing the representation of the one-flavor partition function as a unitary integral, see Eq. (4), we perform the integral over $\widehat{m}_6$. Thus, $\rho_\chi(\widehat{x})|_{\nu=1}$ can be expressed as

$$\rho_\chi(\widehat{x})|_{\nu=1} = \frac{1}{\sqrt{16\pi(\widehat{a}_8^2+\widehat{a}_6^2)}} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \exp\left[\frac{(\widehat{x}+8\widehat{a}_8^2\sin\theta)^2}{\widehat{a}_8^2+\widehat{a}_6^2}\right]. \tag{88}$$

Let us come back to the general result (87). At small lattice spacing, $0 < \widehat{a} \ll 1$, the distribution ρ_χ as well as the integration variables $s_{1/2}$ are of order $\widehat{a}$. Since $I_j(8\widehat{a}_7^2) \propto \widehat{a}_7^{2j}$, the leading order term is given by $j = \nu$ in the sum over

j. Thus we have

$$\rho_\chi(\widehat{x}) \overset{\widehat{a}\ll 1}{=} \frac{1}{16\pi\widehat{a}_8^2}\int\limits_{\mathbb{R}^2} \exp\left[-\frac{1}{16\widehat{a}_8^2}\left((s_1-\widehat{x})^2+(s_2+\imath\widehat{x})^2\right)+\frac{\widehat{a}_6^2+\widehat{a}_7^2}{16\widehat{a}_8^4}(s_1-\imath s_2)^2\right]$$
$$\times\frac{(-|s_1|)^\nu\delta^{(\nu-1)}(s_1)}{(\nu-1)!}Z^\nu_{1/1}\left(s_1,\imath s_2;\widehat{a}=0\right)\frac{ds_1 ds_2}{s_1-\imath s_2}. \tag{89}$$

In the small $\widehat{a}$ limit we can replace $Z^\nu_{1/1}\left(s_1,\imath s_2;\widehat{a}=0\right)\to(\imath s_2/|s_1|)^\nu$. The result becomes a polynomial in $\widehat{x}^2$ times a Gaussian of width $\sqrt{32(\widehat{a}_8^2+\widehat{a}_6^2+\widehat{a}_7^2)}$. Notice that the polynomial is not the one of a GUE anymore as in the case of $\widehat{a}_6=\widehat{a}_7=0$ [10]. For $\nu=1$, ρ_χ is a pure Gaussian,

$$\rho_\chi^{\nu=1}(\widehat{x}) \overset{\widehat{a}\ll 1}{=} \frac{1}{\sqrt{16\pi(\widehat{a}_8^2+\widehat{a}_6^2+\widehat{a}_7^2)}}\exp\left[-\frac{\widehat{x}^2}{16(\widehat{a}_8^2+\widehat{a}_6^2+\widehat{a}_7^2)}\right], \tag{90}$$

and for $\nu=2$ it is given by

$$\rho_\chi^{\nu=2}(\widehat{x}) \overset{\widehat{a}\ll 1}{=} \frac{1}{\sqrt{16\pi(\widehat{a}_8^2+\widehat{a}_6^2+\widehat{a}_7^2)^3}}\left[\widehat{a}_8^2+2(\widehat{a}_6^2+\widehat{a}_7^2)+\frac{\widehat{a}_8^2}{8(\widehat{a}_8^2+\widehat{a}_6^2+\widehat{a}_7^2)}\widehat{x}^2\right]\exp\left[-\frac{\widehat{x}^2}{16(\widehat{a}_8^2+\widehat{a}_6^2+\widehat{a}_7^2)}\right]. \tag{91}$$

At small lattice spacing, ρ_χ only depends on the combinations $\widehat{a}_8^2$ and $(\widehat{a}_6^2+\widehat{a}_7^2)$. Therefore it is in principle possible to determine the two following combinations of low energy constants, W_8 and W_6+W_7, by fitting ρ_χ to lattice results. For example the second moment (variance) of ρ_χ given by

$$\frac{1}{\nu}\int\limits_{-\infty}^{\infty}\rho_\chi(\widehat{x})\widehat{x}^2 d\widehat{x} \overset{\widehat{a}\ll 1}{=} 8(\nu\widehat{a}_8^2+\widehat{a}_6^2+\widehat{a}_7^2) = 8V\widetilde{a}^2(\nu W_8-W_6-W_7),\ \nu>0, \tag{92}$$

at small lattice spacing can be used to fit the combinations $\nu W_8-W_6-W_7$. The statistical error in this quantity scales with the inverse square root of the number of configurations with the index ν. The ensemble of configurations generated in Ref. [7] yields a statistical error of about two to three percent. The statistics can be drastically increased by performing a fit of the variance of ρ_χ to a linear function in the index ν, cf. Eq. (92). The slope is then determined by W_8 and the off-set by W_6+W_7 yielding two important quantities.

In Appendix D 4 we calculate ρ_χ in the limit of large lattice spacing. Then the distribution of chirality over the real eigenvalues has a support on the scale of $\widehat{a}^2$. The function ρ_χ reads

$$\rho_\chi(\widehat{x}) \overset{\widehat{a}\gg 1}{=} \begin{cases} \dfrac{\nu}{\pi}\dfrac{\Theta(8\widehat{a}_8^2-|\widehat{x}|)}{\sqrt{(8\widehat{a}_8^2)^2-\widehat{x}^2}}, & \widehat{a}_8>0, \\ \dfrac{\nu}{\sqrt{16\pi(\widehat{a}_6^2+\widehat{a}_7^2)}}\exp\left[-\dfrac{\widehat{x}^2}{16(\widehat{a}_6^2+\widehat{a}_7^2)}\right], & \widehat{a}_8=0. \end{cases} \tag{93}$$

Interestingly, the low energy constants $W_{6/7}$ have no effect on the behavior of ρ_χ in this limit if $\widehat{a}_8\neq 0$ which is completely different in comparison to $\rho_{\rm r}$ and $\rho_{\rm c}$. The square root singularities at the boundary of the support are unexpected and were already mentioned in Ref. [23].

V. CONCLUSIONS

Starting from RMT for the Wilson Dirac operator, we have derived the microscopic limit of the spectral density and the distribution of the chiralities over the Dirac spectrum. We have focused on the quenched theory, but all arguments can be simply extended to dynamical Wilson fermions. Wilson RMT is equivalent to the ϵ-limit of the Wilson chiral Lagrangian and describes the Wilson QCD partition function and Dirac spectra in this limit. The starting point of our analytical calculations is the joint probability density of the random matrix ensemble for the non-Hermitian Wilson-Dirac operator $D_{\rm W}$. This density was first obtained in Ref. [23], but a detailed derivation is given in this paper, see Appendix A.

More importantly, we studied in detail the effect of the three low energy constants, $W_{6/7/8}$, on the quenched microscopic level density of the complex eigenvalues, the additional real eigenvalues and the distribution of chirality over the real eigenvalues. In terms of the effect on the spectrum of $D_{\rm W}$, the low energy constants W_6 and W_7 are structurally different from W_8. The first two can be interpreted in terms of "collective" fluctuations of the eigenvalues, whereas a non-zero W_8 induces stochastic interactions between all modes, particularly those with different chiralities.

Therefore, the effect of a non-zero W_6 and W_7 at $W_8 = 0$ is just a Gaussian broadening of the Dirac spectrum on the scale of $\widehat{a}$. When $a^2 V W_8 \gg 1$ the interactions between the modes result in a strip of Dirac eigenvalues in the complex plane with real part inside the interval $[-8VW_8\widetilde{a}^2, 8VW_8\widetilde{a}^2]$. The structure along the imaginary axis is on the scale $\widehat{a}$. As was already discussed in Ref. [13], in the mean field limit, the lattice spacing $\widetilde{a}^2 V$ and the eigenvalues $V\widetilde{z}$ fixed, this structure becomes a box-like strip with hard edges at the boundary of the support and with height $1/(16\pi VW_8\widetilde{a}^2)$.

We also discussed the limit of small lattice spacing, i.e. the limit $|VW_{6/7/8}|\widetilde{a}^2 \ll 1$. In practice, this limit is already reached when $|VW_{6/7/8}|\widetilde{a}^2 \leq 0.1$. Such values can be indeed achieved via clover improvement as discussed in Ref. [7]. In the small $\widehat{a}$ limit we have identified several quantities that are suitable to fit the four low energy constants, $W_{6/7/8}$ and Σ, to lattice simulations and our analytical results.

Several promising quantities are (applicable **only** at small lattice spacing):

- According to the Banks-Casher formula we have

$$\Delta \overset{\widehat{a}\ll 1}{=} \frac{\pi}{\Sigma V}. \tag{94}$$

  for the average spacing Δ of the imaginary part of the eigenvalues several eigenvalue spacings from the origin.

- The average number of the additional real modes for $\nu = 0$:

$$N_{\rm add}^{\nu=0} \overset{\widehat{a}\ll 1}{=} 2V\widetilde{a}^2(W_8 - 2W_7). \tag{95}$$

- The width of the Gaussian shaped strip of complex eigenvalues:

$$\frac{\sigma^2}{\Delta^2} \overset{\widehat{a}\ll 1}{=} \frac{4}{\pi^2}\widetilde{a}^2 V(W_8 - 2W_6). \tag{96}$$

- The variance of the distribution of chirality over the real eigenvalues:

$$\frac{\langle \widetilde{x}^2\rangle_{\rho_\chi}}{\Delta^2} \overset{\widehat{a}\ll 1}{=} \frac{8}{\pi^2} V\widetilde{a}^2(\nu W_8 - W_6 - W_7),\ \nu > 0. \tag{97}$$

These quantities are easily accessible in lattice simulations. We believe they will lead to an improvement of the fits performed in Refs. [4, 6, 7]. Note that ρ_χ is close to the density of the real eigenvalues in the limit of small lattice spacing (again we mean by this $|VW_{6/7/8}|\widetilde{a} \approx 0.1$ and smaller). This statement is not true in the limit of large lattice spacing where the density of the additional real modes dominates the density of the real eigenvalues.

The relations (94-97) are an over-determined set for the low energy constants $W_{6/7/8}$ and Σ^2 and are only consistent if we have relations between these quantities. This can be seen by writing the relations as

$$\widetilde{a}^2 V \begin{bmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} W_6 \\ W_7 \\ W_8 \end{bmatrix} = \frac{\pi^2}{8} \begin{bmatrix} 4N_{\rm add}^{\nu=0}/\pi^2 \\ 2\sigma^2/\Delta^2 \\ \langle \widetilde{x}^2\rangle_{\rho_\chi}^{\nu=1}/\Delta^2 \\ \langle \widetilde{x}^2\rangle_{\rho_\chi}^{\nu=2}/\Delta^2 \end{bmatrix}. \tag{98}$$

The first three relations are linearly dependent, but none of the other triplets are. We thus have the consistency relation

$$\frac{\langle \widetilde{x}^2\rangle_{\rho_\chi}^{\nu=1}}{\Delta^2} = \frac{\sigma^2}{\Delta^2} + \frac{2}{\pi^2} N_{\rm add}^{\nu=0}. \tag{99}$$

There are more relations like Eqs. (94-97) which can be derived from our analytical results. The only assumption is a sufficiently small lattice spacing.

The value of W_8 follows immediately from the ν dependence of $\langle \widetilde{x}^2\rangle_{\rho_\chi}$. If there are additional real modes, it cannot be that W_7 and W_8 are both equal to zero. In Ref. [7] it was found $W_8 = 0$ (with clover improvement) and results were fitted as a function of W_6 with $W_7 = 0$. Our prediction is that the number of additional real modes is zero and it would be interesting if the authors of Ref. [7] could confirm that.

The non-trivial effect of W_7 on the quenched spectrum was a surprise for us. In Ref. [13] it was argued that W_7 does not affect the phase structure of the Dirac spectrum. Indeed, we found that the complex eigenvalue density only

shows a weak dependence on W_7, and actually becomes W_7 independent in the small $\widetilde{a}^2V$-limit. Such a dependence on W_7 can be found in the large $\widetilde{a}^2V$-limit but vanishes again in the thermodynamic limit. Since in the thermodynamic limit the number of real eigenvalues is suppressed as $1/\sqrt{V}$ with respect to the number of complex eigenvalues, W_7 will not affect the phase structure of the partition function. However, a non-zero value of W_7 significantly changes the density of the real eigenvalues. In particular, in the large $\widetilde{a}^2V$-limit, we find a square root singularity at the boundary of the support of the additional real eigenvalues if $W_7 \neq 0$, while it is a uniform density for $W_7 = 0$, see Ref. [23]. Nevertheless, we expect in the case of dynamical fermions that the discussion of Ref. [13] also applies to the real spectrum of $D_{\rm W}$.

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Appendix A: Derivation of the Joint Probability Density

In this appendix, we derive the joint probability density in three steps. In Appendix A 1, following the derivation for the joint probability density of the Hermitian Dirac operator [20] we introduce an auxiliary Gaussian integral such that we obtain a Harish-Chandra-Itzykson-Zuber like integral that mixes two different types of variables. In Appendix A 2 this problem is reduced to a Harish-Chandra-Itzykson-Zuber like integral considered in a bigger framework. We derive an educated guess which fulfills a set of differential equations and a boundary value problem. The asymptotics of the integral for large arguments serves as the boundary. In Appendix A 2 b we perform a stationary phase approximation which already yields the full solution implying that the semi-classical approach is exact and the Duistermaat-Heckman localization theorem [45] applies. In the last step we plug the result of Appendix A 2 into the original problem, see Appendix A 2 c, and integrate over the remaining variables to arrive at the result for the joint probability density given in the main text.

1. Introducing auxiliary Gaussian integrals

We consider the functional $I[f]$, see Eq. (11), with an integrable test-function f invariant under $\mathrm{U}(n, n+\nu)$. The idea is to rewrite the exponent of the probability density $P(D_{\rm W})$ as the sum of a $\mathrm{U}(n, n+\nu)$ invariant term $\mathrm{Tr}D_{\rm W}^2$ and a symmetry breaking term which is linear in $D_{\rm W}$. This is achieved by introducing two Gaussian distributed Hermitian matrices $S_{\rm r}$ and $S_{\rm l}$ with dimensions $n \times n$ and $(n+\nu) \times (n+\nu)$, respectively, i.e.

$$I[f] = (2\pi\sqrt{1+a^2})^{-n^2-(n+\nu)^2} \left(-\frac{n}{2\pi}\right)^{n(n+\nu)} \exp\left[-\frac{a^2}{2}\left(\mu_{\rm r}^2 + \frac{n+\nu}{n}\mu_{\rm l}^2\right)\right] \int d[D_{\rm W}]f(D_{\rm W}) \int d[S_{\rm r}, S_{\rm l}]$$
$$\times \exp\left[\frac{n}{2}\mathrm{tr}\, D_{\rm W}^2 + \imath \mathrm{tr}\, D_{\rm W}\, \mathrm{diag}\,(S_{\rm r}, S_{\rm l}) - \frac{a^2}{2n(1+a^2)}\left(\mathrm{tr}\,(S_{\rm r} + \imath\mu_{\rm r}\mathbb{1}_n)^2 + \mathrm{tr}\,(S_{\rm l} + \imath\mu_{\rm l}\mathbb{1}_{n+\nu})^2\right)\right]. \qquad \text{(A1)}$$

The matrix $\mathrm{diag}(S_{\rm r}, S_{\rm l})$ is a block-diagonal matrix with $S_{\rm r}$ and $S_{\rm l}$ on the diagonal blocks. The measure for $S_{\rm r/l}$ is

$$d[S_{\rm r}, S_{\rm l}] = \prod_{j=1}^{n} dS_{jj}^{(\rm r)} \prod_{1\leq i<j\leq n} 2\, d\,\mathrm{Re}\, S_{ij}^{(\rm r)} d\,\mathrm{Im}\, S_{ij}^{(\rm r)} \prod_{j=1}^{n+\nu} dS_{jj}^{(\rm l)} \prod_{1\leq i<j\leq n+\nu} 2\, d\,\mathrm{Re}\, S_{ij}^{(\rm l)} d\,\mathrm{Im}\, S_{ij}^{(\rm l)}. \qquad \text{(A2)}$$

Then the non-compact unitary matrix diagonalizing $D_{\rm W}$ only appears quadratically in the exponent. Notice that we have to integrate first over the Hermitian matrices $S_{\rm r/l}$ and have to be careful when interchanging integrals with integrals over $D_{\rm W}$. Obviously the integrations over the eigenvalues of $D_{\rm W}$ are divergent without performing the $S_{\rm r/l}$ integrals first and cannot be interchanged with these integrals. Also the coset integrals over $\mathbb{G}_l = \mathrm{U}(n, n+\nu)/[\mathrm{U}^{\,2n+\nu-l}(1) \times \mathrm{O}^{\,l}(1,1)]$, cf. Eq. (11), are not absolutely convergent. However we can understand them in a weak way and, below, we will find Dirac delta functions resulting from the non-compact integrals.

Diagonalizing the matrices $D_l = UZ_lU^{-1}$ and $S_{\rm r/l} = V_{\rm r/l}\, s_{\rm r/l}\, V_{\rm r/l}^{\dagger}$ with $s_{\rm r} = \mathrm{diag}\,(s_1^{(\rm r)}, \ldots, s_n^{(\rm r)})$ and $s_{\rm l} = \mathrm{diag}\,(s_1^{(\rm l)}, \ldots, s_{n+\nu}^{(\rm l)})$ we can absorb the integrals over $V_{\rm r}$ and $V_{\rm l}$ in the $U \in \mathbb{G}_l$ integral. Then we end up with

the integral

$$I[f] = \frac{C}{n!(n+\nu)!}\sum_{l=0}^{n}\frac{1}{2^l(n-l)!l!(n+\nu-l)!}\int\limits_{\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l} d[Z_l]|\Delta_{2n+\nu}(Z_l)|^2\int\limits_{\mathbb{R}^{2n+\nu}} d[s_{\rm r}\,,s_{\rm l}]\Delta_n^2(s_{\rm r})\Delta_{n+\nu}^2(s_{\rm l})f(Z_l) \quad \text{(A3)}$$
$$\times\exp\left[\frac{n}{2}\mathrm{tr}\,Z_l^2-\frac{a^2}{2n(1+a^2)}\left(\mathrm{tr}\,(s_{\rm r}+\imath\mu_{\rm r}\mathbb{1}_n)^2+\mathrm{tr}\,(s_{\rm l}+\imath\mu_{\rm l}\mathbb{1}_{n+\nu})^2\right)\right]\int\limits_{\mathbb{G}_l}\exp\left[\imath\mathrm{tr}\,UZ_lU^{-1}\mathrm{diag}\,(s_{\rm r}\,,s_{\rm l}\,)\right]d\mu_{\mathbb{G}_l}(U)$$

and the normalization constant

$$C=\left(-\frac{n}{2\pi}\right)^{n(n+\nu)}\prod_{j=0}^{n-1}\frac{(2\pi)^j\exp\left[-a^2\mu_{\rm r}^2/2n\right]}{j!(2\pi\sqrt{1+a^2})^{2j+1}}\prod_{j=0}^{n+\nu-1}\frac{(2\pi)^j\exp\left[-a^2\mu_{\rm l}^2/2n\right]}{j!(2\pi\sqrt{1+a^2})^{2j+1}}. \quad \text{(A4)}$$

See Sec. II B for a discussion of the prefactors in the sum.

2. The Harish-Chandra-Itzykson-Zuber integral over the non-compact coset $\mathbb{G}_l$

In the next step we calculate the integral

$$\mathcal{I}_l(Z_l,s)=\int\limits_{\mathbb{G}_l}\exp\left[\imath\mathrm{tr}\,UZ_lU^{-1}s\right]d\mu_{\mathbb{G}_l}(U). \quad \text{(A5)}$$

with $s=\mathrm{diag}\,(s_{\rm r}\,,s_{\rm l}\,)$. For $l=0$ this integral was derived in Ref. [46].

We calculate this integral by determining a complete set of functions and expanding the integral for asymptotically large s in this set. In this limit it can be calculated by a stationary phase approximation. It turns out that this integral, as is the case with the usual Harish-Chandra-Itzykson-Zuber integral, is semi-classically exact.

a. Non-compact Harish-Chandra-Itzykson-Zuber Integral

Let us consider the non-compact integral

$$\mathcal{I}_l(Z_l,Z'_{l'})=\int\limits_{\mathbb{G}_l}\exp\left[\imath\mathrm{tr}\,UZ_lU^{-1}Z'_{l'}\right]d\mu_{\mathbb{G}_l}(U) \quad \text{(A6)}$$

in a bigger framework where $Z'_{l'}$ is a quasi-diagonal matrix with l' complex conjugate eigenvalue pairs. The integral is invariant under the Weyl group $\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l)\times\mathbb{Z}_2^l$ in Z_l. To make the integral well-defined we have to assume that $l\geq l'$ otherwise the integral is divergent since the non-compact subgroup $\mathrm{O}^{l'-l}(1,1)\subset\mathbb{G}_l$ commutes with $Z'_{l'}$.

The integral (A6) should be contrasted with the well-known compact Harish-Chandra-Itzykson-Zuber integral [43, 44]

$$\mathcal{I}^{\rm com}(X,X')=\int\limits_{\mathrm{U}\,(2n+\nu)}\exp\left[\imath\mathrm{tr}\,UXU^{-1}X'\right]d\mu_{\mathrm{U}\,(2n+\nu)}(U)=\frac{(-2\pi\imath)^{\nu(\nu-1)/2}}{\Delta_\nu(x)\Delta_\nu(x')}\det\left[\exp\left(\imath x_ix'_j\right)\right]_{1\leq i,j\leq\nu} \quad \text{(A7)}$$

with Weyl group $\mathbf{S}(2n+\nu)$. Moreover the compact case is symmetric when interchanging X with X'. This symmetry is broken in Z_l and $Z'_{l'}$ due to the coset $\mathbb{G}_l$.

For a γ_5-Hermitian matrix V with eigenvalues Z_l, we can rewrite the integral (A6) as

$$\mathcal{I}_l(Z_l,Z'_{l'})=\mathcal{I}_l(V,Z'_{l'})=\int\limits_{\mathbb{G}_l}\exp\left[\imath\mathrm{tr}\,UVU^{-1}Z'_{l'}\right]d\mu_{\mathbb{G}_l}(U). \quad \text{(A8)}$$

This trivially satisfies the Sekigushi-like differential equation [47, 48]

$$\det\left(\frac{\partial}{\partial V_{kl}}+u\mathbb{1}_{2n+\nu}\right)\mathcal{I}_l(V,Z'_{l'})=\det\left(\imath Z'_{l'}+u\mathbb{1}_{2n+\nu}\right)\mathcal{I}_l(V,Z'_{l'})\text{ for all }u\in\mathbb{C}. \quad \text{(A9)}$$

This equation is written in terms of the independent matrix elements of V and, hence, is independent of the fact to which sector l the matrix V can be quasi-diagonalized.

We would like to rewrite Eq. (A9) in terms of derivatives with respect to the eigenvalues [62]. Because of the coefficients that enter after applying the chain rule when changing coordinates, the derivatives do not commute and a direct evaluation of the determinant is cumbersome. Therefore we will calculate $\mathcal{I}_l(Z_l, Z'_{l'})$ in an indirect way. We will do this by constructing a complete set of $\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l)\times\mathbb{Z}_2^l$ symmetric functions in the space of the $\{Z_l\}$ with the $\{Z'_{l'}\}$ as quantum numbers which have to be $\mathbf{S}(n-l')\times\mathbf{S}(l')\times\mathbf{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}$ symmetric. Then we expand $\mathcal{I}_l(Z_l, Z'_{l'})$ in this set of functions and determine the coefficients for asymptotic large $\{Z_l\}$ where the integral can be evaluated by a stationary phase approximation.

To determine the complete set of functions, we start from the usual Harish-Chandra-Itzykson-Zuber integral over the compact group $\mathrm{U}(2n+\nu)$. This integral is well-known and satisfies the Sekigushi-like differential equation [47, 48] with

$$\frac{1}{\Delta_{2n+\nu}(X)}\det\left(\frac{\partial}{\partial X}+u\mathbb{1}_{2n+\nu}\right)\Delta_{2n+\nu}(X)\mathcal{I}^{\mathrm{com}}(X,X')=\det\left(\imath X'+u\mathbb{1}_{2n+\nu}\right)\mathcal{I}^{\mathrm{com}}(X,X') \tag{A10}$$

in terms of the $(2n+\nu)$ real eigenvalues $X=\mathrm{diag}\,(x_1,\ldots,x_{2n+\nu})$ with

$$\det\left(\frac{\partial}{\partial X}+u\mathbb{1}_{2n+\nu}\right)=\prod_{j=1}^{2n+\nu}\left(\frac{\partial}{\partial x_j}+u\right). \tag{A11}$$

The expansion in powers of u gives the complete set of $2n+\nu$ independent Casimir operators on the Cartan subspace of $\mathrm{U}(2n+\nu)$, so that the Sekigushi equation determines a complete set of functions $\mathcal{I}_l(Z_l, Z'_{l'})$ up to the Weyl group. Since the non-compact group $\mathrm{U}(n+\nu,n)$ shares the same complexified Lie algebra as $\mathrm{U}(2n+\nu)$ the Casimir operators are the same, i.e. the corresponding operator for $\mathrm{U}(n+\nu,n)$ to the one in Eq. (A10) is

$$D_{Z_l}(u)=\frac{1}{\Delta_{2n+\nu}(Z_l)}\det\left(\frac{\partial}{\partial Z_l}+u\mathbb{1}_{2n+\nu}\right)\Delta_{2n+\nu}(Z_l) \tag{A12}$$

with

$$\det\left(\frac{\partial}{\partial Z_l}+u\mathbb{1}_{2n+\nu}\right)=\prod_{j=1}^{n-l}\left(\frac{\partial}{\partial x_j^{(1)}}+u\right)\prod_{j=1}^{l}\left(\frac{\partial}{\partial z_j^{(2)}}+u\right)\left(\frac{\partial}{\partial z_j^{(2)*}}+u\right)\prod_{j=1}^{n+\nu-l}\left(\frac{\partial}{\partial x_j^{(3)}}+u\right). \tag{A13}$$

In the compact case, the Sekigushi-like equation (A10) follows from Eq. (A9) by transforming the equation in terms of the eigenvalues and eigenvectors of $V=UXU^{-1}$, see Ref. [48]. The only difference in the non-compact case is that the parameters of U as well as some of the eigenvalues x become complex, but the algebraic manipulations to obtain the Sekigushi-like differential equation in terms of eigenvalues remain the same. Let f be an integrable test-function on the Cartan-subset $\mathbb{R}^{2n+\nu-2l'}\times\mathbb{C}^{l'}$. Then the non-compact integral (A6) satisfies the weak Sekigushi-like equation

$$D_{Z_l}(u)\int\limits_{\mathbb{R}^{2(n-l')+\nu}\times\mathbb{C}^{l'}}d[Z'_{l'}]f(Z'_{l'})\mathcal{I}_l(Z_l,Z'_{l'})=\int\limits_{\mathbb{R}^{2(n-l')+\nu}\times\mathbb{C}^{l'}}d[Z'_{l'}]f(Z'_{l'})\det\left(\imath Z'_{l'}+u\mathbb{1}_{2n+\nu}\right)\mathcal{I}_l(Z_l,Z'_{l'}), \tag{A14}$$

and solutions of this equation yield a complete set of functions for the non-compact case as well. The only difference is the corresponding Weyl group. The completeness can be seen because we can generate any polynomial of order $k\in\mathbb{N}_0$ (the non-negative integers) in $Z'_{l'}$ symmetric under $\mathbf{S}(n-l')\times\mathbf{S}(l')\times\mathbf{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}$ via the differential operator $\prod_{j=1}^k D_{Z_l}(u_j)$. Since those polynomials are dense in the space of $\mathbf{S}(n-l')\times\mathbf{S}(l')\times\mathbf{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}$ invariant functions, it immediately follows that if a function is in the kernel of $D_{Z_l}(u)$ for all u it is zero, i.e.

$$D_{Z_l}(u)F(Z_l)=0\ \forall u\in\mathbb{C}\Leftrightarrow F(Z_l)=0. \tag{A15}$$

Therefore if we found a solution for Eq. (A14) for an arbitrary test-function f we found $\mathcal{I}_l(Z_l, Z'_{l'})$ up to the normalization which can be fixed in the large $\mathrm{tr}\,Z_lZ_l^\dagger$-limit.

Some important remarks about Eq. (A14) are in order. The Vandermonde determinant $\Delta_{2n+\nu}(Z_l)$ enters in a trivial way in the operator $D_{Z_l}(u)$ and the remaining operator has plane waves as eigenfunctions which indeed build a complete set of functions. Thus a good ansatz of $\mathcal{I}_l(Z_l, Z'_{l'})$ is

$$\mathcal{I}_l(Z_l,Z'_{l'})=\frac{1}{\Delta_{2n+\nu}(Z_l)}\prod_{j=1}^{l}\frac{y_j^{(2)}}{|y_j^{(2)}|}\sum_{\omega\in\mathbf{S}}c_\omega^{(ll')}(Z'_{l'})\exp\left[\imath\mathrm{tr}\,\Pi_\omega Z_l\Pi_\omega^{-1}\widehat{Z}'_{l'}\right], \tag{A16}$$

where the coefficients $c_\omega^{(ll')}(Z'_{l'})$ have to be determined. The factors $y_j^{(2)}/|y_j^{(2)}|$ guarantee the invariance under complex conjugation of each complex eigenvalue pair of Z_l. We sum over the permutation group ω and Π_ω is its standard representation in terms of $(2n+\nu)\times(2n+\nu)$ matrices. The $\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l)\times\mathbb{Z}_2^l$ invariance in Z_l and the $\mathbf{S}(n-l')\times\mathbf{S}(l')\times\mathbf{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}$ invariance in $Z'_{l'}$ carry over to the coefficients $c_\omega^{(ll')}(Z'_{l'})$. Hence, we can reduce all coefficients to coefficients independent of ω,

$$\mathcal{I}_l(Z_l,Z'_{l'})=\frac{1}{\Delta_{2n+\nu}(Z_l)}\prod_{j=1}^{l}\frac{y_j^{(2)}}{|y_j^{(2)}|}\sum_{\substack{\omega\in\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l)\times\mathbf{Z}_2^l\\ \omega'\in\mathbf{S}(n-l')\times\mathbf{S}(l')\times\mathbf{S}(n+\nu-l')\times\mathbf{Z}_2^{l'}}}\operatorname{sign}\omega\; c^{(ll')}(Z'_{l'\omega'})\exp\left[\imath\operatorname{tr}Z_{l\omega}Z'_{l'\omega'}\right] \tag{A17}$$

where we employ the abbreviation

$$Z_{l\omega}=\Pi_\omega Z_l\Pi_\omega^{-1}\quad\text{and}\quad Z'_{l'\omega'}=\Pi_{\omega'}Z'_{l'}\Pi_{\omega'}^{-1}. \tag{A18}$$

The sign of elements in the group $\mathbb{Z}_2$ generating the complex conjugation of single complex conjugated pairs is always $+1$. Moreover, any element in the permutation group $\mathbf{S}(l)$ is an even permutation since it interchanges a complex conjugate pair with another one and, thus, always yields a positive sign. Hence the sign of the permutation ω is the product of the sign of the permutations in $\mathbf{S}(n-l)$ and in $\mathbf{S}(n+\nu-l)$.

Solving the weak Sekigushi-like equation (A14) for the general case $l\neq l'$ is quite complicated but as we will show below for $l=l'$ the ansatz

$$\begin{aligned}\mathcal{I}_l(Z_l,Z'_l)=&\frac{(-2\pi\imath)^{(2n+\nu)(2n+\nu-1)/2}}{\Delta_{2n+\nu}(Z_l)\Delta_{2n+\nu}(Z'_l)}\det\left[\exp\left(\imath x_i^{(1)}x_j'^{(1)}\right)\right]_{1\le i,j\le n-l}\det\left[\exp\left(\imath x_i^{(3)}x_j'^{(3)}\right)\right]_{1\le i,j\le n+\nu-l}\\ &\times\operatorname{perm}\left[\frac{y_i^{(2)}y_j'^{(2)}}{|y_i^{(2)}y_j'^{(2)}|}\left(\exp\left[2\imath\operatorname{Re}\,z_i^{(2)}z'^{(2)}_j\right]+\exp\left[2\imath\operatorname{Re}\,z_i^{(2)*}z'^{(2)}_j\right]\right)\right]_{1\le i,j\le l},\end{aligned} \tag{A19}$$

i.e. $c^{(ll)}(Z'_{l\omega'})\propto(\prod_{j=1}^{l'}y'^{(2)}_j/|y'^{(2)}_j|)/\Delta_{2n+\nu}(Z'_{l\omega'})$, does the job. Note that we have again the symmetry when interchanging Z_l with Z'_l since both matrices are in the Cartan subspace corresponding to $\mathbb{G}_l$. The constant can be fixed by a stationary phase approximation when taking $\operatorname{tr}Z_lZ_l^\dagger\to\infty$. The function "perm" is the permanent which is defined analogously to the determinant but without the sign-function in the sum over the permutations. It arises because the Vandermonde determinants are even under the interchange of a complex pair with another one, i.e. it is the $\mathbf{S}(l)$-invariance of the corresponding Weyl-group. It can be explicitly shown that the ansatz (A19) satisfies the completeness relation in the space of functions on $\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l$ invariant under $\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l)\times\mathbb{Z}_2^l$ and with the measure $|\Delta_{2n+\nu}(Z_l)|^2d[Z_l]$, i.e.

$$\begin{aligned}&\int\limits_{\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l}\mathcal{I}_l(Z_l,Z'_l)\mathcal{I}_l(Z''_l,Z_l)|\Delta_{2n+\nu}(Z_l)|^2d[Z_l]\\ &\propto\frac{1}{\Delta_{2n+\nu}(Z'_l)\Delta_{2n+\nu}(Z''_l)}\det\left[\delta\left(x_i'^{(1)}-x_j''^{(1)}\right)\right]_{1\le i,j\le n-l}\det\left[\delta\left(x_i'^{(3)}-x_j''^{(3)}\right)\right]_{1\le i,j\le n+\nu-l}\\ &\quad\times\operatorname{perm}\left[\frac{y_i'^{(2)}y_j''^{(2)}}{|y_i'^{(2)}y_j''^{(2)}|}\delta\left(|y_i'^{(2)}|-|y_j''^{(2)}|\right)\delta\left(x_i'^{(2)}-x_j''^{(2)}\right)\right]_{1\le i,j\le l}.\end{aligned} \tag{A20}$$

Therefore, for given $l'=l$ and $Z'_{l'}$ the ansatz (A19) for $\mathcal{I}_l(Z_l,Z'_l)$ is the unique solution of the Sekiguchi-like equation (A14). One has only to show that the global prefactor is correct, see A 2 b.

What happens in the general case $l\neq l'$? The ansatz (A17) can only fulfill the Sekigushi-like differential equation (A14) if we assume that the coefficient $c^{(ll')}(Z'_{l'\omega'})$ restricts the matrix $Z'_{l'}$ to a matrix in the sector with l complex conjugate eigenvalue pairs (notice that Z_l has the representation given in Eq. (7)). This is only possible on the boundary of the Cartan subsets $\mathbb{R}^{2(n-l)+\nu}\times\mathbb{C}^l$ and $\mathbb{R}^{2(n-l')+\nu}\times\mathbb{C}^{l'}$, i.e. the coefficient has to be proportional to Dirac delta functions

$$c^{(ll')}(Z'_{l'\omega'})\propto\prod_{j=1}^{l-l'}\delta\left(x'^{(1)}_{\omega'(n-l+j)}-x'^{(3)}_{\omega'(j)}\right). \tag{A21}$$

The reason for this originates in the fact that not all complex pairs of Z_l can couple with a complex eigenvalue pair in $Z'_{l'}$ and, hence, $\operatorname{tr}Z_{l\omega}Z'_{l'\omega'}$ does not depend on the combinations $x'^{(1)}_{\omega'(n-l+j)}-x'^{(3)}_{\omega'(j)}$. Therefore we would miss it in

the determinant $\det(Z'_{l'}+u\mathbb{1}_{2n+\nu})$ generated by the differential operator $D_{Z_l}(u)$. To cure this we have to understand $\mathcal{I}_l(Z_l,Z'_{l'})$ as a distribution where the Dirac delta functions set these missing terms to zero. In A 2 b we show that the promising ansatz

$$\mathcal{I}_l(Z_l,Z'_{l'}) = \frac{c^{(ll')}}{\Delta_{2n+\nu}(Z_l)\Delta_{2n+\nu}(Z'_{l'})}\prod_{j=1}^{l}\frac{y_j^{(2)}}{|y_j^{(2)}|}\prod_{j=1}^{l'}\frac{y'^{(2)}_j}{|y'^{(2)}_j|} \tag{A22}$$

$$\times \sum_{\substack{\omega\in\mathbb{S}(n-l)\times\mathbb{S}(l)\times\mathbb{S}(n+\nu-l)\times\mathbb{Z}_2^l\\ \omega'\in\mathbb{S}(n-l')\times\mathbb{S}(l')\times\mathbb{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}}} \operatorname{sign}\omega\omega' \exp\left(\imath \operatorname{tr} Z_{l\omega}Z'_{l'\omega'}\right)\prod_{j=1}^{|l-l'|}\left(x'^{(1)}_{\omega'(n-l+j)}-x'^{(3)}_{\omega'(j)}\right)\delta\left(x'^{(1)}_{\omega'(n-l+j)}-x'^{(3)}_{\omega'(j)}\right)$$

is indeed the correct result.

Note that the ansatz (A22) agrees with the solution (A19) for the case $l=l'$. Furthermore one can easily verify that it also solves the weak Sekiguchi-like differential equation (A14). Indeed, the ansatz is trivially invariant under the two Weyl groups $\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l)\times\mathbb{Z}_2^l$ and $\mathbf{S}(n-l')\times\mathbf{S}(l')\times\mathbf{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}$ due to the sum. The global prefactor $1/\Delta_{2n+\nu}(Z'_{l'})$ reflects the singularities when an eigenvalue in $x'^{(1)}$ agrees with one in $x'^{(3)}$ as well as a complex eigenvalue pair in $x'^{(2)}$ degenerates with another eigenvalue in $Z'_{l'}$, namely then $Z'_{l'}$ commutes with some non-compact subgroups in $\mathbb{G}_l$. Hereby the eigenvalues which have to degenerate via the Dirac delta functions are excluded.

In the next section we calculate the global coefficients in Eq. (A22). For this we consider the stationary phase approximation which fixes this coefficient.

b. The stationary phase approximation of $\mathcal{I}_l(Z_l,Z'_{l'})$

Let us introduce a scalar parameter t as a small parameter in the integral $\mathcal{I}_l(t^{-1}Z_l,Z'_{l'})$ as a bookkeeping device for the expansion around the saddlepoints. Taking $t\to 0$ the group integral (A6) can be evaluated by a stationary phase approximation. The saddlepoint equation is given by

$$\operatorname{tr} dUU^{-1}[UZ_lU^{-1},Z'_{l'}]_- = 0. \tag{A23}$$

If $l\neq l'$ this equation cannot be satisfied in all directions. The reason is that the quasi-diagonal matrix $Z'_{l'}$ will never commute with a γ_5-Hermitian matrix with exactly $l\neq l'$ complex conjugate eigenvalue pairs since UZ_lU^{-1} can be at most quasi-diagonalized by $\mathrm{U}(n,n+\nu)$ and generically $[Z_l,Z'_{l'}]_-\neq 0$. This means that we can only expand the sub-Lie-algebra $\mathrm{o}^{l-l'}(1/1)$ to the linear order while the remaining massive modes are expanded to the second order. The extrema are given by

$$U_0 = \Pi'\Phi\Pi \in \mathbb{G}_l \tag{A24}$$

where the permutations are

$$\begin{aligned}\Pi &\in \mathbf{S}(n-l)\times[\mathbf{S}(l)/[\mathbf{S}(l')\times\mathbf{S}(l-l')]]\times\mathbf{S}(n+\nu-l),\\ \Pi' &\in [\mathbf{S}(n-l')/\mathbf{S}(n-l)]\times\mathbf{S}(l')\times[\mathbf{S}(n+\nu-l')/\mathbf{S}(n+\nu-l)]\times\mathbb{Z}^{l'},\end{aligned} \tag{A25}$$

and a block-diagonal matrix

$$\Phi = \left[\begin{array}{c|c|c|c|c|c} \mathbb{1}_{n-l} & 0 & 0 & 0 & 0 & 0\\ \hline 0 & \exp[\imath\widehat{\Phi}] & 0 & 0 & 0 & 0\\ \hline 0 & 0 & \mathbb{1}_{l'} & 0 & 0 & 0\\ \hline 0 & 0 & 0 & \mathbb{1}_{l'} & 0 & 0\\ \hline 0 & 0 & 0 & 0 & \exp[-\imath\widehat{\Phi}] & 0\\ \hline 0 & 0 & 0 & 0 & 0 & \mathbb{1}_{n+\nu-l}\end{array}\right], \tag{A26}$$

where the diagonal matrix of angles is $\widehat{\Phi}=\operatorname{diag}(\varphi_1,\ldots,\varphi_{l-l'})\in[0,\pi]^{l-l'}$. The matrix Φ describes the set $\mathrm{U}^{\,l-l'}(1)$ ($l-l'$ unit circles in the complex plane) which commutes with $Z'_{l'}$ and is a subgroup of $\mathbb{G}_l$. Note that other rotations commuting with $Z'_{l'}$ are already divided out in $\mathbb{G}_l$. The matrix of phases already comprises the complex conjugation of the complex eigenvalues represented by the finite group $\mathbb{Z}_2^{l-l'}$, choosing $\varphi_j=\pi/2$ switches the sign of the imaginary

part y'_j. However we have to introduce the complex conjugation for those complex conjugated pairs in $Z'_{l'}$ which couple with pairs in Z_l, cf. the group $\mathbb{Z}^{l'}$ in Π'.

The expansion of U reads

$$U = \Pi' \Phi \left(\mathbb{1}_{2n+\nu} - tH_1 - \sqrt{t} H_2 + \frac{t}{2} H_2^2 \right) \Pi. \tag{A27}$$

We employ the notation (A18) for the action of $\omega \in \mathbf{S}(n-l) \times \mathbf{S}(l) \times \mathbf{S}(n+\nu-l) \times \mathbb{Z}^l$ and $\omega' \in \mathbf{S}(n-l') \times \mathbf{S}(l') \times \mathbf{S}(n+\nu-l') \times \mathbb{Z}^{l'}$ on the matrices $Z_{l\omega}$ and $Z'_{l'\omega'}$, respectively. Note that the matrix Φ commutes with $Z'_{l'\omega'}$ for any ω' and, hence, only yields an overall prefactor $\pi^{l-l'}$. The matrix H_1 spans the Lie algebra $\mathrm{o}^{l-l'}(1,1)$ and is embedded as

$$H_1 = \left(\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & h & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & h & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \text{with } h = \mathrm{diag}\,(h_1, \ldots, h_{l-l'}) \in \mathbb{R}^{l-l'}. \tag{A28}$$

The matrix H_2 is in the tangent space of the coset $\mathbb{G}_l/[\mathrm{U}^{\,l-l'}(1) \times \mathrm{O}^{l-l'}(1,1)] = \mathrm{U}\,(n, n+\nu)/[\mathrm{U}^{\,2n+\nu-2l+l'}(1) \times \mathrm{O}^{l'}(1,1) \times \mathrm{U}^{\,l-l'}(1,1)]$ and has the form

$$H_2 = \left(\begin{array}{c|c||c|c|c||c} H_{11} & H_{12} & H_{13} & H_{14} & H_{15} & H_{16} \\ \hline\hline -H_{12}^\dagger & H_{22} & H_{23} & H_{24} & H_{25} & H_{26} \\ \hline -H_{13}^\dagger & -H_{23}^\dagger & H_{33} & H_{34} & H_{35} & H_{36} \\ H_{14}^\dagger & H_{24}^\dagger & H_{34}^\dagger & H_{44} & H_{45} & H_{46} \\ \hline H_{15}^\dagger & H_{25}^\dagger & H_{35}^\dagger & -H_{45}^\dagger & H_{55} & H_{56} \\ \hline\hline H_{16}^\dagger & H_{26}^\dagger & H_{36}^\dagger & -H_{46}^\dagger & -H_{56}^\dagger & H_{66} \end{array} \right), \tag{A29}$$

where H_{11}, H_{22}, H_{55}, and H_{66} are anti-Hermitian matrices without diagonal elements since they are divided out in the coset $\mathbb{G}_l$ or are lost to Φ. The two matrices H_{33} and H_{44} are anti-Hermitian matrices whose diagonal elements are the same with opposite sign which is also because of the subgroup we divide out in $\mathbb{G}_l$. The matrices H_{12}, H_{13}, H_{14}, H_{15}, H_{16}, H_{23}, H_{24}, H_{26}, H_{35}, H_{36}, H_{45}, H_{46}, and H_{56} are arbitrary complex matrices. Since we have to remove the degrees of freedom already included in H_1 and in the subgroups quotient out in $\mathbb{G}_l$ the matrix H_{25} is a complex matrix with all $l-l'$ diagonal elements removed and H_{34} is a complex matrix whose diagonal entries are real. The sizes of the blocks of H_1 and H_2 correspond to the sizes shown in the diagonal matrix of phases Φ, see Eq. (A26). The double lines in the matrix (A29) shall show the decomposition of Z_l in its real and complex eigenvalues whereas the single lines represent the decomposition for $Z'_{l'}$.

The exponent in the coset integral (A6) takes the form

$$\mathrm{tr}\, U Z_l U^{-1} Z'_{l'} = \mathrm{tr}\, Z_{l\omega} Z'_{l'\omega'} - t\mathrm{tr}\,[Z_{l\omega}, Z'_{l'\omega'}]_- H_1 - \frac{t}{2}\mathrm{tr}\,[Z_{l\omega}, H_2]_- [Z'_{l'\omega'}, H_2]_-. \tag{A30}$$

The measure for H_1 and H_2 is the induced Haar measure, i.e.

$$\mathrm{tr}\,[U^{-1}dU, Z_{l\omega}]_-^2 = \mathrm{tr}\,[\Phi^\dagger d\Phi, Z_{l\omega}]_-^2 + t^2 \mathrm{tr}\,[dH_1, Z_{l\omega}]_-^2 + t\mathrm{tr}\,[dH_2, Z_{l\omega}]_-^2 \tag{A31}$$

which gives

$$d\mu_{\mathbb{G}}(U) = t^{(2n+\nu)(2n+\nu-1)/2} d[H_1] d[\Phi] d[H_2] = (-1)^{n(n+\nu)} \left(\frac{2}{\imath}\right)^{l'} \prod_{k=1}^{l-l'} \frac{4t}{\imath} d\varphi_k dh_k \prod_{j,i} 2t\, d\mathrm{Re}\,(H_2)_{ij} d\mathrm{Im}\,(H_2)_{ij}. \tag{A32}$$

The product over the two indices i and j is over all independent matrix elements of H_2.

We emphasize again that the integrand in $\mathcal{I}_l(t^{-1}Z_l, Z'_{l'})$ does not depend on Φ making this integration trivial and yielding the prefactor $\pi^{l-l'}$. The integral over H_1 yields the $l-l'$ Dirac delta functions mentioned in Eq. (A21), i.e. it yields

$$(2\pi)^{l-l'} \prod_{j=1}^{l-l'} \delta\left(2y^{(2)}_{\omega(j)} \left[x'^{(1)}_{\ \omega'(n-l+j)} - x'^{(3)}_{\ \omega'(j)} \right] \right) = \prod_{j=1}^{l-l'} \frac{\pi \delta\left(x'^{(1)}_{\ \omega'(n-l+j)} - x'^{(3)}_{\ \omega'(j)} \right)}{|y^{(2)}_{\omega(j)}|}. \tag{A33}$$

Notice that the other term in the expansion of the Dirac delta function does not contribute because of the order of the integrations [63].

The integrals over H_2 are simple Gaussian integrals resulting in the main result of this section,

$$\mathcal{I}_l(t^{-1}Z_l, Z'_{l'}) = \frac{(-2\pi\imath)^{l-l'}}{l'!(l-l')!(n-l)!(n+\nu-l)!2^l} \frac{(-2\pi\imath)^{(2n+\nu)(2n+\nu-1)/2}}{\Delta_{2n+\nu}(t^{-1}Z_l)\Delta_{2n+\nu}(Z'_{l'})} \prod_{j=1}^{l} \frac{y_j^{(2)}}{|y_j^{(2)}|} \prod_{j=1}^{l'} \frac{y'^{(2)}_j}{|y'^{(2)}_j|} \tag{A34}$$

$$\times \sum_{\substack{\omega\in\mathbb{S}(n-l)\times\mathbb{S}(l)\times\mathbb{S}(n+\nu-l)\times\mathbb{Z}_2^l \\ \omega'\in\mathbb{S}(n-l')\times\mathbb{S}(l')\times\mathbb{S}(n+\nu-l')\times\mathbb{Z}_2^{l'}}} \operatorname{sign}\omega\omega' \exp\left(\frac{\imath}{t}\operatorname{tr} Z_{l\omega}Z'_{l'\omega'}\right) \prod_{j=1}^{|l-l'|} \left(x'^{(1)}_{\omega'(n-l+j)} - x'^{(3)}_{\omega'(j)}\right) \delta\left(x'^{(1)}_{\omega'(n-l+j)} - x'^{(3)}_{\omega'(j)}\right).$$

The overall coefficient $c^{(ll')}$ in Eq. (A22) can be easily read off. Thereby the numerator of the first factor results from the integral over H_1 and is related to the $l-l'$ Dirac delta functions. The denominator is the volume of the finite group $\mathbf{S}(n-l)\times\mathbf{S}(l')\times\mathbf{S}(l-l')\times\mathbf{S}(n+\nu-l)\times\mathbb{Z}^l$ which we extend to summing over the full Weyl groups for Z_l and $Z'_{l'}$. We recall that the sum over permutations in $\mathbf{S}(l)$ and $\mathbf{S}(l')$ describe the interchange of complex pairs which are even permutations because we interchange both z_k and z_k^* with another pair. The numerator of the term with the Vandermonde determinants essentially result from the Gaussian integrals and always appears independent of how many complex pairs Z_l and $Z'_{l'}$ have. The factors of t^{-1} appear as prefactors of Z_l and can be omitted again since they have done their job as bookkeeping device.

Let us summarize what we have found. Comparing the result (A34) with the Z_l dependence of the ansatz $\mathcal{I}(Z_l, Z'_l)$ given in Eq. (A22), we observe that they are exactly the same. This implies that the asymptotic large Z_l result for the integral (A6) is actually equal to the exact result. We conclude that the non-compact Harish-Chandra-Itzykson-Zuber integral is semi-classically exact and seems to fulfill the conditions of the Duistermaat-Heckman theorem [45].

Let us consider two particular cases. For $l = l'$ we sum over all permutations in $\mathbf{S}(l)$ which yields the permanent in Eq. (A20), whereas the sum over permutations in $\mathbf{S}(n+\nu-l)$ and $\mathbf{S}(n-l)$ gives determinants and, thus, agrees. The special case $n = 0$ yields the original Harish-Chandra-Itzykson-Zuber integral [43, 44], see Eq. (A7).

c. The joint probability density

We explicitly write out Z_l and apply Eq. (A34) for $Z'_{l'} = s$. Then, we find for our original non-compact group integral

$$\mathcal{I}_l(Z_l, s) = \frac{(-2\pi\imath)^{(2n+\nu)(2n+\nu-1)/2}}{(n-l)!l!(n+\nu-l)!} \frac{(-2\pi\imath)^l}{\Delta_{2n+\nu}(Z_l)\Delta_{2n+\nu}(s)} \sum_{\substack{\omega'\in\mathbf{S}(n-l)\times\mathbf{S}(l)\times\mathbf{S}(n+\nu-l) \\ \omega\in\mathbf{S}(n)\times\mathbf{S}(n+\nu)}} \operatorname{sign}(\omega\omega') \prod_{j=1}^{n-l} \exp\left(\imath x^{(1)}_{\omega'(j)} s^{(\mathrm{r})}_{\omega(j)}\right) \tag{A35}$$

$$\times \prod_{j=1}^{n+\nu-l} \exp\left(\imath x^{(3)}_{\omega'(j)} s^{(\mathrm{l})}_{\omega(l+j)}\right) \prod_{j=1}^{l} \frac{y^{(2)}_{\omega'(j)}}{|y^{(2)}_{\omega'(j)}|} \left(s^{(\mathrm{r})}_{\omega(n-l+j)} - s^{(\mathrm{l})}_{\omega(j)}\right) \delta\left(s^{(\mathrm{r})}_{\omega(n-l+j)} - s^{(\mathrm{l})}_{\omega(j)}\right) \exp\left(\imath x^{(2)}_{\omega'(j)} \left(s^{(\mathrm{r})}_{\omega(n-l+j)} + s^{(\mathrm{l})}_{\omega(j)}\right)\right).$$

Now we are ready to integrate over s.

We plug Eq. (A35) into the integral (A3). The sum over the permutations can be absorbed by the integral due to

relabelling resulting in

$$
\begin{aligned}
I[f] = C \sum_{l=0}^{n} \frac{(-2\pi\imath)^{(2n+\nu)(2n+\nu-1)/2+l}}{2^l (n-l)!l!(n+\nu-l)!} & \int\limits_{\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l} d[Z_l]\Delta_{2n+\nu}(Z_l^*) f(Z_l) \prod_{j=1}^{l} \frac{y_j^{(2)}}{|y_j^{(2)}|} \int\limits_{\mathbb{R}^{2n+\nu}} d[s_{\mathrm{r}}, s_{\mathrm{l}}] \frac{\Delta_n^2(s_{\mathrm{r}})\Delta_{n+\nu}^2(s_{\mathrm{l}})}{\Delta_{2n+\nu}(s)} \\
& \times \prod_{j=1}^{n-l} \exp\left[\frac{n}{2}(x_j^{(1)})^2 + \imath x_j^{(1)} s_j^{(\mathrm{r})} - \frac{a^2}{2n(1+a^2)}(s_j^{(\mathrm{r})} + \imath\mu_{\mathrm{r}})^2\right] \\
& \times \prod_{j=1}^{n+\nu-l} \exp\left[\frac{n}{2}(x_j^{(3)})^2 + \imath x_j^{(3)} s_{l+j}^{(\mathrm{l})} - \frac{a^2}{2n(1+a^2)}(s_{l+j}^{(\mathrm{l})} + \imath\mu_{\mathrm{l}})^2\right] \\
& \times \prod_{j=1}^{l} \left(s_{n-l+j}^{(\mathrm{r})} - s_j^{(\mathrm{l})}\right) \delta\left(s_{n-l+j}^{(\mathrm{r})} - s_j^{(\mathrm{l})}\right) \exp\left[\frac{a^2}{4n(1+a^2)}(\mu_{\mathrm{r}} - \mu_{\mathrm{l}})^2\right] \\
& \times \exp\left[n((x_j^{(2)})^2 - (y_j^{(2)})^2) + 2\imath x_j^{(2)} s_j^{(\mathrm{l})} - \frac{a^2}{n(1+a^2)}\left(s_j^{(\mathrm{l})} + \imath\frac{\mu_{\mathrm{r}} + \mu_{\mathrm{l}}}{2}\right)^2\right].
\end{aligned} \tag{A36}
$$

The quotient of the Vandermonde determinants is

$$
\frac{\Delta_n^2(s_{\mathrm{r}})\Delta_{n+\nu}^2(s_{\mathrm{l}})}{\Delta_{2n+\nu}(s)} = (-1)^{n(n-1)/2+\nu(\nu-1)/2} \det\begin{bmatrix} \left\{\dfrac{1}{s_i^{(\mathrm{r})} - s_j^{(\mathrm{l})}}\right\}_{\substack{1\le i\le n \\ 1\le j\le n+\nu}} \\ \left\{(s_j^{(\mathrm{l})})^{i-1}\right\}_{\substack{1\le i\le \nu \\ 1\le j\le n+\nu}} \end{bmatrix}. \tag{A37}
$$

This determinant also appears in the supersymmetry method of RMT [48, 49] and is a square root of a Berezinian (the supersymmetric analogue of the Jacobian).

Expanding the determinant (A37) in the first l columns not all terms will survive. Only those terms which cancel the prefactor of the Dirac delta functions do not vanish. The integration over $\mathrm{diag}\,(s_{n-l+1}^{(\mathrm{r})}, \ldots, s_n^{(\mathrm{r})}, s_1^{(\mathrm{l})}, \ldots, s_l^{(\mathrm{l})})$ yields

$$
\begin{aligned}
I[f] = C \sum_{l=0}^{n} \frac{(-2\pi\imath)^{(2n+\nu)(2n+\nu-1)/2+l}(-1)^{n(n-1)/2+\nu(\nu-1)/2+(n+l)l}}{2^l (n-l)!l!(n+\nu-l)!} & \int\limits_{\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l} d[Z_l]\Delta_{2n+\nu}(Z_l^*) f(Z_l) \int\limits_{\mathbb{R}^{\nu+2(n-l)}} d[s_{\mathrm{r}}, s_{\mathrm{l}}] \\
& \times \det\begin{bmatrix} \left\{\dfrac{1}{s_i^{(\mathrm{r})} - s_{l+j}^{(\mathrm{l})}}\right\}_{\substack{1\le i\le n-l \\ 1\le j\le n+\nu-l}} \\ \left\{(s_{l+j}^{(\mathrm{l})})^{i-1}\right\}_{\substack{1\le i\le \nu \\ 1\le j\le n+\nu-l}} \end{bmatrix} \prod_{j=1}^{n-l} \exp\left[\frac{n}{2}(x_j^{(1)})^2 + \imath x_j^{(1)} s_j^{(\mathrm{r})} - \frac{a^2}{2n(1+a^2)}(s_j^{(\mathrm{r})} + \imath\mu_{\mathrm{r}})^2\right] \\
& \times \prod_{j=1}^{n+\nu-l} \exp\left[\frac{n}{2}(x_j^{(3)})^2 + \imath x_j^{(3)} s_{l+j}^{(\mathrm{l})} - \frac{a^2}{2n(1+a^2)}(s_{l+j}^{(\mathrm{l})} + \imath\mu_{\mathrm{l}})^2\right] \prod_{j=1}^{l} \sqrt{\frac{n\pi(1+a^2)}{a^2}} \frac{y_j^{(2)}}{|y_j^{(2)}|} \exp\left[\frac{a^2}{4n(1+a^2)}(\mu_{\mathrm{r}} - \mu_{\mathrm{l}})^2\right] \\
& \times \exp\left[-\frac{n}{a^2}(x_j^{(2)})^2 - n(y_j^{(2)})^2 + x_j^{(2)}(\mu_{\mathrm{r}} + \mu_{\mathrm{l}})\right].
\end{aligned} \tag{A38}
$$

The other exponential functions as well as the remaining integrations over s_{r} and s_{l} can be pulled into the determinant. The integrals in the ν bottom rows yield harmonic oscillator wave function. These can be reordered into monomials

times a Gaussian. This results in

$$I[f] = C\sum_{l=0}^{n} \frac{(-2\pi\imath)^{(2n+\nu)(2n+\nu-1)/2+l}(-1)^{n(n-1)/2+\nu(\nu-1)/2+(n+l)l}}{2^l(n-l)!l!(n+\nu-l)!}(2\pi)^{\nu/2}\imath^{\nu(\nu-1)/2}\left(\frac{n(1+a^2)}{a^2}\right)^{\nu^2/2} \tag{A39}$$

$$\times \int\limits_{\mathbb{R}^{\nu+2(n-l)}\times\mathbb{C}^l} d[Z_l]\Delta_{2n+\nu}(Z_l^*)f(Z_l)\det\begin{bmatrix} \left\{\widetilde{G}(x_i^{(1)},x_j^{(3)})\right\}_{\substack{1\le i\le n-l\\ 1\le j\le n+\nu-l}} \\ \left\{(x_j^{(3)})^{i-1}\exp\left[-\frac{n}{2a^2}(x_j^{(3)})^2+\mu_{\rm l}x_j^{(3)}\right]\right\}_{\substack{1\le i\le \nu\\ 1\le j\le n+\nu-l}} \end{bmatrix}$$

$$\times\prod_{j=1}^{l}\sqrt{\frac{n\pi(1+a^2)}{a^2}}\frac{y_j^{(2)}}{|y_j^{(2)}|}\exp\left[\frac{a^2}{4n(1+a^2)}(\mu_{\rm r}-\mu_{\rm l})^2-\frac{n}{a^2}(x_j^{(2)})^2-n(y_j^{(2)})^2+x_j^{(2)}(\mu_{\rm r}+\mu_{\rm l})\right].$$

What remains is to simplify the function

$$\widetilde{G}(x_i^{(1)},x_j^{(3)}) = \int\limits_{\mathbb{R}^2} ds_{\rm r}\,ds_{\rm l}\,\frac{\exp\left[x_i^{(1)}\mu_{\rm r}+x_j^{(3)}\mu_{\rm l}\right]}{s_{\rm r}-s_{\rm l}+\imath n(1+a^2)(x_i^{(1)}-x_j^{(3)})/a^2} \tag{A40}$$

$$\times\exp\left[-\frac{n}{2a^2}\left((x_i^{(1)})^2+(x_j^{(3)})^2\right)-\frac{a^2}{2n(1+a^2)}\left((s_{\rm r}+\imath\mu_{\rm r})^2+(s_{\rm l}+\imath\mu_{\rm l})^2\right)\right].$$

We use the difference $x_i^{(1)}-x_j^{(3)}$ as a regularization of the integral. This works because generically this difference is not equal to zero. Then we can express the denominator as an exponential function. Let $\beta=(x_i^{(1)}-x_j^{(3)})/|x_i^{(1)}-x_j^{(3)}|$ be the sign of this difference. The integral (A40) can be written as

$$\begin{aligned}
\widetilde{G}(x_i^{(1)},x_j^{(3)}) &= \frac{\beta}{\imath}\exp\left[x_i^{(1)}\mu_{\rm r}+x_j^{(3)}\mu_{\rm l}-\frac{n}{2a^2}\left((x_i^{(1)})^2+(x_j^{(3)})^2\right)\right]\int\limits_0^\infty dt\int\limits_{\mathbb{R}^2} ds_{\rm r}\,ds_{\rm l}\\
&\quad\times\exp\left[-\frac{n(1+a^2)}{a^2}|x_i^{(1)}-x_j^{(3)}|t+\imath\beta(s_{\rm r}-s_{\rm l})t-\frac{a^2}{2n(1+a^2)}\left((s_{\rm r}+\imath\mu_{\rm r})^2+(s_{\rm l}+\imath\mu_{\rm l})^2\right)\right]\\
&= \frac{-2\pi\imath n(1+a^2)}{a^2}\beta\exp\left[x_i^{(1)}\mu_{\rm r}+x_j^{(3)}\mu_{\rm l}-\frac{n}{2a^2}\left((x_i^{(1)})^2+(x_j^{(3)})^2\right)\right]\\
&\quad\times\int\limits_0^\infty\exp\left[-\frac{n(1+a^2)}{a^2}t^2+\left(\beta(\mu_{\rm r}-\mu_{\rm l})-\frac{n(1+a^2)}{a^2}|x_i^{(1)}-x_j^{(3)}|\right)t\right]dt\\
&= -\pi\imath\sqrt{\frac{\pi n(1+a^2)}{a^2}}\beta\exp\left[-\frac{n}{4a^2}\left(x_i^{(1)}+x_j^{(3)}\right)^2+\frac{n}{4}\left(x_i^{(1)}-x_j^{(3)}\right)^2+\frac{1}{2}\left(x_i^{(1)}+x_j^{(3)}\right)(\mu_{\rm r}+\mu_{\rm l})\right]\\
&\quad\times\exp\left[\frac{a^2}{4n(1+a^2)}(\mu_{\rm r}-\mu_{\rm l})^2\right]\operatorname{erfc}\left[\sqrt{\frac{n(1+a^2)}{4a^2}}|x_i^{(1)}-x_j^{(3)}|-\beta\sqrt{\frac{a^2}{4n(1+a^2)}}(\mu_{\rm r}-\mu_{\rm l})\right].
\end{aligned} \tag{A41}$$

Plugging this result into Eq. (A39) we get the joint probability density for a fixed number of real eigenvalues given in Eq. (24). Moreover one can perform the sum over l to find the joint probability density of all eigenvalues given in Eq. (17).

Appendix B: Two useful Integral Identities

In this appendix we evaluate two integrals that have been used to simplify the expression for $\rho_{\rm r}$ and $\rho_{\rm c}$.

1. Convolution of a Gaussian with an error function

Let $\operatorname{Re}\gamma^2>-1$. We consider the integral

$$I(\alpha,\gamma)=\int\limits_{\mathbb{R}}\exp[-(x+\alpha)^2]\operatorname{erf}(\gamma x)dx. \tag{B1}$$

The solution can be obtained by constructing an initial value problem. Since the Gaussian is symmetric and the error function anti-symmetric around the origin we have

$$I(0,\gamma)=0. \tag{B2}$$

The derivative is

$$\partial_\alpha I(\alpha,\gamma)=\int\limits_{\mathbb{R}}\mathrm{erf}(\gamma x)\partial_x\exp[-(x+\alpha)^2]dx=-\frac{2\gamma}{\sqrt{\pi}}\int\limits_{\mathbb{R}}\exp[-(x+\alpha)^2-\gamma^2x^2]dx=-\frac{2\gamma}{\sqrt{\gamma^2+1}}\exp\left[-\frac{\gamma^2\alpha^2}{\gamma^2+1}\right]. \tag{B3}$$

Integrating the derivative from 0 to α we find the desired result

$$\int\limits_{\mathbb{R}}\exp[-(x+\alpha)^2]\mathrm{erf}(\gamma x)dx=-\sqrt{\pi}\,\mathrm{erf}\left(\frac{\gamma\alpha}{\sqrt{\gamma^2+1}}\right). \tag{B4}$$

This integral is needed to simplify the term (42).

Another integral identity which is used for the derivation of the level density of the real eigenvalues with positive chirality is given by

$$\begin{aligned}&\int\limits_{\mathbb{R}^2}\exp(-\alpha_1x_1^2-\alpha_2x_2^2+\beta_1x_1+\beta_2x_2)\mathrm{erf}\left(\frac{x_1+\delta x_2}{\gamma}+\epsilon\right)dx_1dx_2 \\ &=\frac{\pi}{\sqrt{\alpha_1\alpha_2}}\exp\left[\frac{1}{4}\left(\frac{\beta_1^2}{\alpha_1}+\frac{\beta_2^2}{\alpha_2}\right)\right]\mathrm{erf}\left(\frac{\alpha_2\gamma\beta_1+\alpha_1\gamma\delta\beta_2+2\alpha_1\alpha_2\gamma^2\epsilon}{2\sqrt{\alpha_1\alpha_2\gamma^2(\alpha_1\alpha_2\gamma^2+\alpha_1\delta^2+\alpha_2)}}\right).\end{aligned} \tag{B5}$$

This identity is a direct consequence of the identity (B4). The constants α_i (with $\mathrm{Re}\,\alpha_i>0$), β_i, $\gamma\neq0$, δ and ϵ are arbitrary.

2. Convolution of a Gaussian with a *sinus cardinalis*

The second integral enters in the simplification of the asymptotic behavior of ρ_{c}. It is the convolution integral

$$\widetilde{I}(\alpha,\gamma)=\int\limits_{\mathbb{R}}dx\exp[-(x+\alpha)^2]\mathrm{sinc}(\gamma x). \tag{B6}$$

To evaluate this integral we introduce an auxiliary integral to obtain a Fourier transform of a Gaussian, i.e.

$$\widetilde{I}(\alpha,\gamma)=\frac{1}{\gamma}\int\limits_0^\gamma d\widetilde{\gamma}\int\limits_{\mathbb{R}}dx\exp[-(x+\alpha)^2]\cos(\widetilde{\gamma}x). \tag{B7}$$

First we integrate over x and then over $\widetilde{\gamma}$ to obtain an expression in terms of error functions,

$$\widetilde{I}(\alpha,\gamma)=\frac{\pi}{\gamma}\exp(-\alpha^2)\mathrm{Re}\;\mathrm{erf}\left(\frac{\gamma}{2}+\imath\alpha\right). \tag{B8}$$

Appendix C: The $Z^\nu_{1/1}$–Partition Function

In this appendix we evaluate the partition function $Z^\nu_{1/1}$ which enters in the expression for the distribution of the chiralities over the real eigenvalues of D_{W}. The derivation below is along the lines given in Ref. [42].

We employ the parametrization (52) to evaluate

$$\begin{aligned}&\lim_{\varepsilon\to0}\mathrm{Im}\int\frac{\det(D_{\mathrm{W}}-\widehat{z}_1/(2n)\mathbb{1}_{2n+\nu})}{\det(D_{\mathrm{W}}-\widehat{x}_2/(2n)\mathbb{1}_{2n+\nu}\mp\imath\varepsilon\gamma_5)}P(D_{\mathrm{W}})d[D_{\mathrm{W}}] \\ &\overset{n\gg1}{=}-\lim_{\varepsilon\to0}\mathrm{Im}\int\frac{de^{\imath\varphi}}{2\pi\imath}de^{\vartheta}d\eta d\eta^*\mathrm{Sdet}^{\,\nu}U\exp[-\widehat{a}_8^2\mathrm{Str}\,(U^2+U^{-2})] \\ &\quad\times\exp\left[\pm\frac{\imath}{2}\mathrm{Str}\,\mathrm{diag}\,(\widehat{x}_2-\widehat{m}_6,\widehat{z}_1-\widehat{m}_6)(U-U^{-1})-\left(\varepsilon\pm\frac{\imath\widehat{\lambda}_7}{2}\right)\mathrm{Str}\,(U+U^{-1})\right].\end{aligned} \tag{C1}$$

We employ the same trick as in Ref. [42] to linearize the exponent in U and U^{-1} by introducing an auxiliary Gaussian integral over a supermatrix, i.e.

$$\exp[-\widehat{a}_8^2 \mathrm{Str}\,(U^2+U^{-2})] = \int d[\sigma] \exp\left[-\frac{1}{16\widehat{a}_8^2}\mathrm{Str}\,\sigma^2 \pm \frac{\imath}{2}\mathrm{Str}\,\sigma(U-U^{-1})\right] \tag{C2}$$

with

$$\sigma = \begin{bmatrix} \sigma_1 & \eta_\sigma \\ \eta_\sigma^* & \imath\sigma_2 \end{bmatrix} \text{ and } d[\sigma] = d\sigma_1 d\sigma_2 d\eta_\sigma d\eta_\sigma^*. \tag{C3}$$

After plugging Eq. (C2) in Eq. (C1) we diagonalize $\sigma = V\mathrm{diag}\,(s_1,s_2)V^\dagger$ and integrate over $V \in \mathrm{U}\,(1/1)$. We obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\mathrm{Im}\int \frac{\det(D_\mathrm{W} - \widehat{z}_1/(2n)\mathbb{1}_{2n+\nu})}{\det(D_\mathrm{W} - \widehat{x}_2/(2n)\mathbb{1}_{2n+\nu} \mp \imath\varepsilon\gamma_5)} P(D_\mathrm{W})d[D_\mathrm{W}] \qquad \text{(C4)}\\ &\overset{n\gg 1}{=} \frac{1}{16\pi\widehat{a}_8^2}(\widehat{z}_1-\widehat{x}_2)\lim_{\varepsilon\to 0}\mathrm{Im}\int \frac{ds_1 ds_2}{s_1-\imath s_2}\left(\frac{s_1-\widehat{\lambda}_7 \pm \imath\varepsilon}{s_1+\widehat{\lambda}_7 \mp \imath\varepsilon}\frac{\imath s_2+\widehat{\lambda}_7 \mp \imath\varepsilon}{\imath s_2-\widehat{\lambda}_7 \pm \imath\varepsilon}\right)^{\nu/2}\\ &\quad\times \exp\left[-\frac{1}{16\widehat{a}_8^2}\left((s_1-\widehat{x}_2+\widehat{m}_6)^2-(s_2+\imath\widehat{z}_1-\imath\widehat{m}_6)^2\right)\right] Z^\nu_{1/1}\left(\sqrt{s_1^2-(\widehat{\lambda}_7\mp\imath\varepsilon)^2},\imath\sqrt{s_2^2+(\widehat{\lambda}_7\mp\imath\varepsilon)^2};\widehat{a}=0\right), \end{aligned}$$

which expresses the partition function at non-zero lattice spacing in terms of an integral over the partition function with one bosonic and one fermionic flavor at zero lattice spacing (55).

The resolvent $G_{1/1}$ is given by the derivative with respect to $\widehat{z}_1$, see Eq. (48). To obtain a non-zero result we necessarily have to differentiate the prefactor $(\widehat{z}_1-\widehat{x}_2)$. The distribution of the chiralities of the real eigenvalues of D_W follows from the imaginary part of the resolvent. The Efetov-Wegner term [57, 58] appearing after diagonalizing σ is the normalization $Z^\nu_{1/1}(1,1)=1$ and vanishes when taking the imaginary part.

Two terms contribute to the imaginary part of the resolvent. First, the imaginary part of

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\mathrm{Im}\left[\frac{1}{(s_1+\widehat{\lambda}_7-\imath\varepsilon)^\nu}\right] = \frac{(-1)^{\nu-1}}{(\nu-1)!}\delta^{(\nu)}(s_1+\widehat{\lambda}_7) \tag{C5}$$

is the ν-th derivative of the Dirac delta-function. Second, when $|s_1|<|\widehat{\lambda}_7|$, the imaginary part arising from the logarithmic contribution of $K_\nu(z)$, i.e.

$$K_\nu(z) = (-1)^{\nu+1}I_\nu(z)\log z + \frac{1}{z^\nu}\sum_{k=0}^\infty a_k z^k \tag{C6}$$

also contributes to the imaginary part of the resolvent. The Bessel functions of the imaginary part of $Z^\nu_{1/1}(x_1,x_2,\widehat{a}=0)$ combine into the two-flavor partition function $Z^\nu_{2/0}(x_1,x_2,\widehat{a}=0)$. Adding both contributions we arrive at the result

$$\begin{aligned} &\lim_{\varepsilon\to 0}\mathrm{Im}\int \frac{\det(D_\mathrm{W} - z_1\mathbb{1}_{2n+\nu})}{\det(D_\mathrm{W} - x_2\mathbb{1}_{2n+\nu} \mp \imath\varepsilon\gamma_5)} P(D_\mathrm{W})d[D_\mathrm{W}] \qquad \text{(C7)}\\ &\overset{n\gg 1}{=} \frac{1}{16\pi\widehat{a}_8^2}(\widehat{z}_1-\widehat{x}_2)\lim_{\varepsilon\to 0}\mathrm{Im}\int \frac{ds_1 ds_2}{s_1-s_2}(\imath s_2+\widehat{\lambda}_7)^\nu(s_1+\widehat{\lambda}_7)^\nu \exp\left[-\frac{1}{16\widehat{a}_8^2}\left((s_1-\widehat{x}_2+\widehat{m}_6)^2-(s_2+\imath\widehat{z}_1-\imath\widehat{m}_6)^2\right)\right]\\ &\times\left[\frac{\delta^{(\nu-1)}(s_1+\widehat{\lambda}_7)}{(\nu-1)!(s_1-\widehat{\lambda}_7)^\nu}\left(\frac{s_1^2-\widehat{\lambda}_7^2}{s_2^2+\widehat{\lambda}_7^2}\right)^{\nu/2} Z^\nu_{1/1}\left(\sqrt{s_1^2-\widehat{\lambda}_7^2},\imath\sqrt{s_2^2+\widehat{\lambda}_7^2};\widehat{a}=0\right)\right.\\ &\qquad\left. -\mathrm{sign}(\widehat{\lambda}_7)\Theta(|\widehat{\lambda}_7|-|s_1|)\left(s_1^2+s_2^2\right)\frac{Z^\nu_{2/0}\left(\sqrt{s_1^2-\widehat{\lambda}_7^2},\imath\sqrt{s_2^2+\widehat{\lambda}_7^2};\widehat{a}=0\right)}{[(s_1^2-\widehat{\lambda}_7^2)(s_2^2+\widehat{\lambda}_7^2)]^{\nu/2}}\right] \end{aligned}$$

yielding Eq. (83).

Appendix D: Derivations of the Asymptotic Results given in Sec. IV

The derivation of asymptotic limits of the spectral density can be quite non-trivial because of cancellations of the leading contributions so that a naive saddle point approximation cannot be used. In the subsections below, we derive asymptotic expressions for the average number of additional real modes (Appendix D 1), the level density of the right handed modes (Appendix D 2) and the level density of the complex modes (Appendix D 3). In Appendix D 4 we consider the distribution of chirality over the real modes.

1. The average number of additional real modes

The limit of small lattice spacing is obvious and will not be discussed here. At large lattice spacing we rewrite Eq. (73) as

$$N_{\rm add} = \int\limits_{[0,2\pi]^2} \frac{d\Phi d\varphi}{8\pi^2} \cos[2\nu\Phi] \frac{1-\exp\left[-8(\widehat{a}_8^2\sin^2\varphi + 2\widehat{a}_7^2\cos^2\varphi)\sin^2\Phi\right]}{\sin^2\Phi}. \tag{D1}$$

Since $\widehat{a}_{7/8}$ are large we expand the angle Φ around the origin, in particular

$$\Phi = \frac{\delta\Phi}{\sqrt{\widehat{a}_8^2\sin^2\varphi + 2\widehat{a}_7^2\cos^2\varphi}} \ll 1. \tag{D2}$$

Note that we have two equivalent saddlepoints at 0 and at π. We thus have

$$N_{\rm add} = \int\limits_{\mathbb{R}\times[0,2\pi]} \frac{d\delta\Phi d\varphi}{4\pi^2} \frac{1-\exp\left[-8\delta\Phi^2\right]}{\delta\Phi^2} \sqrt{\widehat{a}_8^2\sin^2\varphi + 2\widehat{a}_7^2\cos^2\varphi}. \tag{D3}$$

The integral over $\delta\Phi$ is equal to $\sqrt{32\pi}$, and the integral over φ is the elliptic integral of the second kind. Hence we obtain the result (74).

2. The density of the additional real modes

We have two different cases for the behavior of $\rho_{\rm r}$ at large lattice spacing. To derive the large $\widehat{a}$ asymptotics in the case $\widehat{a}_8^2 = 0$ we rewrite Eq. (69) as a group integral, i.e.

$$\begin{aligned}\rho_{\rm r}(\widehat{x}) = {} & \frac{1}{2^{15/2}\pi\widehat{a}_7\sqrt{\widehat{a}_8^2+2\widehat{a}_6^2}} \int\limits_{\mathbb{R}^2} d\widehat{\lambda}_7 d\widetilde{x} \int\limits_{\rm U\,(2)} d\mu(U) \det{}^{\nu} U \exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}\right] \left[\mathrm{sign}(\widetilde{x}-\widehat{x}) - \mathrm{erf}\left(\frac{\widetilde{x}-\widehat{x}+2\widehat{\lambda}_7}{\sqrt{32\widehat{a}_8^2}}\right)\right](\widetilde{x}-\widehat{x}) \\ & \times \exp\left[-\widehat{a}_8^2 \mathrm{tr}\left(U+U^{-1}-\frac{\widetilde{x}+\widehat{x}}{8\widehat{a}_8{}^2}\mathbb{1}_2\right)^2 + \frac{\widehat{a}_6^2\widehat{a}_8^2}{\widehat{a}_8^2+2\widehat{a}_6^2}\mathrm{tr}^{\,2}\left(U+U^{-1}-\frac{\widetilde{x}+\widehat{x}}{8\widehat{a}_8{}^2}\mathbb{1}_2\right)\right] \\ & \times \exp\left[\frac{\widetilde{x}-\widehat{x}}{4}\mathrm{tr}\,\mathrm{diag}\,(1,-1)(U+U^{-1}) + \frac{\widehat{\lambda}_7}{2}\mathrm{tr}\,(U-U^{-1})\right]. \end{aligned} \tag{D4}$$

For $\widehat{a}_8^2 = 0$ Eq. (D4) simplifies to

$$\begin{aligned}\rho_{\mathrm{r}}(\widehat{x}) &= \frac{1}{256\pi\widehat{a}_7\widehat{a}_6}\int\limits_{\mathbb{R}^2} d\widehat{\lambda}_7 d\widetilde{x} \int\limits_{\mathrm{U}\,(2)} d\mu(U)\det{}^{\nu}U \exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}\right]\Big[\mathrm{sign}(\widetilde{x}-\widehat{x})-\mathrm{sign}(\widetilde{x}-\widehat{x}+2\widehat{\lambda}_7)\Big](\widetilde{x}-\widehat{x})\\ &\quad\times\exp\left[-\frac{(\widetilde{x}+\widehat{x})^2}{64\widehat{a}_6^2}+\frac{\widetilde{x}-\widehat{x}}{4}\mathrm{tr}\,\mathrm{diag}\,(1,-1)(U+U^{-1})+\frac{\widehat{\lambda}_7}{2}\mathrm{tr}\,(U-U^{-1})\right]\\ &= \frac{1}{32\pi\widehat{a}_7\widehat{a}_6}\int\limits_{\mathbb{R}} d\widetilde{x}\left(\Theta(\widetilde{x})\int\limits_{-\infty}^{-\widetilde{x}}-\Theta(-\widetilde{x})\int\limits_{-\widetilde{x}}^{\infty}\right)d\widehat{\lambda}_7\int\limits_{\mathrm{U}\,(2)} d\mu(U)\det{}^{\nu}U\\ &\quad\times\widetilde{x}\exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}-\frac{(\widetilde{x}+\widehat{x})^2}{16\widehat{a}_6^2}-\frac{\sqrt{\widehat{\lambda}_7^2-\widetilde{x}^2}}{2}\mathrm{tr}\,(U-U^{-1})\right].\end{aligned}\tag{D5}$$

For the second equality we substituted $\widetilde{x}\to 2\widetilde{x}+\widehat{x}$ and replaced the sign functions by the integration domains of $\widehat{\lambda}_7$. Moreover we used the fact that the group integral only depends on $\sqrt{\widehat{\lambda}_7^2-\widetilde{x}^2}$.

The saddlepoint equation of the U integral in Eq. (D5) gives four saddle points,

$$U=\pm\imath\mathbb{1}_2,\ \text{and}\ U=\pm\imath\mathrm{diag}\,(1,-1).\tag{D6}$$

The saddlepoints which are proportional to unity are algebraically suppressed while the contribution of the other two saddle points is the same. We thus find

$$\rho_{\mathrm{r}}(\widehat{x})=\frac{1}{8\pi^2\widehat{a}_7\widehat{a}_6}\int\limits_0^{\infty}d\widetilde{x}\int\limits_{\widetilde{x}}^{\infty}d\widehat{\lambda}_7\frac{\widetilde{x}}{\sqrt{\widehat{\lambda}_7^2-\widetilde{x}^2}}\cosh\left(\frac{\widetilde{x}\widehat{x}}{8\widehat{a}_6^2}\right)\exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}-\frac{\widetilde{x}^2+\widehat{x}^2}{16\widehat{a}_6^2}\right].\tag{D7}$$

After substituting $\widehat{\lambda}_7\to\widetilde{x}\cosh\vartheta$ the integral over ϑ yields the first case of Eq. (78).

For $\widehat{a}_8\neq 0$ we again start with Eq. (69). The integration over the two error functions, see Eq. (70), makes it difficult to evaluate the result directly, particularly when $\widehat{a}_7\neq 0$. As long as $\widehat{a}_7$ is finite, the second error function does not yield anything apart from giving a Gaussian cut-off to the integral. The imaginary part of the argument of the second error function shows strong oscillations resulting in cancellations. These oscillations also impede a numerical evaluation of the integrals for large lattice spacing.

Let $\widehat{a}_6=0$ to begin with. A non-zero value of $\widehat{a}_6$ can be introduced later by a convolution with a Gaussian in $\widehat{x}$. To obtain the correct contribution from the first term we consider a slight modification of ρ_{r},

$$I(X,\alpha)=\int\limits_{[0,2\pi]^2}d\varphi_1 d\varphi_2\sin^2\left[\frac{\varphi_1-\varphi_2}{2}\right]e^{\imath\nu(\varphi_1+\varphi_2)}\frac{\widehat{k}(X,\varphi_1,\varphi_2)-\widehat{k}(X,\varphi_2,\varphi_1)}{\cos\varphi_2-\cos\varphi_1}\tag{D8}$$

with

$$\begin{aligned}\widehat{k}(X,\varphi_1,\varphi_2)&=\exp\left[4\widehat{a}_8^2\left(\cos\varphi_1-X\right)^2-4\widehat{a}_8^2\left(\cos\varphi_2-X\right)^2-4\widehat{a}_7^2(\sin\varphi_1+\sin\varphi_2)^2\right]\\ &\quad\times\left[\mathrm{erf}\left[\sqrt{8}\widehat{a}_8(X-\cos\varphi_1)\right]+\mathrm{erf}\left[2\widehat{a}_8\alpha(\cos\varphi_1-X)\right]\right].\end{aligned}\tag{D9}$$

The variable X plays the role of $\widehat{x}/(8\widehat{a}_8^2)$. The error function with the constant α replaces the second error function in Eq. (70) and is of order one in the limit $\widehat{a}\to\infty$. It regularizes the integral and its contribution will be removed at the end. However it has to fulfill some constraints to guarantee the existence of the saddlepoints

$$\varphi_1^{(0)},\varphi_2^{(0)}\in\{\pm\arccos X\}\ \text{with}\ X\in[-1,1].\tag{D10}$$

Nevertheless these saddlepoints are independent of α. The saddle point $\varphi_1^{(0)}=\varphi_2^{(0)}$ is algebraically suppressed in comparison to $\varphi_1^{(0)}=-\varphi_2^{(0)}$ due to the $\sin^2$ factor in the measure. Expanding about the saddlepoints yields

$$\begin{aligned}I(X,\alpha)&\propto\frac{\Theta(1-|X|)}{\widehat{a}_8}\int\limits_{\mathbb{R}^2}\frac{d\delta\varphi_1 d\delta\varphi_2}{\delta\varphi_1+\delta\varphi_2}\exp\left[-\frac{\widehat{a}_7^2}{\widehat{a}_8^2}\gamma^2(\delta\varphi_1+\delta\varphi_2)^2\right]\\ &\quad\times\left[\exp(\delta\varphi_1^2-\delta\varphi_2^2)\left(\mathrm{erf}(\sqrt{2}\delta\varphi_1)-\mathrm{erf}(\alpha\delta\varphi_1)\right)+\exp(\delta\varphi_2^2-\delta\varphi_1^2)\left(\mathrm{erf}(\sqrt{2}\delta\varphi_2)-\mathrm{erf}(\alpha\delta\varphi_2)\right)\right]\end{aligned}\tag{D11}$$

with

$$\gamma(X) = \frac{X}{\sqrt{1-X^2}}. \tag{D12}$$

In the next step we change the coordinates to center-of-mass-relative coordinates, i.e. $\Phi = \delta\varphi_1 + \delta\varphi_2$ and $\Delta\varphi = \delta\varphi_1 - \delta\varphi_2$, and find

$$\begin{aligned} I(X,\alpha) \propto \frac{\Theta(1-|X|)}{\widehat{a}_8} \int\limits_{\mathbb{R}^2} \frac{d\Phi d\Delta\varphi}{\Phi} \exp\left[-\frac{\widehat{a}_7^2}{\widehat{a}_8^2}\gamma^2(X)\Phi^2\right] \\ \times \left[\exp(\Phi\Delta\varphi)\left(\operatorname{erf}\left(\frac{\Phi+\Delta\varphi}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\alpha}{2}(\Phi+\Delta\varphi)\right)\right) + \exp(-\Phi\Delta\varphi)\left(\operatorname{erf}\left(\frac{\Phi-\Delta\varphi}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\alpha}{2}(\Phi-\Delta\varphi)\right)\right)\right]. \end{aligned} \tag{D13}$$

We perform an integration by parts in $\Delta\varphi$ yielding Gaussian integrals in $\Delta\varphi$ which evaluate to

$$I(X,\alpha) \propto \frac{\Theta(1-|X|)}{\widehat{a}_8} \int\limits_{\mathbb{R}} \frac{d\Phi}{\Phi^2} \exp\left[-\frac{\widehat{a}_7^2}{\widehat{a}_8^2}\gamma^2(X)\Phi^2\right]\left[\exp\left(-\frac{\Phi^2}{2}\right) - \exp\left(-\frac{1+\alpha^2}{\alpha^2}\Phi^2\right)\right]. \tag{D14}$$

The $1/\Phi^2$ term can be exponentiated by introducing an auxiliary integral and the resulting Gaussian over Φ can be performed. We obtain

$$\begin{aligned} I(X,\alpha) &\propto \Theta(1-|X|) \int\limits_{(1+\alpha^2)/\alpha^2}^{1/2} \frac{dt}{\sqrt{\widehat{a}_7^2\gamma^2(X) + \widehat{a}_8^2 t}} \\ &\propto \frac{\Theta(1-|X|)}{\widehat{a}_8^2}\left(\sqrt{\widehat{a}_7^2\gamma^2(X) + \frac{\widehat{a}_8^2}{2}} - \sqrt{\widehat{a}_7^2\gamma^2(X) + \frac{1+\alpha^2}{\alpha^2}\widehat{a}_8^2}\right). \end{aligned} \tag{D15}$$

The contribution of the artificial term depending on α can be readily read off, but it fixes the integral only up to an additive constant. This constant can be determined by integrating the result over $\widehat{x}$ which has to agree with the large $\widehat{a}$ limit of N_{add}, cf. Eq. (74). It turns out that this constant is equal to zero. The overall constant is also obtained by comparing to N_{add}.

The convolution with the Gaussian distribution generating $\widehat{a}_6$ does not give something new in the limit of large lattice spacing. The width of this Gaussian scales with $\widehat{a}$ while the density ρ_{r} has support on $\widehat{a}^2$, so that it becomes a Dirac delta-function in the large $\widehat{a}$ limit.

3. The density of the complex eigenvalues

Let $\widehat{a}_8 > 0$ and $\widehat{a} \gg 1$. Then we perform a saddlepoint approximation of Eq. (79) in the integration variables $\varphi_{1/2}$. The saddlepoints are given by

$$\varphi_1^{(0)} = -\varphi_2^{(0)} = \pm\arccos\left(\frac{\widehat{x}}{8\widehat{a}_8^2}\right) \quad \text{with} \quad \widehat{x} \in [-8\widehat{a}_8^2, 8\widehat{a}_8^2]. \tag{D16}$$

We have also the saddlepoints $\varphi_1^{(0)} = \varphi_2^{(0)}$ if $\widehat{a}_7 = 0$. However they are algebraically suppressed due to the Haar measure. Notice the two saddlepoints in Eq. (D16) yield the same contribution. After the integration over the massive modes about the saddlepoint we find the first case of Eq. (82). In the calculation we used the convolution integral derived in Appendix B 2.

Let us now look at the case with $\widehat{a}_8 = 0$. Then we have

$$\begin{aligned} \rho_{\text{c}}(\widehat{z}) \overset{\widehat{a}\gg 1}{=} \frac{|\widehat{y}|}{4\pi^2\sqrt{16\pi\widehat{a}_6^2}\sqrt{16\pi\widehat{a}_7^2}} \exp\left[-\frac{\widehat{x}^2}{16\widehat{a}_6^2}\right] \int\limits_{\mathbb{R}} d\widehat{\lambda}_7 \exp\left[-\frac{\widehat{\lambda}_7^2}{16\widehat{a}_7^2}\right] \\ \times \int\limits_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1-\varphi_2}{2}\right] \cos[\nu(\varphi_1+\varphi_2)] \operatorname{sinc}\left[\widehat{y}(\cos\varphi_1 - \cos\varphi_2)\right] \exp\left[\imath\widehat{\lambda}_7(\sin\varphi_1 + \sin\varphi_2)\right]. \end{aligned} \tag{D17}$$

The integrals over the angles can be rewritten as a group integral over $\mathrm{U}(2)$,

$$\int\limits_{[0,2\pi]^2} d\varphi_1 d\varphi_2 \sin^2\left[\frac{\varphi_1-\varphi_2}{2}\right] e^{\nu\imath(\varphi_1+\varphi_2)}\mathrm{sinc}\left[\widehat{y}(\cos\varphi_1-\cos\varphi_2)\right]\exp\left[\imath\widehat{\lambda}_7(\sin\varphi_1+\sin\varphi_2)\right]$$
$$= 2\pi^2\int\limits_{\mathrm{U}(2)} d\mu(U)\det{}^{\nu}U\exp\left[\frac{1}{2}\mathrm{tr}\left(\Lambda U-\Lambda^* U^\dagger\right)\right] \tag{D18}$$

with $\Lambda = \mathrm{diag}\,(\widehat{\lambda}_7+\imath\widehat{y},\widehat{\lambda}_7-\imath\widehat{y})$. This integral only depends on the quantity $\sqrt{\widehat{\lambda}_7^2+\widehat{y}^2}$ because the angle of the combined complex variable $\widehat{\lambda}+\imath\widehat{y}$ can be absorbed into U, i.e.

$$\int\limits_{\mathrm{U}(2)} d\mu(U)\det{}^{\nu}U\exp\left[\frac{1}{2}\mathrm{tr}\left(\Lambda U-\Lambda^* U^\dagger\right)\right] = \int\limits_{\mathrm{U}(2)} d\mu(U)\det{}^{\nu}U\exp\left[\frac{\sqrt{\widehat{\lambda}_7^2+\widehat{y}^2}}{2}\mathrm{tr}\left(U-U^\dagger\right)\right]. \tag{D19}$$

The variable $\widehat{y}$ as well as the integration variable $\widehat{\lambda}_7$ are of the order $\widehat{a}$. Therefore we can perform a saddlepoint approximation and end up with

$$\rho_{\mathrm{c}}(\widehat{z}) \overset{\widehat{a}\gg 1}{=} \frac{|\widehat{y}|}{\pi\sqrt{16\pi\widehat{a}_6^2}\sqrt{16\pi\widehat{a}_7^2}}\exp\left[-\frac{\widehat{x}^2}{16\widehat{a}_6^2}\right]\int\limits_{\mathbb{R}} d\widehat{\lambda}_7\frac{\exp\left[-\widehat{\lambda}_7^2/(16\widehat{a}_7^2)\right]}{\sqrt{\widehat{\lambda}_7^2+\widehat{y}^2}}. \tag{D20}$$

resulting in the second case of Eq. (82).

4. The distribution of chirality over the real eigenvalues

In this Appendix we derive the large $\widehat{a}$ limit of $\rho_\chi(\widehat{x})$ for $\widehat{a}_8>0$ given in Eq. (93). The case $\widehat{a}_8=0$ reduces $\rho_\chi(\widehat{x})$ to the result (59) and will not be discussed in this section. We set $\widehat{a}_{6/7}=0$ to begin with and introduce them later on.

The best way to obtain the asymptotics for large lattice spacing is to start with Eq. (C1) with $\widehat{m}_6=\widehat{\lambda}_7=\varepsilon=0$. The integral does not need a regularization since the $\widehat{a}_8$-term guarantees the convergence. We also omit the sign in front of the linear trace terms in the Lagrangian because we can change $U\to -U$.

In the first step we substitute $\eta\to e^{\imath\varphi}\eta$ and $\eta^*\to e^{\vartheta}\eta^*$. Then the measure is $d\varphi d\vartheta d\eta d\eta^*$ and the parametrization of U is given by

$$U=\begin{bmatrix} e^{\vartheta} & 0\\ 0 & e^{\imath\varphi}\end{bmatrix}\begin{bmatrix} 1 & \eta^*\\ \eta & 1\end{bmatrix},\; U^{-1}=\begin{bmatrix} 1+\eta^*\eta & -\eta^*\\ -\eta & 1-\eta^*\eta\end{bmatrix}\begin{bmatrix} e^{-\vartheta} & 0\\ 0 & e^{-\imath\varphi}\end{bmatrix}. \tag{D21}$$

There are two saddlepoints in the variables ϑ and φ, i.e.

$$e^{\vartheta_0}=-\frac{\imath\widehat{x}_2}{8\widehat{a}_8^2}+\sqrt{1-\left(\frac{\widehat{x}_2}{8\widehat{a}_8^2}\right)^2},\quad e^{\imath\varphi_0}=-\frac{\imath\widehat{z}_1}{8\widehat{a}_8^2}+L\sqrt{1-\left(\frac{\widehat{z}_1}{8\widehat{a}_8^2}\right)^2} \tag{D22}$$

with $L=\pm 1$. Moreover, the variables $\widehat{z}_1,\widehat{x}_2$ have to be in the interval $[-8\widehat{a}_8^2,8\widehat{a}_8^2]$ else the contributions will be exponentially suppressed. We have no second saddlepoint for the variable ϑ since the real part of the exponential has to be positive definite. Other saddlepoints which can be reached by shifting φ and ϑ by $2\pi\imath$ independently are forbidden since they are not accessible in the limit $\widehat{a}_8\to\infty$. Notice that the saddlepoint solutions (D22) are phases, i.e. $|e^{\vartheta_0}|=|e^{\imath\varphi_0}|=1$.

In the second step we expand the integration variables

$$e^{\vartheta}=e^{\vartheta_0}\left(1+\frac{\delta\vartheta}{\sqrt{(8\widehat{a}_8^2)^2-\widehat{x}_2^2}}\right),\quad e^{\imath\varphi}=e^{\imath\varphi_0}\left(1+\frac{\imath\delta\varphi}{\sqrt{(8\widehat{a}_8^2)^2-\widehat{z}_1^2}}\right). \tag{D23}$$

All terms in front of the exponential as well as of the Grassmann variables are replaced by the saddlepoint solutions ϑ_0 and φ_0. The resulting Gaussian integrals over the variables $\delta\vartheta$ and $\delta\varphi$ yield

$$\begin{aligned}
&\mathrm{Im}\int d\mu(U)\mathrm{Sdet}^{\nu}U\exp\left[-\widehat{a}_8^2\mathrm{Str}\,(U-U^{-1})^2+\frac{\imath}{2}\mathrm{Str}\,\mathrm{diag}\,(\widehat{x}_2,\widehat{z}_1)(U-U^{-1})\right]\\
&\quad\propto\sum_{L\in\{\pm1\}}\mathrm{Im}\,\frac{\exp[\nu(\vartheta_0-\imath\varphi_0)]}{\sqrt{(8\widehat{a}_8^2)^2-\widehat{x}_2^2}\sqrt{(8\widehat{a}_8^2)^2-\widehat{z}_1^2}}\exp\left[-\frac{\widehat{x}_2^2-\widehat{z}_1^2}{16\widehat{a}_8^2}\right]\\
&\quad\times\int d\eta d\eta^*(1-\eta^*\eta)^{\nu}\exp[-2\widehat{a}_8^2(e^{\vartheta_0}+e^{-\imath\varphi_0})(e^{-\vartheta_0}+e^{\imath\varphi_0})\eta^*\eta].
\end{aligned}\tag{D24}$$

After the integration over the Grassmann variables we have two terms, one is of order one, and the other one of order $\widehat{a}_8^2$ which exceeds the first term for $\widehat{a}_8\gg1$. Hence we end up with

$$\begin{aligned}
&\mathrm{Im}\int d\mu(U)\mathrm{Sdet}^{\nu}U\exp\left[-\widehat{a}_8^2\mathrm{Str}\,(U-U^{-1})^2+\frac{\imath}{2}\mathrm{Str}\,\mathrm{diag}\,(\widehat{x}_2,\widehat{z}_1)(U-U^{-1})\right]\\
&\quad\propto\sum_{L\in\{\pm1\}}\mathrm{Im}\,\frac{1}{\sqrt{(8\widehat{a}_8^2)^2-\widehat{x}_2^2}\sqrt{(8\widehat{a}_8^2)^2-\widehat{z}_1^2}}\left(\frac{-\imath\widehat{x}_2+\sqrt{(8\widehat{a}_8^2)^2-\widehat{x}_2^2}}{-\imath\widehat{z}_1+L\sqrt{(8\widehat{a}_8^2)^2-\widehat{z}_1^2}}\right)^{\nu}\\
&\quad\times\left[\left(\frac{\widehat{z}_1-\widehat{x}_2}{8\widehat{a}_8^2}\right)^2+\left(\sqrt{1-\left(\frac{\widehat{x}_2}{8\widehat{a}_8^2}\right)^2}+L\sqrt{1-\left(\frac{\widehat{z}_1}{8\widehat{a}_8^2}\right)^2}\right)^2\right]\exp\left[-\frac{\widehat{x}_2^2-\widehat{z}_1^2}{16\widehat{a}_8^2}\right].
\end{aligned}\tag{D25}$$

Notice that both saddlepoints, $L\in\{\pm1\}$, give a contribution for independent variables $\widehat{z}_1$ and $\widehat{x}_2$. To obtain the resolvent we differentiate this expression with respect to $\widehat{z}_1$ and put $\widehat{z}_1=\widehat{x}_1$ afterwards. The first term between the large brackets and the second term for $L=-1$ are quadratic in $\widehat{z}_1-\widehat{x}_2$ and do not contribute to the resolvent. For $L=+1$ we obtain

$$\mathrm{Im}\,\partial_{\widehat{z}_1}|_{\widehat{z}_1=\widehat{x}_2}\int d\mu(U)\mathrm{Sdet}^{\nu}U\exp\left[-\widehat{a}_8^2\mathrm{Str}\,(U-U^{-1})^2+\frac{\imath}{2}\mathrm{Str}\,\mathrm{diag}\,(\widehat{x}_2,\widehat{z}_1)(U-U^{-1})\right]\propto\nu\frac{\Theta(8\widehat{a}_8^2-|\widehat{x}_2|)}{2\widehat{a}_8^2\sqrt{64\widehat{a}_8^4-\widehat{x}_2^2}}\tag{D26}$$

This limit yields the square root singularity. The normalization of ρ_χ to ν yields an overall normalization constant of $2\widehat{a}_8^2/\pi$.

The effect of $\widehat{a}_6$ is introduced by the integral

$$\begin{aligned}
\frac{1}{4\widehat{a}_6\sqrt{\pi}}\int_{\mathbb{R}}\exp\left[-\frac{\widehat{m}_6^2}{16\widehat{a}_6^2}\right]\rho_\chi(\widehat{x}-\widehat{m}_6)|_{\widehat{a}_6=0}d\widehat{m}_6&=\frac{1}{4\widehat{a}_6\sqrt{\pi}}\int_{\mathbb{R}}\exp\left[-\frac{\widehat{m}_6^2}{16\widehat{a}_6^2}\right]\frac{\Theta(8\widehat{a}_8^2-|\widehat{x}-\widehat{m}_6|)}{\pi\sqrt{(8\widehat{a}_8^2)^2-(\widehat{x}-\widehat{m}_6)^2}}d\widehat{m}_6\\
&=\frac{1}{4\widehat{a}_6\pi^{3/2}}\int_0^{\pi}\exp\left[-\frac{4\widehat{a}_8^4}{\widehat{a}_6^2}\left(\cos\varphi+\frac{\widehat{x}}{8\widehat{a}_8^2}\right)^2\right]d\varphi.
\end{aligned}\tag{D27}$$

In the large $\widehat{a}$ limit this evaluates to

$$\frac{1}{4\widehat{a}_6\sqrt{\pi}}\int_{\mathbb{R}}\exp\left[-\frac{\widehat{m}_6^2}{16\widehat{a}_6^2}\right]\frac{\Theta(8\widehat{a}_8^2-|\widehat{x}-\widehat{m}_6|)}{\pi\sqrt{(8\widehat{a}_8^2)^2-(\widehat{x}-\widehat{m}_6)^2}}d\widehat{m}_6\overset{\widehat{a}\gg1}{=}\nu\frac{\Theta(8\widehat{a}_8^2-|\widehat{x}|)}{\pi\sqrt{64\widehat{a}_8^4-\widehat{x}^2}},\tag{D28}$$

which is exactly the same Heaviside distribution with the square root singularities in the interval $[-8\widehat{a}_8^2,8\widehat{a}_8^2]$ of Eq. (D26). The introduction of $\widehat{a}_7$ follows from Eq. (87). We have to replace $\widehat{a}_6^2\to\widehat{a}_6^2+\widehat{a}_7^2$ and sum the result over the index j with the prefactor $\exp(-8\widehat{a}_7^2)[I_{j-\nu}(8\widehat{a}_7^2)-I_{j-\nu}(8\widehat{a}_7^2)]$. The intermediate result (D28) is independent of $\widehat{a}_7$ and linear in the index, in the sum this index is j. The sum over j can be performed according to

$$\sum_{j=1}^{\infty}j\left(I_{j-\nu}(8\widehat{a}_7^2)-I_{j+\nu}(8\widehat{a}_7^2)\right)=\nu\exp(8\widehat{a}_7^2)\tag{D29}$$

resulting in the asymptotic result (93).

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[60] Notice that in this subsection the terms proportional to $\widehat{m}_6$ and $\widehat{\lambda}_7$ are explicitly included in the Dirac operator rather than in the probability distribution as in the earlier sections. Moreover the operators D and $D_{\rm W}$ are already multiplied with ΣV such that we consider the dimensionless, rescaled spectrum of the Dirac operator.
[61] The authors of Ref. [7] fitted some RMT results with their own lattice data and obtained a dimensionless lattice spacing of the order $\widehat{a}_{6/7/8} \approx 0.1$ and less. The number of their configurations with index $\nu = 0$ was about 1000. Therefore our result estimates the number of additional real modes for the full ensemble they generated with $1000 \times N_{\rm add}^{\nu=0} \approx 10$ with a statistical error of about thirty percent. Increasing the number of configurations by a factor ten would already yield a statistical error of only ten percent.
[62] Note that $\mathcal{I}_l(Z_l, Z'_{l'})$ does not depend on the unitary transformation that diagonalizes V.
[63] We integrate first over $Z'_{l'}$ and then over Z_l.

Dirac Spectra of 2-dimensional QCD-like theories

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We analyze Dirac spectra of two-dimensional QCD like theories both in the continuum and on the lattice and classify them according to random matrix theories sharing the same global symmetries. The classification is different from QCD in four dimensions because the anti-unitary symmetries do not commute with γ_5. Therefore in a chiral basis, the number of degrees of freedom per matrix element are not given by the Dyson index. Our predictions are confirmed by Dirac spectra from quenched lattice simulations for QCD with two or three colors with quarks in the fundamental representation as well as in the adjoint representation. The universality class of the spectra depends on the parity of the number of lattice points in each direction. Our results show an agreement with random matrix theory that is qualitatively similar to the agreement found for QCD in four dimensions. We discuss the implications for the Mermin-Wagner-Coleman theorem and put our results in the context of two-dimensional disordered systems.

I. INTRODUCTION

It has been well established that chiral symmetry is spontaneously broken in strongly interacting systems of quarks and gluons for a wide range of parameters such as the temperature, the chemical potential, the number of colors, the number of flavors, the representation of the gauge group. In the broken phase the corresponding low energy effective theory is given by a weakly interacting system of pseudo-Goldstone bosons with a Lagrangian that is determined by the pattern of chiral symmetry breaking. In lattice QCD the spontaneous breaking of chiral symmetry is studied by evaluating the Euclidean partition function which is the average of the determinant of the Euclidean Dirac operator weighted by the Euclidean Yang-Mills action. Its low energy limit is given by the partition function of the Euclidean chiral Lagrangian. This theory simplifies drastically [1, 2] in the limit that the pion Compton wave-length is much larger than the size of the box. Then the partition function factorizes into a part comprising the modes with zero momentum and a part describing the modes with non-zero momentum. It turns out that the zero momentum part is equivalent to a random matrix theory with the same global symmetries of QCD [3].

A particular useful way to study chiral symmetry breaking is to analyze the properties of the eigenvalues of the Dirac operator. Because of the Banks-Casher formula [4] the chiral condensate $\Sigma = |\langle\bar{\psi}\psi\rangle|$, the order parameter for the spontaneous breaking of chiral symmetry, is given by the average spectral density (denoted by $\rho(\lambda)$) near zero of the Dirac operator per unit of the space-time volume V,

$$\Sigma \equiv |\langle\bar{\psi}\psi\rangle| = \lim_{a\to 0}\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\int_{-\infty}^{\infty}\frac{2m\rho(\lambda)d\lambda}{\lambda^2+m^2} = \lim_{a\to 0}\lim_{\lambda\to 0}\lim_{V\to\infty}\frac{\pi}{V}\rho(\lambda). \tag{1}$$

Here, a is the lattice spacing which provides the ultraviolet cut-off. The order of the limits is critical and a different order gives a different result. A better understanding of these limits can be obtained from the behavior of the eigenvalue density of the Dirac operator on the scale of the smallest eigenvalues which according to the Banks-Casher formula is given by

$$\Delta\lambda = \frac{1}{\rho(0)} = \frac{\pi}{\Sigma V}. \tag{2}$$

The so called microscopic spectral density is defined by [5]

$$\rho_s(x) = \lim_{V\to\infty}\frac{1}{\Sigma V}\rho\left(\frac{x}{\Sigma V}\right). \tag{3}$$

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If the Compton wavelength associated with the Dirac eigenvalues, λ, is much larger than the size of the box, L, then the partition function that generates the Dirac spectrum factorizes into a zero momentum part and a nonzero momentum part. The zero momentum part is completely determined by the global symmetries of the QCD-(like) partition function and is equivalent to a random matrix theory with the same global symmetries. The Compton wave length associated with the Dirac eigenvalues is the Compton wave-length of the corresponding pseudo-Goldstone modes and is given by

$$\frac{2\pi}{m_\pi} = \frac{2\pi F_\pi}{\sqrt{2\lambda\Sigma}}, \tag{4}$$

where F_π is the pion decay constant. The condition $1/m_\pi \gg L$ implies

$$\lambda \ll \frac{F_\pi^2}{2L^2\Sigma} = \lambda_L, \tag{5}$$

meaning that λ_L is the characteristic eigenvalue scale corresponding to the size of the box. In d dimensions the Euclidean volume is $V = L^d$ so that the average number of eigenvalues in the universal domain scales as

$$E_{\rm Th} \propto \frac{\lambda_L}{\Delta\lambda} = 2\pi F_\pi^2 L^{d-2}. \tag{6}$$

This scale is also known as the Thouless energy $E_{\rm Th}$. In two dimensions, the number of eigenvalues in the universal domain remains of $O(1)$ in the thermodynamic limit.

Arguments have been made that in one dimensional systems all states become localized for an arbitrary small amount of disorder. The two dimensional case is marginal. For site disorder all states are exponentially localized whereas for link disorder the situation is less clear [6]. For systems that are both rotational invariant and time reversal invariant (denoted by the Dyson index $\beta_{\rm D} = 1$) all states seem to be localized. In the case of rotational invariant systems with broken time reversal invariance (denoted by the Dyson index $\beta_{\rm D} = 2$) states in the center of the band seem to be delocalized and the localization length may be very large in a region around the band center. For non-rotational invariant spin 1/2 systems (denoted by the Dyson index $\beta_{\rm D} = 4$) states are delocalized for a substantial range of disorder and energies [7–9].

The connection between localization and Goldstone bosons was most clearly formulated by McKane and Stone [10]. They argued that a nonzero density of states around the origin for a disordered system may either indicate the presence of Goldstone bosons or may be due to a nonzero density of the localized states. For dynamical quarks the second alternative is not possible. The reason is that the eigenvalues of localized states are uncorrelated so that the partition function

$$Z(m) = \left\langle \prod_k (i\lambda_k + m) \right\rangle \tag{7}$$

factorizes into single eigenvalue partition functions resulting in a vanishing chiral condensate [11].

If we take the results from the condensed matter literature at face value, also in two dimensional systems, there may be a finite region of extended states around zero with correlations that are described by chiral random matrix theory or alternatively a partition function with spontaneously broken chiral symmetry. In more than two dimensions we expect that these correlations will remain in the presence of a fermion determinant and the corresponding partition function will be the zero momentum part of a chiral Lagrangian. In two dimensions, the presence of a fermion determinant may push these states beyond the Thouless energy so that *all* states become localized. This would reconcile the numerical results for $\beta_{\rm D} = 4$ with the Mermin-Wagner-Coleman theorem which states that a continuous symmetry cannot be broken spontaneously in two or less dimensions in systems with sufficiently short-range interactions. In terms of the supersymmetric formulation of the quenched limit, the Mermin-Wagner-Coleman theorem could be evaded because the symmetry group is non-compact. This has been shown for hyperbolic spin models [12, 13]. This opens the possibility to have extended states and universal spectral correlations also in two dimensions.

Another interpretation is possible if the localization length ξ is large so that we can consider the limiting case

$$1 \ll L \ll \xi. \tag{8}$$

Then the states behave as extended states with eigenvalues that are described by random matrix theory up to the Thouless energy. In this case the problems with the Mermin-Wagner-Coleman theorem can be avoided and a transition to a localized phase only takes place when $L \sim \xi$. In such scenario the scalar correlation function may drop off at a

similar rate, so that chiral symmetry appears to be broken in the regime (8). This correlation function was studied for the $N_{\rm c}$-color Thirring model [14] with a drop-off of $1/x^{1/N_{\rm c}}$.

A two-dimensional model for which Dirac spectra have been studied in great detail, both analytically and numerically, is the Schwinger model. The eigenvalue correlations of the one-flavor Schwinger model are given by random matrix theory as was shown numerically [15, 16] and analytically by calculating the Leutwyler-Smilga sum rules [17]. The two flavor Schwinger model was analyzed in great detail in [18], and after rescaling the eigenvalues by the average level spacing excellent agreement with chiral random matrix theory is observed. No agreement with chiral random matrix theory is found for the quenched Schwinger model [16], while the spectral density seems to diverge for $\lambda \to 0$. The repulsion between the eigenvalues seems to be greatly suppressed indicating that the states are localized.

Let us consider a theory where the mass dependent chiral condensate scales with the quark mass as

$$\Sigma(m) \sim m^{\alpha}. \tag{9}$$

In the Schwinger model we have that [19]

$$\alpha = \frac{N_{\rm f}-1}{N_{\rm f}+1}, \tag{10}$$

but the argument given in this paragraph is more general. According to the Gell-Mann-Oakes-Rennner relation the mass of the "pions" associated with this condensate is given by

$$m_\pi^2 \sim m^{\alpha+1}. \tag{11}$$

Using the relation (1) between the spectral density and the chiral condensate we find that the eigenvalue density behaves as

$$\rho(\lambda) \sim V\Sigma(\lambda) \sim V\lambda^{\alpha}. \tag{12}$$

The Thouless energy is given by the scale for which the pion Compton wavelength is equal to the size of the box, i.e. $m_\pi \propto 1/L$. Employing the relation (11) we find the mass associated to the Thouless energy,

$$m_{\rm th} \sim L^{-2/(\alpha+1)}. \tag{13}$$

The integrated spectral density is given by

$$N(\lambda) = \int_{-\lambda}^{\lambda} \rho(\lambda')d\lambda' \sim V\lambda^{\alpha+1}, \tag{14}$$

so that the average number of eigenvalues below the Thouless energy is proportional to

$$N_{\rm th} = N(m_{\rm th}) \sim L^{d-2}. \tag{15}$$

due to combination of Eqs. (13) and (14). Remarkably, the number of eigenvalues described by random matrix theory does not depend on α. In two dimensions this number is constant in the thermodynamic limit but the agreement with chiral random matrix theory seems to improve with larger volumes for the Schwinger model [16]. The corollary of this argument is that correlations of low-lying Dirac eigenvalues in conformal QCD-like theories are given by chiral random matrix theory after unfolding the eigenvalues, i.e. $\lambda_k' = \lambda_k^{\alpha+1}$.

The eigenvalues scale with the volume as

$$\lambda \sim V^{-1/(\alpha+1)} \tag{16}$$

via the relation (14) when keeping the average number of eigenvalues fixed. This scaling was studied in [18] where a volume scaling of $V^{-5/8}$ is observed for two almost massless flavors, c.f. Eq. (10). This would correspond to $N_{\rm f} = 4$, cf. Eq. (10). This is actually correct because the lattice Dirac operator couples only even and odd sites doubling the number of flavors. Apparently, we need exact massless quarks to push the states in the localized domain.

Another important difference between QCD in four dimensions and QCD in lower dimensions is the index of the Dirac operator. In four dimensions the index is equal to the topological charge of the gauge field configurations. In three dimensions the index is not defined. In two dimensions topology is defined for U (1) and can for example be studied for the Schwinger model [17, 20]. However, for higher dimensional gauge groups the index of the Dirac operator is zero [19, 21, 22] although unstable instantons do exist [23, 24].

In this paper we consider the quenched two-dimensional QCD Dirac operator in the strong coupling limit with the gauge fields distributed according to the Haar measure. Both the continuum limit and the lattice QCD Dirac operator will be discussed. For the lattice Dirac operator we employ naive fermions. Our original motivation for this choice was to understand the transition between different symmetry classes when taking the continuum limit which was observed for staggered fermions in three [25] as well as in four [26] dimensions, but this issue is not addressed in this paper.

The strong coupling lattice model is expected to be equivalent to an interacting theory of mesons and/or baryons. For U(1) gauge theories in two dimensions this has been shown explicitly [27] by means of a color-flavor transformation [28–30], where a gradient expansion generates the various terms of a chiral Lagrangian. In this paper, we do not perform the continuum limit, so that the lattice theory is equivalent to an unrenormalized chiral Lagragian, and the usual arguments, that the states below the Thouless energy are correlated according to random matrix theory, apply. In the continuum limit and two dimensions, the fluctuations of the hadronic fields will dominate the chiral condensate and the theory renormalizes to a trivial phase without Goldstone bosons. In other words, the theory renormalizes to a localized phase.

The symmetry breaking pattern for the continuum limit in any dimension was discussed in [31] and in the context of topological insulators in [32] and goes back to what is known as Bott-periodicity. The dimensional dependence of the symmetry breaking pattern (see Table I) has its origins in the structure of Clifford algebras. The four dimensional symmetry breaking pattern and its description in terms of random matrix theory has been known for a long time [33–35]. Because of the absence of the γ_5 Dirac matrix in three dimensions the symmetry breaking pattern is different [36–38]. In two dimensions, the γ_5 matrix is replaced by the third Pauli matrix, σ_3. This matrix also anticommutes with the Dirac operator, but it does not commute with the charge conjugation matrix given by σ_2, which leads to a different symmetry breaking pattern [31].

The symmetries of the Dirac operator also depend on the parity of the lattice. If the lattice size is even in both directions, even lattice sites are only coupled to odd lattice sites resulting in a "lattice chiral symmetry". Having a lattice that is odd in one direction and even in the other one also puts global constraints on the Dirac operator resulting in different symmetry properties. In this paper we classify lattice theories in terms of random matrix theory. In total we can distinguish 9 classes. First of all they differ in their anti-unitary symmetries, namely with no anti-unitary symmetry, with an anti-unitary symmetry that squares to 1, and with an anti-unitary symmetry that squares to -1. Moreover for each of these three classes we can have an even-even, an even-odd or an odd-odd lattice. For all nine classes we give the spectral properties in the microscopic domain and compare them with lattice simulations of the corresponding lattice theory in the strong coupling limit.

In Section 2 we discuss the microscopic Dirac spectrum and chiral symmetry breaking pattern for the continuum limit of two-dimensional QCD. The two dimensional lattice gauge theory for three different values of the Dyson index is analyzed in section 3, and concluding remarks are made in Section 4. In the appendices we derive several random matrix results that have been used in the main text.

II. CONTINUUM DIRAC OPERATOR

The Euclidean Dirac operator of QCD-like theories is given by

$$\mathcal{D} = \gamma^\mu(\partial_\mu + iA^a_\mu\lambda_a), \tag{17}$$

where A^a_μ are the gauge fields, γ_μ are the Euclidean γ-matrices and λ_a are the generators of the gauge group. For an even number of dimensions the Dirac operator in a chiral basis reduces to a 2×2 block structure

$$\mathcal{D}^{(2/4)} = \begin{bmatrix} 0 & \mathcal{W}^{(2/4)} \\ -\mathcal{W}^{(2/4)\,\dagger} & 0 \end{bmatrix}, \tag{18}$$

where in two dimensions the operator $\mathcal{W}^{(2)}$ is given by

$$\mathcal{W}^{(2)} = \partial_1 + i\partial_2 + (iA^a_1 - A^a_2)\lambda_a, \tag{19}$$

and in four dimensions the operator $\mathcal{W}^{(4)}$ can be written as

$$\mathcal{W}^{(4)} \;=\; i\sigma_\mu(\partial_\mu + iA^a_\mu\lambda_a) \tag{20}$$

employing the standard chiral representation of Euclidean γ-matrices with $\sigma_\mu = (\sigma_k, -i\mathbf{1}_4)$ and σ_k the Pauli matrices.

d	Gauge theory	$\beta_{\rm D}$	$\alpha_{\rm D}$	Symmetry Breaking Pattern	RMT
2	$N_{\rm c}=2$, fund.	1	1	$\mathrm{USp}\,(2N_{\rm f})\times\mathrm{USp}\,(2N_{\rm f})\to\mathrm{USp}\,(2N_{\rm f})$	(CI)
2	$N_{\rm c}\geq 3$, fund.	2	1	$\mathrm{U}\,(N_{\rm f})\times\mathrm{U}\,(N_{\rm f})\to\mathrm{U}\,(N_{\rm f})$	chGUE (AIII)
2	$N_{\rm c}\geq 2$, adj.	4	1, 5	$\mathrm{O}\,(2N_{\rm f})\times\mathrm{O}\,(2N_{\rm f})\to\mathrm{O}\,(2N_{\rm f})$	(DIII)
3	$N_{\rm c}=2$, fund.	1	0	$\mathrm{USp}\,(4N_{\rm f})\to\mathrm{USp}\,(2N_{\rm f})\times\mathrm{USp}\,(2N_{\rm f})$	GOE (AI)
3	$N_{\rm c}\geq 3$, fund.	2	0	$\mathrm{U}\,(2N_{\rm f})\to\mathrm{U}\,(N_{\rm f})\times\mathrm{U}\,(N_{\rm f})$	GUE (A)
3	$N_{\rm c}\geq 2$, adj.	4	0	$\mathrm{O}\,(2N_{\rm f})\to\mathrm{O}\,(N_{\rm f})\times\mathrm{O}\,(N_{\rm f})$	GSE (AII)
4	$N_{\rm c}=2$, fund.	1	ν	$\mathrm{U}\,(2N_{\rm f})\to\mathrm{USp}\,(2N_{\rm f})$	chGOE (BDI)
4	$N_{\rm c}\geq 3$, fund.	2	$2\nu+1$	$\mathrm{U}\,(N_{\rm f})\times\mathrm{U}\,(N_{\rm f})\to\mathrm{U}\,(N_{\rm f})$	chGUE (AIII)
4	$N_{\rm c}\geq 2$, adj.	4	$4\nu+3$	$\mathrm{U}\,(2N_{\rm f})\to\mathrm{O}\,(2N_{\rm f})$	chGSE (CII)

TABLE I: Symmetry breaking patterns in two ($d=2$), three ($d=3$), and four ($d=4$) dimensions for different gauge theories and their associated Dyson index $\beta_{\rm D}$ which is equal to the level repulsion. The corresponding random matrix theory sharing the same symmetry breaking pattern and its classificatrion according to symmetric spaces is indicated in the last column. The repulsion of the levels from the origin, $\lambda^{\alpha_{\rm D}}$, depends on the topological charge ν for QCD-like theories in four dimensions. The case of the two-dimensional $\mathrm{SU}\,(N_{\rm c})$ theory with the fermions in the adjoint representation is particular since the index of the Dirac operator is either 0 or 1 depending on the parity of the dimensions of the Dirac matrix. This results in a repulsion that is either λ or λ^5. The corresponding random matrix theory consists of anti-symmetric off-diagonal blocks so that depending on the dimensionality we have either no or one pair of generic zero modes, respectively. For a discussion of the classification of random matrix theories in terms of symmetric spaces we refer to Refs. [39, 40]. In this table we do not include the breaking of the axial symmetry.

In three dimensions, the Dirac operator is given by

$$\mathcal{D}=\sum_{k=1}^{3}\sigma^k(\partial_k+iA_k^a\lambda_a). \tag{21}$$

There is no involution that anti-commutes with the Dirac operator so that there is no chiral block structure. This is the crucial difference with even dimensional theories and was already studied in Ref. [41].

In addition to chiral symmetry the Dirac operator has other symmetries depending on the representation of the gauge group which is discussed in the ensuing subsections. In subsection II A we recall the discussion of the global symmetries of the QCD Dirac operator in three and four dimensions and extend it to the two dimensional theory as well. This symmetry classification is summarized in table I. In subsection II B we discuss the corresponding random matrix theories. Thereby we summarize the classification of the random matrix theories for three- and four-dimensional continuum QCD and supplement this with the random matrix theories for two-dimensional QCD. In subsection II C we recall the symmetry breaking patterns.

A. Anti-unitary symmetries of the QCD Dirac operator

The anti-unitary symmetries of the Dirac operator depend on the representation of the generators λ_a of the gauge group $\mathrm{SU}\,(N_{\rm c})$. We consider three different gauge theories, namely with the gauge group $\mathrm{SU}\,(N_{\rm c}=2)$ and fermions in the fundamental representation denoted by the Dyson index $\beta_{\rm D}=1$, with the gauge group $\mathrm{SU}\,(N_{\rm c}>2)$ and fermions in the fundamental representation which is $\beta_{\rm D}=2$, and with the gauge group $\mathrm{SU}\,(N_{\rm c}\geq 2)$ and fermions in the adjoint representation labelled by $\beta_{\rm D}=4$.

1. $\beta_{\rm D}=1$

Let us consider the first case which is QCD with two colors ($N_{\rm c}=2$) and fermions in the fundamental representation. Then the λ_a are given by the three Pauli matrices τ_a acting in color space. Hence each covariant derivative

$$\mathcal{D}_\mu=\partial_\mu+iA_\mu^a\tau_a \tag{22}$$

is pseudo-real (quaternion) and anti-Hermitian, i.e.

$$\mathcal{D}_\mu^\dagger=-\mathcal{D}_\mu \quad\text{and}\quad [\mathcal{D}_\mu,\tau_2K]_-=\mathcal{D}_\mu\tau_2K-\tau_2K\mathcal{D}_\mu=0 \tag{23}$$

with K the complex conjugation operator. The corresponding Dirac operator has the anti-unitary symmetry

$$[i\mathcal{D}^{(d)}, \tau_2 C K]_- = 0, \tag{24}$$

where C is the charge conjugation matrix. In four dimensions the charge conjugation matrix reads $C \equiv \gamma_2\gamma_4$ and in two and three dimensions it is given by $C = \sigma_2$.

A crucial point is that the anti-unitary operator satisfies

$$(C\tau_2 K)^2 = 1. \tag{25}$$

Therefore one can always find a gauge field independent basis for which the Dirac operator is real [35, 39]. This is the reason why this case is denoted by the Dyson index $\beta_{\rm D} = 1$ (one degree of freedom per matrix element). Collecting everything, the continuum Euclidean QCD Dirac operator for QCD with two fundamental fermions fulfills three global symmetries in four and two dimensions namely anti-Hermiticity, chiral symmetry, and a reality condition, i.e.

$$\mathcal{D}^{(4)\,\dagger} = -\mathcal{D}^{(4)}, \quad [\mathcal{D}^{(4)}, \gamma_5]_+ = 0, \quad \text{and} \quad [i\mathcal{D}^{(4)}, \tau_2\gamma_2\gamma_4 K]_- = 0 \tag{26}$$

for four dimensions, see Ref. [35], and

$$\mathcal{D}^{(2)\,\dagger} = -\mathcal{D}^{(2)}, \quad [\mathcal{D}^{(2)}, \sigma_3]_+ = 0, \quad \text{and} \quad [i\mathcal{D}^{(2)}, \tau_2\sigma_2 K]_- = 0 \tag{27}$$

for two dimensions. For three dimensions there is no chiral symmetry but the rest remains the same as in the even dimensional case

$$\mathcal{D}^{(3)\,\dagger} = -\mathcal{D}^{(3)} \quad \text{and} \quad [i\mathcal{D}^{(3)}, \tau_2\sigma_2 K]_- = 0, \tag{28}$$

see Ref. [41]. Next we discuss the implications of these symmetries.

In four dimensions, Eq. (26) implies that we can construct a gauge field independent basis for which the Dirac operator decomposes into a chiral block structure or a basis for which the Dirac operator becomes real. This can be done at the same time if the projection onto a chiral basis commutes with the anti-unitary symmetry. This is the case in four dimensions where

$$\left[\frac{1 \pm \gamma_5}{2}, \tau_2\gamma_2\gamma_4 K\right]_- = 0. \tag{29}$$

The corresponding random matrix ensemble is the chiral Gaussian orthogonal ensemble (chGOE), see Refs. [35].

Equation (29) does not carry over to the two-dimensional theory. In this case the projectors onto a chiral basis are given by $(1 \pm \sigma_3)/2$, playing the role of $(1 \pm \gamma_5)/2$, but the commutator with the anti-unitary operator does not vanish,

$$\left[\frac{1 \pm \sigma_3}{2}, \tau_2\sigma_2 K\right]_- \neq 0. \tag{30}$$

Therefore, one cannot find a basis for which the two-dimensional Dirac operator decomposes into real chiral blocks.

Choosing the chiral basis for $\mathcal{D}^{(2)}$ the anti-unitary symmetry yields a different condition

$$\left[\begin{pmatrix} 0 & i\tau_2 K \\ -i\tau_2 K & 0 \end{pmatrix}, \begin{pmatrix} 0 & i\mathcal{W}^{(2)} \\ -i\mathcal{W}^{(2)\,\dagger} & 0 \end{pmatrix}\right]_- = 0, \tag{31}$$

which is equivalent to

$$\mathcal{W}^{(2)} = -\tau_2 \mathcal{W}^{(2)\,T} \tau_2. \tag{32}$$

Thus the operator is anti-self-dual and complex since we have no additional symmetries. After a unitary transformation one obtains an equivalent Dirac operator with an off-diagonal block $\tau_2\mathcal{W}^{(2)}$ which is complex symmetric. The corresponding random matrix is known as the first Bogolyubov-de Gennes ensemble denoted by the Cartan symbol CI, see Ref. [40], and has been applied to the normal-superconducting transitions in mesoscopic physics [42].

In three dimensions we can construct a gauge field independent basis for which the matrix elements of the operator $i\mathcal{D}^{(3)}$ become real symmetric. The corresponding random matrix ensemble is the Gaussian orthogonal ensemble (GOE), see Refs. [41].

2. $\beta_{\rm D} = 2$

In the case of three or more colors ($N_{\rm c} \geq 3$) with the fermions in the fundamental representation the symmetry under complex conjugation (23) is lost. Only anti-Hermiticity and, for even dimensions, chiral symmetry survive. The global symmetries of the Dirac operator are

$$\mathcal{D}^{(2)\,\dagger} = -\mathcal{D}^{(2)} \quad \text{and} \quad [\mathcal{D}^{(2)}, \sigma_3]_+ = 0 \tag{33}$$

in two dimensions and

$$\mathcal{D}^{(4)\,\dagger} = -\mathcal{D}^{(4)} \quad \text{and} \quad [\mathcal{D}^{(4)}, \gamma_5]_+ = 0 \tag{34}$$

in four dimensions. Since there are no anti-unitary symmetries the operator $\mathcal{W}^{(2/4)}$ is generically complex both in two and four dimensions. This is the reason why we denote this case by the Dyson index $\beta_{\rm D} = 2$. Therefore the random matrix ensemble corresponding to the Dirac operator $\mathcal{D}^{(2)}$ as well as $\mathcal{D}^{(4)}$ is given by an ensemble of chiral, complex, anti-Hermitian random matrices which can be chosen with Gaussian weights. This ensemble is known as the chiral Gaussian Unitary Ensemble (chGUE), see Refs. [35].

In three dimensions we only have the anti-Hermiticity condition,

$$\mathcal{D}^{(3)\,\dagger} = -\mathcal{D}^{(3)}. \tag{35}$$

Hence the operator $i\mathcal{D}^{(3)}$ is Hermitian and its analogue in random matrix theory is the Gaussian Unitary Ensemble (GUE). The three dimensional case was discussed in Ref. [41] and its predictions for the microscopic Dirac spectrum have been confirmed by various lattice simulations [25].

3. $\beta_{\rm D} = 4$

The third case is for fermions in the adjoint representation with two or more colors ($N_{\rm c} \geq 2$). In this case the generators of the gauge group are anti-symmetric and purely imaginary. This results in two symmetry relations for the covariant derivatives

$$\mathcal{D}_\mu^\dagger = -\mathcal{D}_\mu \quad \text{and} \quad [K, i\mathcal{D}_\mu]_- = 0. \tag{36}$$

The corresponding Dirac operator fulfills the anti-unitary symmetry

$$[i\mathcal{D}^{(d)}, CK]_- = 0, \tag{37}$$

where the anti-unitary operator satisfies

$$(CK)^2 = -1 \tag{38}$$

for all dimensions. This allows us to construct a gauge field independent basis for which the matrix elements of the Dirac operator can be grouped into real quaternions. This case is denoted by the Dyson index $\beta_{\rm D} = 4$.

Collecting all global symmetries of the Dirac operator we have

$$\mathcal{D}^{(4)\,\dagger} = -\mathcal{D}^{(4)}, \quad [\mathcal{D}^{(4)}, \gamma_5]_+ = 0, \quad \text{and} \quad [i\mathcal{D}^{(4)}, \gamma_2\gamma_4 K]_- = 0 \tag{39}$$

for four dimensions,

$$\mathcal{D}^{(2)\,\dagger} = -\mathcal{D}^{(2)}, \quad [\mathcal{D}^{(2)}, \sigma_3]_+ = 0, \quad \text{and} \quad [i\mathcal{D}^{(2)}, \sigma_2 K]_- = 0 \tag{40}$$

for two dimensions, and

$$\mathcal{D}^{(3)\,\dagger} = -\mathcal{D}^{(3)} \quad \text{and} \quad [i\mathcal{D}^{(3)}, \sigma_2 K]_- = 0 \tag{41}$$

for three dimensions. The last case is the simplest. There is no chiral symmetry, but we can construct a basis for which the matrix elements of the Hermitian operator $i\mathcal{D}^{(3)}$ can be grouped into real quaternions. The associated random matrix ensemble is the Gaussian Symplectic Ensemble (GSE) pointed out for the first time in Ref. [41].

In two and four dimensions we have again to consider the commutator of the projection operators onto the eigenspaces of γ_5 and the anti-unitary operator. As is the case for $\beta_{\rm D} = 1$, the commuator vanishes in four dimensions,

$$\left[\frac{\mathbb{1} \pm \gamma_5}{2}, \gamma_2\gamma_4 K\right]_- = 0. \tag{42}$$

Therefore we can construct a basis for which $\mathcal{D}^{(4)}$ decomposes into chiral blocks with quaternion real elements. Therefore, such Dirac operators are in the universality class of the chiral Gaussian Symplectic Ensemble (chGSE), see Refs. [35].

In two dimensions the commutator of the anti-unitary symmetry and the chiral projector does not vanish, i.e.

$$\left[\frac{\mathbb{1} \pm \sigma_3}{2}, \sigma_2 K\right]_- \neq 0. \tag{43}$$

Therefore, there is no gauge field independent basis for which $\mathcal{D}^{(2)}$ decomposes into quaternion real chiral blocks. Nevertheless, we can find a basis for which one of these properties holds. In a chiral basis the anti-unitary symmetry (37) reads

$$\left[\begin{pmatrix} 0 & iK \\ -iK & 0 \end{pmatrix}, \begin{pmatrix} 0 & i\mathcal{W}^{(2)} \\ -i\mathcal{W}^{(2)\,\dagger} & 0 \end{pmatrix}\right]_- = 0 \tag{44}$$

and results into

$$\mathcal{W}^{(2)\,T} = -\mathcal{W}^{(2)}. \tag{45}$$

Thus the operator $\mathcal{W}^{(2)}$, see Eq. (19), is complex anti-symmetric. In random matrix theory this symmetry class is known as the second Bogolyubov-de Gennes ensemble denoted by the Cartan symbol DIII [40]. This ensemble also plays an important role in mesoscopic physics [42].

B. Random Matrix Theory for continuum QCD

As was outlined in Refs. [5, 35] a random matrix theory for the Dirac operator is obtained by replacing its matrix elements by random numbers while maintaining the global unitary and anti-unitary symmetries of the QCD(-like) theory. Within a wide class, the distribution of the eigenvalues on the scale of the average level spacing does not depend on the probability distribution of the matrix elements. This allows us to choose the probability distribution to be Gaussian. The random matrix partition function is thus given by

$$Z^{\nu}_{N_{\rm f}} = \int d[D] \exp\left[-\frac{n\beta_{\rm D}}{2} {\rm tr}\, D^\dagger D\right] \prod_{k=1}^{N_{\rm f}} \det(D + m_k \mathbb{1}). \tag{46}$$

In even dimensions, in particular for $d = 2, 4$, the Dirac operator has the chiral block structure

$$D = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}, \tag{47}$$

while in three dimensions the Dirac operator is still anti-Hermitian but the block structure is absent. The mass matrix for the $N_{\rm f}$ quarks is given by $M = {\rm diag}\,(m_1, \ldots, m_{N_{\rm f}})$. The measure $d[D]$ is the product of all real independent differentials of the matrix elements of D.

In three dimensions, the random matrix ensemble is $n \times n$ dimensional for $\beta_{\rm D} = 1, 2$ and $2n \times 2n$ dimensional for $\beta_{\rm D} = 4$. The random matrix iD is either real symmetric ($\beta_{\rm D} = 1$), Hermitian ($\beta_{\rm D} = 2$), or Hermitian self-dual ($\beta_{\rm D} = 4$). From the corresponding joint probability density of the eigenvalues [43],

$$p_{d=3}(\Lambda) \prod_{1 \le j \le n} d\lambda_j \propto |\Delta_n(\Lambda)|^{\beta_{\rm D}} \prod_{1 \le j \le n} \exp\left[-\frac{n\beta_{\rm D}}{2}\lambda_j^2\right] d\lambda_j, \tag{48}$$

one can already read off many important spectral properties of the QCD-Dirac operator $\mathcal{D}^{(3)}$ in the microscopic limit, cf. table I. Recall the Vandermonde determinant

$$\Delta_n(\Lambda) = \prod_{1\leq a<b\leq n} (\lambda_a - \lambda_b) = (-1)^{n(n-1)/2} \det\left[\lambda_a^{b-1}\right]_{1\leq a,b\leq n}. \tag{49}$$

Thus, in three dimensions the eigenvalues are not degenerate apart from the Kramers degeneracy of QCD with adjoint fermions. Moreover, the eigenvalues of $\mathcal{D}^{(3)}$ repel each other like $|\lambda_a-\lambda_b|^{\beta_{\rm D}}$ and have no repulsion from the origin [41].

In four dimensions, the operator $\mathcal{W}^{(4)}$ is replaced by an $n\times(n+\nu)$ real ($\beta_{\rm D}=1$) or complex ($\beta_{\rm D}=2$) random matrix W or a $2n\times 2(n+\nu)$ quaternion matrix for $\beta_{\rm D}=4$. Then the Dirac operator has exactly ν and 2ν zero modes for $\beta_{\rm D}=1,2$ and $\beta_{\rm D}=4$, respectively. Therefore, ν is identified as the index of the Dirac operator. Due to the axial symmetry the nonzero eigenvalues always come in pairs $\pm i\lambda$. Moreover, because of the quaternion structure, the eigenvalues of $\mathcal{D}^{(4)}$ as well as of the corresponding random matrix Dirac operator are degenerate for QCD with adjoint fermions. The joint probability density of the eigenvalues of the random matrix D reads [35]

$$p_\chi(\Lambda) \prod_{1\leq j\leq 2n} d\lambda_j \propto |\Delta_{2n}(\Lambda^2)|^{\beta_{\rm D}} \prod_{1\leq j\leq 2n} \exp\left[-\frac{n\beta_{\rm D}}{2}\lambda_j^2\right] \lambda_j^{\alpha_{\rm D}} d\lambda_j, \tag{50}$$

cf. table I. Again we can read off the behavior of the eigenvalues of D which, in the microscopic limit, are shared with the behavior of the low-lying eigenvalues of the QCD Dirac operator. The eigenvalues again repel each other as $|\lambda_a-\lambda_b|^{\beta_{\rm D}}$. The difference with the three dimensional case is the level repulsion from the origin $\lambda_a^{\alpha_{\rm D}} = \lambda_a^{\beta_{\rm D}(\nu+1)-1}$ which results from the generic zero modes and the chiral structure of the Dirac operator. The global symmetries of the four-dimensional QCD Dirac operator and their impact on the microscopic spectrum were discussed in Refs. [5, 35].

In two dimensions, rather than choosing a basis for which the Dirac operator becomes real or quaternion real for $\beta_{\rm D}=1$ and $\beta_{\rm D}=4$, respectively, we insist on a chiral basis that preserves the chiral block structure of the Dirac operator. This results in a random matrix theory for which the matrix $\tau_2 W$ is complex symmetric for $\beta_{\rm D}=1$, $\tau_2 W = (\tau_2 W)^T \in \mathbb{C}^{2n\times 2n}$ and complex anti-symmetric for $\beta_{\rm D}=4$, $W=-W^T\in\mathbb{C}^{n\times n}$, cf. Eqs. (32) and (45), respectively. For QCD with three or more colors and the fermions in the fundamental representation ($\beta_{\rm D}=2$), the two-dimensional Dirac operator has the same symmetries as the four-dimensional theory resulting in the same random matrix theory.

Another important difference between two and four dimensions is the topology of the gauge field configurations. For QCD with fundamental fermions the homotopy class is $\Pi_1(\mathrm{SU}(2))=0$. Hence, no stable instanton solutions exist [22, 45] (unstable instanton solutions are still possible [23, 24, 44]). Also the index of the Dirac operator is necessarily zero. Suppose that the two-dimensional Dirac operator has an exact zero mode

$$\mathcal{D}^{(2)}\phi = 0 \tag{51}$$

with definite chirality

$$\sigma_3\phi = \pm\phi. \tag{52}$$

Then, because of the anti-unitary symmetry, we also have that

$$\mathcal{D}^{(2)}\sigma_2\tau_2 K\phi = 0, \tag{53}$$

which generates another zero mode unless $\sigma_2\tau_2 K\phi$ and ϕ are linearly dependent. This exactly happens in the four-dimensional theory. However in two dimensions ϕ and $\sigma_2\tau_2 K\phi$ have opposite chiralities

$$\sigma_3\sigma_2\tau_2 K\phi = -\sigma_2\tau_2 K\sigma_3\phi = \mp\sigma_2\tau_2 K\phi \tag{54}$$

implying that they have to be linearly independent states. We conclude that the index of the Dirac operator is zero for two-dimensional QCD in the fundamental representation and with two colors.

Although the index is trivial we still have a linear repulsion of the spectrum from the origin resulting from the chiral structure of $\mathcal{D}$. The joint probability density of the corresponding random matrix ensemble was first derived in the context of mesoscopic phyiscs [42] and is given by Eq. (50). For completeness we give a derivation of this result in appendix A 1. Since we have a linear repulsion from the origin we have $\alpha_{\rm D}=1$. The level repulsion is also linear, i.e. $\sim|\lambda_a-\lambda_b|$, and the eigenvalues show no generic degeneracy.

For quarks in the adjoint representation the gauge group is given by $\mathrm{SU}(N_{\rm c})/\mathbb{Z}_{N_{\rm c}}$ with the homotopy group $\Pi_1(\mathrm{SU}(N_{\rm c})/\mathbb{Z}_{N_{\rm c}}) = \mathbb{Z}_{N_{\rm c}}$ [45]. If ϕ is a zero mode with positive chirality, then $\sigma_2 K\phi$ is a zero mode with negative chirality. Therefore, the index of the Dirac operator is zero. Using a bosonization approach it can be shown that

the chiral condensate is nonzero for all N_c [46], which is consistent with having at most one pair of zero modes. Indeed, in a chiral basis, the nonzero off-diagonal block of the Dirac matrix is a square anti-symmetric matrix, and generically has one zero mode if the matrix is odd-dimensional and no zero modes if the matrix is even dimensional. In Ref. [45], in the sector of topological charge $k = 0, \ldots, N_c - 1$, a total of $2k(N_c - k)$ zero modes are found, half of them right-handed and the other half left-handed. However, these zero modes are only obtained after complexifying the $\mathrm{SU}(N_c)$ algebra and are irrelevant in the present context. The corresponding random matrix theory for this universality class also has an anti-symmetric off-diagonal block with no zero modes or one zero mode.

The joint probability density of the eigenvalues is given by the form (50) where the level repulsion is $|\lambda_a - \lambda_b|^4$ since all eigenvalues are Kramers degenerate (because the anti-unitary symmetry operator satisfies $(\sigma_2 K)^2 = -1$). We rederive this joint probability density in appendix A 2 and relate it to the QCD Dirac operator in the microscopic limit. The repulsion of the eigenvalues from the origin is either linear ($\alpha_D = 1$) for an even dimensional W or quintic ($\alpha_D = 5$) for an odd-dimensional W. We emphasize that the pair of zero modes for odd-dimensional matrices is not related to topology.

C. Symmetry Breaking Pattern

In table I, we also summarize the symmetry breaking patterns for continuum QCD in two, three, and four dimensions (see [31] for a discussion of general dimensions). We recall the results for the cases considered in our work and show that they also apply to the random matrix ensembles introduced in the previous subsection. We restrict ourselves to the two-dimensional case with the Dyson index $\beta_D = 1, 4$. The other symmetry breaking patterns and their relation to random matrix theory were extensively discussed in Refs. [35, 41].

For $\beta_D = 1$, the off-diagonal block is symmetric after a unitary transformation, $(\tau_2 \mathcal{W}^{(2)})^T = \tau_2 \mathcal{W}^{(2)}$. Then we have

$$\bar{\psi}_R \tau_2 \mathcal{W}^{(2)} \psi_R = \frac{1}{2} \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}^T \begin{pmatrix} 0 & \tau_2 \mathcal{W}^{(2)} \\ -\tau_2 \mathcal{W}^{(2)} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}, \tag{55}$$

where $\psi_R = (\mathbb{1} + \sigma_3)\psi/2$ is the right handed component of a quark field ψ. We obtain a similar expression for the other off-diagonal block, $\mathcal{W}^{(2)\,\dagger}$, of the Dirac operator $\mathcal{D}^{(2)}$ with $\psi_R \to \psi_L = (\mathbb{1} - \sigma_3)\psi/2$, i.e.

$$\bar{\psi}_L (\tau_2 \mathcal{W}^{(2)})^\dagger \psi_L = \frac{1}{2} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}^T \begin{pmatrix} 0 & (\tau_2 \mathcal{W}^{(2)})^\dagger \\ -(\tau_2 \mathcal{W}^{(2)})^\dagger & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}, \tag{56}$$

Therefore, the chiral symmetry is $\mathrm{USp}(2N_f) \times \mathrm{USp}(2N_f)$ and acts on the doublets via the transformation $(\bar{\psi}_R, \psi_R^T) \to (\bar{\psi}_R, \psi_R^T) U_R$ and $(\bar{\psi}_L, \psi_L^T) \to (\bar{\psi}_L, \psi_L)^T U_L$ with $U_{R/L} \in \mathrm{USp}(2N_f)$. In terms of these doublets the chiral condensate can be written as

$$\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R = \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}. \tag{57}$$

A non-zero expectation value of the chiral condensate requires that the unitary symplectic matrices fulfill the constraint

$$U_R \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} U_L^T = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \tag{58}$$

so that the chiral symmetry is broken to $\mathrm{USp}(2N_f)$.

This result can be derived by an explicit calculation for the corresponding random matrix model, see appendix A 1 b, and was also found in Ref. [31] for general QCD-like theories and in Ref. [40] for random matrix theories.

For two dimensional QCD with adjoint fermions ($\beta_D = 4$) we have that $\mathcal{W}^{(2)\,T} = -W^{(2)}$ is anti-symmetric so that the coupling of the gauge fields and the quarks can be rewritten as

$$\bar{\psi}_R \mathcal{W}^{(2)} \psi_R = \frac{1}{2} \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{W}^{(2)} \\ \mathcal{W}^{(2)} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}, \tag{59}$$

and

$$\bar{\psi}_L \mathcal{W}^{(2)\,\dagger} \psi_L = \frac{1}{2} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{W}^{(2)\,\dagger} \\ \mathcal{W}^{(2)\,\dagger} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}. \tag{60}$$

The corresponding chiral symmetry is $\mathrm{O}(2N_\mathrm{f}) \times \mathrm{O}(2N_\mathrm{f})$ with the transformation $(\bar{\psi}_\mathrm{R}, \psi_\mathrm{R}^T) \to (\bar{\psi}_\mathrm{R}, \psi_\mathrm{R}^T) A O_\mathrm{R} A^{-1}$ and $(\bar{\psi}_\mathrm{L}, \psi_\mathrm{L}^T) \to (\bar{\psi}_\mathrm{L}, \psi_\mathrm{L}^T) A O_\mathrm{L} A^{-1}$ with $O_{\mathrm{R/L}} \in \mathrm{O}(2N_\mathrm{f})$ and

$$A^T \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} A = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{61}$$

Invariance of the non-zero chiral condensate,

$$\bar{\psi}_\mathrm{R}\psi_\mathrm{L} + \bar{\psi}_\mathrm{L}\psi_\mathrm{R} = \begin{pmatrix} \bar{\psi}_\mathrm{R}^T \\ \psi_\mathrm{R} \end{pmatrix}^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_\mathrm{L}^T \\ \psi_\mathrm{L} \end{pmatrix}, \tag{62}$$

requires

$$O_\mathrm{R} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} O_\mathrm{L} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \tag{63}$$

such that the symmetry is broken to $\mathrm{O}(2N_\mathrm{f})$. Also this case agrees with results of Refs. [31, 40].

III. TWO DIMENSIONAL LATTICE QCD WITH NAIVE FERMIONS AT STRONG COUPLING

In this section we consider the microscopic limit of naive fermions in the strong coupling limit and the corresponding random matrix theories. Thus the links, the gauge group elements on the lattice, are distributed according to the Haar-measure of the gauge group. In Secs. III A we discuss the general effect of the parity of the lattice on the global symmetries of the Dirac operator. This discussion is combined with the specific anti-unitary symmetries of the QCD-like theories in Secs. III B, III C, and III D. In particular, we classify each lattice Dirac operator according to a random matrix ensemble, which is summarize in table II together with some spectral properties. These random matrix theory predictions are compared with 2-dim lattice simulations confirming that the parity of the lattice has an important effect on the properties of the smallest eigenvalues. This was observed before in the condensed matter literature [6].

A. General lattice model

The covariant derivatives that enter in the lattice QCD Dirac operator can be readily constructed via the translation matrices. Before doing so we introduce the lattice. Let $|j\rangle$ be the j'th site in one direction of a lattice written in Dirac's bra-ket notation. Then the dual vector is $\langle j|$. The translation matrices of an $L_1 \times L_2$ lattice in the directions $\mu = 1, 2$ are given by

$$T_\mu = \begin{cases} \sum\limits_{\substack{1\le i\le L_1 \\ 1\le j\le L_2}} U_{1ij} \otimes |i\rangle\langle i+1| \otimes |j\rangle\langle j|, \ \ \mu = 1, \\ \sum\limits_{\substack{1\le i\le L_1 \\ 1\le j\le L_2}} U_{2ij} \otimes |i\rangle\langle i| \otimes |j\rangle\langle j+1|, \ \ \mu = 2. \end{cases} \tag{64}$$

The matrices $U_{\mu ij}$ are given in some representation of the special unitary group $\mathrm{SU}(N_\mathrm{c})$ and are weighted with the Haar-measure of $\mathrm{SU}(N_\mathrm{c})$. Hence, the translation matrices T_μ are unitary.

Note that our lattices have a toroidal geometry. We have numerically looked at the effect of periodic and anti-periodic fermionic boundary conditions on the spectrum of the Dirac operator. Indeed, the universality class remains unaffected since the global symmetries are independent of the boundary conditions. Only the Thouless energy marginally changes.

The Dirac operator on a two dimensional lattice is given by

$$\begin{aligned} D &= \sigma_\mu (T_\mu - T_\mu^\dagger) \\ &= \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} = \begin{bmatrix} 0 & T_x - T_x^\dagger + i(T_y - T_y^\dagger) \\ T_x - T_x^\dagger - i(T_y - T_y^\dagger) & 0 \end{bmatrix}. \end{aligned} \tag{65}$$

Due to the lattice structure, an additional symmetry can exist in each direction if the number of sites in a direction is even. Then the matrix elements of the Dirac operator between even and odd sites are non-vanishing while there is no direct coupling between an even and an even lattice site and between an odd and an odd site. For a two dimensional lattice we can distinguish three cases. First, the number of lattice sites L_1 and L_2 are both odd. Then, there are no additional symmetries such that the lattice Dirac operator is in the same symmetry class as the continuum theory. The other two cases are, second, L_1 even and L_2 odd or the reverse, and third, both L_1 and L_2 are even. We analyze these two cases in detail for each anti-unitary symmetry class separately. Thereby we assume that both L_1 and L_2 are larger than 2 because only then the low-lying eigenvalues of the Dirac operator show a generic behavior.

Let us define the operators

$$\Gamma_5^{(\mu)} = \begin{cases} \sum\limits_{\substack{1\le i\le L_1\\ 1\le j\le L_2}} (-1)^i \mathbb{1}_{N_c} \otimes |i\rangle\langle i| \otimes |j\rangle\langle j|, & \mu = 1, \\ \sum\limits_{\substack{1\le i\le L_1\\ 1\le j\le L_2}} (-1)^j \mathbb{1}_{N_c} \otimes |i\rangle\langle i| \otimes |j\rangle\langle j|, & \mu = 2. \end{cases} \tag{66}$$

Then one can show that the operator $\Gamma_5^{(\mu)}$ fulfills the relation

$$\Gamma_5^{(\mu)} T_\omega \Gamma_5^{(\mu)} = (-1)^{\delta_{\mu\omega}} T_\omega \tag{67}$$

if L_μ is even. Hereby we employ the Kronecker symbol $\delta_{\mu\omega}$ in the exponent of the sign.

Let us consider the simplest case where L_1 and L_2 are odd. Then W has no additional symmetries resulting from the lattice structure. Therefore, the Dirac operator will have the same unitary and anti-unitary symmetries as in the continuum limit discussed in section II, in particular it is anti-Hermitian and chirally symmetric,

$$D = -D^\dagger \quad \text{and} \quad [\sigma_3, D]_+ = 0. \tag{68}$$

Therefore the Dirac operator has the structure

$$D = \begin{pmatrix} 0 & W \\ -W^\dagger & 0 \end{pmatrix}, \tag{69}$$

where W may fulfill some additional anti-unitary symmetries because of the representation of the gauge theory.

In the second case, we have in one direction an even number of lattice sites and in the other direction an odd number of lattice sites. Let us assume that without loss of generality $L_1 \in 2\mathbb{N}$ and $L_2 \in 2\mathbb{N}+1$. Then the lattice Dirac operator fulfills the global symmetries

$$D = -D^\dagger, \quad [\sigma_3, D]_+ = 0, \quad \text{and} \quad [\Gamma_5^{(1)}\sigma_2, D]_- = 0 \tag{70}$$

plus possible anti-unitary symmetries depending on the representation of the gauge group. From the first two symmetries it follows that the Dirac operator has the chiral structure (69). The last symmetry relation of Eq. (70) tells us hat the matrix W is $\Gamma_5^{(1)}$-Hermitian, i.e.

$$W^\dagger = \Gamma_5^{(1)} W \Gamma_5^{(1)}. \tag{71}$$

Hence the Dirac operator for this kind of lattices takes the form

$$D = \begin{pmatrix} 0 & \Gamma_5^{(1)} H \\ -H\Gamma_5^{(1)} & 0 \end{pmatrix} = \mathrm{diag}\,(\Gamma_5^{(1)}, \mathbb{1}) \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix} \mathrm{diag}\,(\Gamma_5^{(1)}, \mathbb{1}), \tag{72}$$

with H a Hermitian matrix. This matrix H may be restricted to a subspace of the Hermitian matrices if we take into account the anti-unitary symmetries resulting from the representation of the gauge theory. The unitary matrix $\mathrm{diag}\,(\Gamma_5^{(1)}, \mathbb{1})$ does not change the eigenvalue spectrum of D and can be omitted.

One can also derive the structure (72) by employing the projection operators $(1 \pm \Gamma_5^{(1)})/2$. They project the lattice onto sub-lattices associated to the even and odd lattice sites in the direction $\mu = 1$. In such a basis, the translation matrix T_1 maps the even lattice sites to the odd ones and vice versa while the translation matrix T_2 maps the two sub-lattices onto themselves.

Gauge theory	$\beta_{\rm D}$	Lat.	Sym. Class	$\beta_{\rm D}^{(\rm eff)}$	$\alpha_{\rm eff}$	Deg	ZM	Symmetry Breaking Pattern
$N_{\rm c}=2$, fund.	1	ee	CII	4	3	4	0	$\mathrm{U}(4N_{\rm f}) \to \mathrm{O}(4N_{\rm f})$
$N_{\rm c}=2$, fund.	1	eo	C	2	2	2	0	$\mathrm{USp}(4N_{\rm f}) \to \mathrm{U}(2N_{\rm f})$
$N_{\rm c}=2$, fund.	1	oo	CI	1	1	1	0	$\mathrm{USp}(2N_{\rm f}) \times \mathrm{USp}(2N_{\rm f}) \to \mathrm{USp}(2N_{\rm f})$
$N_{\rm c}>2$, fund.	2	ee	AIII	2	1	2	0	$\mathrm{U}(2N_{\rm f}) \times \mathrm{U}(2N_{\rm f}) \to \mathrm{U}(2N_{\rm f})$
$N_{\rm c}>2$, fund.	2	eo	A	2	0	1	0	$\mathrm{U}(2N_{\rm f}) \to \mathrm{U}(N_{\rm f}) \times \mathrm{U}(N_{\rm f})$
$N_{\rm c}>2$, fund.	2	oo	AIII	2	1	1	0	$\mathrm{U}(N_{\rm f}) \times \mathrm{U}(N_{\rm f}) \to \mathrm{U}(N_{\rm f})$
$N_{\rm c}\geq 2$, adj.	4	ee	BDI	1	0	2	0	$\mathrm{U}(4N_{\rm f}) \to \mathrm{USp}(4N_{\rm f})$
$N_{\rm c}\geq 2$, adj.	4	eo	D	2	0	2	0	$\mathrm{O}(4N_{\rm f}) \to \mathrm{U}(2N_{\rm f})$
$N_{\rm c}\in 2\mathbb{N}+1$, adj.	4	oo	DIII (even-dim)	4	1	2	0	$\mathrm{O}(2N_{\rm f}) \times \mathrm{O}(2N_{\rm f}) \to \mathrm{O}(2N_{\rm f})$
$N_{\rm c}\in 2\mathbb{N}$, adj.	4	oo	DIII (odd-dim)	4	5	2	2	$\mathrm{O}(2N_{\rm f}) \times \mathrm{O}(2N_{\rm f}) \to \mathrm{O}(2N_{\rm f})$

TABLE II: Random matrix theories for the two-dimensional naive lattice QCD Dirac operator with gauge group listed in the first column. The Dyson index $\beta_{\rm D}$ refers to the anti-unitary symmetry of the Dirac operator in the continuum. Because of additional symmetries the power of the Vandermonde determinant, $\beta_{\rm D}^{(\rm eff)}$, is generally different from the continuum theory and thus, the level repulsion as well. Moreover the repulsion of the levels from the origin, namely $\lambda^{\alpha_{\rm eff}}$, the generic degeneracy of the eigenvalues (third to last column, "Deg"), and the number of generic zero modes (second to last column, "ZM") generally change as well. The third column refers to whether L_1 or L_2 are even (e) or odd (o). All discretizations are classified according to the ten-fold classification of random matrix theories (fourth column) which share the same pattern of chiral symmetry breaking with the lattice QCD Dirac operator (we do not consider axial symmetry breaking). Notice that the symmetry breaking pattern and, therefore, the global symmetries of the lattices where L_1 and L_2 are both odd is the same with the two-dimensional QCD Dirac operator in continuum, cf. table I.

In the third case the lattice has an even number of lattice sites in both directions. This is exactly the situation of staggered fermions. The corresponding Dirac operator for naive fermions has the symmetries

$$D=-D^\dagger, \quad [\sigma_3, D]_+=0, \quad [\Gamma_5^{(1)}\sigma_2, D]_-=0, \quad \text{and} \quad [\Gamma_5^{(2)}\sigma_1, D]_-=0. \tag{73}$$

Again the Dirac operator has the chiral structure (69), but the symmetry relation of the matrix W is given by

$$W^\dagger=\Gamma_5^{(1)} W \Gamma_5^{(1)} \quad \text{and} \quad [\Gamma_5^{(1)}\Gamma_5^{(2)}, W]_+=0. \tag{74}$$

The first symmetry restricts W to a $\Gamma_5^{(1)}$-Hermitian matrix whereas the second relation reflects the even-odd symmetry of the Dirac operator. Therefore the lattice Dirac operator has the structure

$$D=\mathrm{diag}\,(\Gamma_5^{(1)},\mathbb{1}) \left(\begin{array}{cc|cc} & & 0 & X \\ & 0 & X^\dagger & 0 \\ \hline 0 & -X & & \\ -X^\dagger & 0 & 0 & \end{array}\right) \mathrm{diag}\,(\Gamma_5^{(1)},\mathbb{1}), \tag{75}$$

where X is a complex matrix that may fulfill anti-unitary symmetries depending on the representation of the gauge fields. The double degeneracy is immediate and is eliminated for staggered fermions.

Again one can also explicitly construct the form of the lattice Dirac operator (75) by employing the four projection operators $(1\pm\Gamma_5^{(1)})/2$ and $(1\pm\Gamma_5^{(2)})/2$. They split the lattice into four sub-lattices which are coupled via the translation matrices $T_{1/2}$.

Adding the anti-unitary symmetries to the symmetries (68), (70), and (73) will give rise to further constraints on W. In table II we summarize these cases for each anti-unitary symmetry class. In general, the symmetry class will differ from the symmetry class in continuum. Therefore the corresponding random matrix ensemble and the symmetry breaking pattern will also change. In particular, one has to replace the indices $\beta_{\rm D}$ (Dyson index = level repulsion) and $\alpha_{\rm D}$ (=repulsion of the levels from the origin) in the joint probability densities of the eigenvalues of the random matrix model, cf. Eqs. (48) and (50), by effective values,

$$\beta_{\rm D} \to \beta_{\rm D}^{(\rm eff)} \quad \text{and} \quad \alpha_{\rm D} \to \alpha_{\rm D}^{(\rm eff)}. \tag{76}$$

This impacts the spectral properties of the Dirac operator in the microscopic limit.

There are additional conditions on the off-diagonal block W of the lattice Dirac operator D which are independent of the gauge configurations. For example the traces of W satisfy the relations

$$\operatorname{tr} W^2 = \operatorname{tr} W^{2l+1} = 0 \quad \text{and} \quad \operatorname{tr} WW^\dagger = \begin{cases} 2N_c L_1 L_2, & \text{fundamental fermions,} \\ 2(N_c^2-1)L_1 L_2, & \text{adjoint fermions} \end{cases} \tag{77}$$

with $l = 0, 1, 2, \ldots$ such that $l \leq \min\{L_1, L_2\}/2 - 1$. They result from the fact that the translation matrices (64) are unitary and have no diagonal elements. The conditions of the kind (77) are expected to have no influence on the microscopic spectrum in the limit of large matrices. Nevertheless, they may give rise to finite volume corrections which turn out to be particularly large for the simulations of SU (3) gauge theory with fermions in the fundamental representation and choosing L_1 even and L_2 odd, see subsection III D 2. The effect of such conditions can also be studied with random matrix theory and we do this for the simplest condition, namely that W is traceless, i.e. $\operatorname{tr} W = 0$.

B. SU (2) and fermions in the fundamental representation

When studying the two-color theory in its fundamental representation the translation matrix fulfills exactly the same anti-unitary symmetry as the covariant derivative in the continuum theory,

$$[iT_\mu, \tau_2 K]_- = 0, \tag{78}$$

cf. Eq. (23). This symmetry carries over to the symmetry

$$[iD, \tau_2\sigma_2 K]_- = 0, \tag{79}$$

for the lattice Dirac operator meaning that there is always a gauge field independent basis where the Dirac operator appears real. However, as is the case in the continnuum theory, the symmetry (79) may not commute with the symmetries (68), (70), and (73). In the continuum theory we showed that the anti-unitary symmetry resuts in a symmetry of the off-diagonal block W,

$$W = -\tau_2 W^T \tau_2. \tag{80}$$

cf. Eq. (32). This carries over to the lattice theory as well and together with the symmetries (68), (70), and (73) yields the symmetry classification given in Table 2. This is worked out in detail in the subsections III B 1, III B 2, and III B 3 for (L_1, L_2) odd-odd, even-odd, and even-even, respectively.

1. *The Odd-Odd Case*

As already discussed before, this case does not have any additional symmetries and the pattern of chiral symmetry breaking as well as the distribution of the eigenvalues in the microscopic domain has to be the same as in the continuum limit which was discussed in section III A. The symmetries of the Dirac operator are summarized in Eq. (27) which translates in terms of the lattice Dirac operator as in Eqs. (68) and (80). That corresponds to a chiral random matrix theory with symmetric complex off-diagonal blocks. In the Cartan classification of symmetric spaces, this is denoted by the symbol CI. The corresponding microscopic level density is given by ($x = \lambda V\Sigma$) [48]

$$\rho(x) = \frac{x}{2}\left[J_1^2(x) - J_0(x)J_2(x)\right] + \frac{1}{2}J_0(x)J_1(x). \tag{81}$$

The symmetry breaking pattern is therefore the same as in the continuum, namely $\mathrm{U}(2N_f) \to \mathrm{O}(2N_f)$.

In Fig. 1a we compare the prediction (81) for the low-lying Dirac spectrum with lattice QCD data at strong coupling. The size of the lattices is quite small. Nevertheless the agreement of the analytical prediction for the microscopic level density and the simulations around the origin is good. In particular, the linear repulsion of the eigenvalues from the origin is confirmed. Also the degree of degeneracy and the number of generic zero modes, which are in this case one and zero, respectively, are confirmed. The lattice results are obtained from an ensemble of about 10^5 independent configurations with the links generated by the Haar measure of the gauge group SU (2).

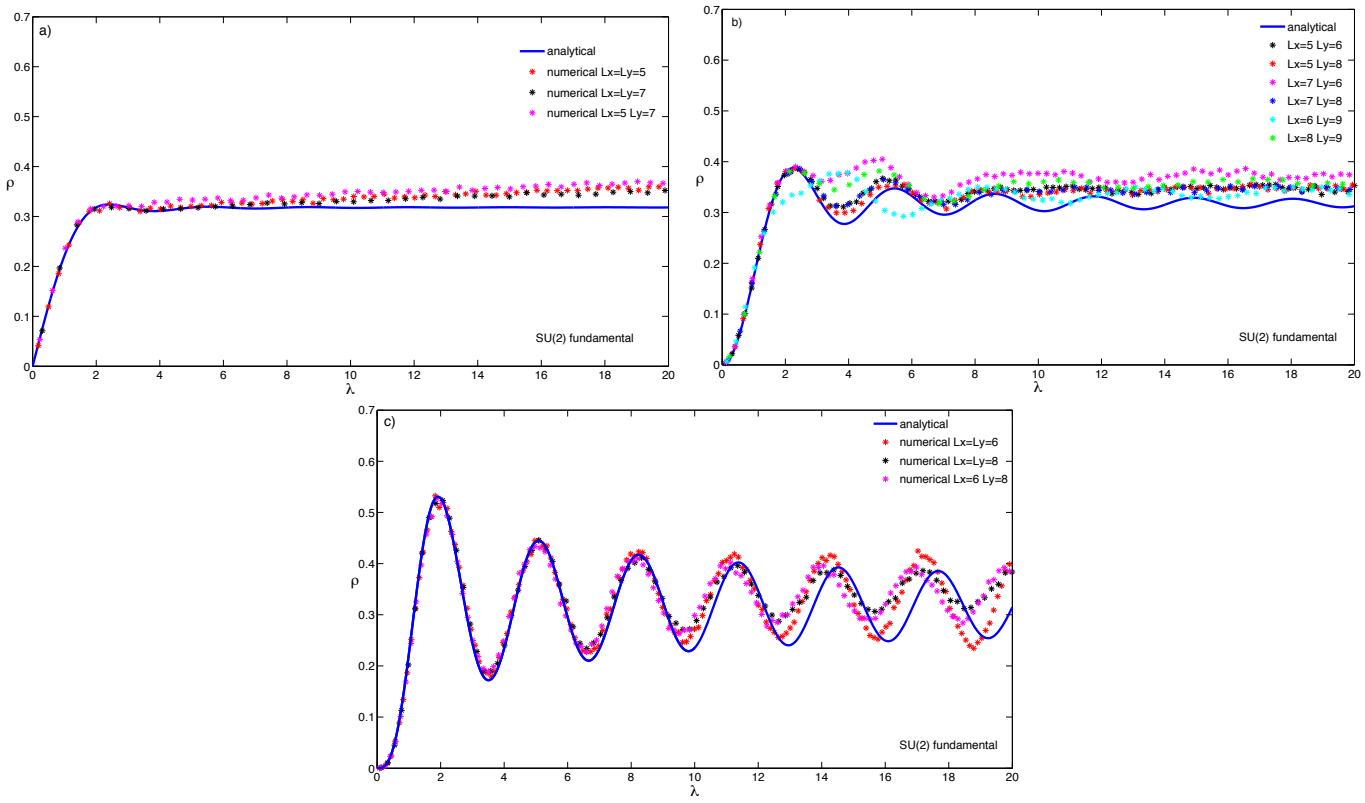

FIG. 1: Comparison of the microscopic level densities of lattice QCD data in the strong coupling limit at various lattice sizes (stars) and the analytical results given by the corresponding random matrix theories (solid curves). The plotted lattice gauge theories are: a) SU (2) fundamental and L_1, L_2 odd, b) SU (2) fundamental and $L_1 + L_2$ odd, and c) SU (2) fundamental and L_1, L_2 even.

2. The Even-Odd Case

For definiteness we choose L_1 even and L_2 odd. Then, the Dirac operator is of the form (72). We combine the intermediate result $W = \Gamma_5^{(1)} H$ with a Hermitian matrix H and the anti-unitary symmetry (80). Therefore we can find a gauge field independent rotation, namely $U_5^{(1)} = \exp[\pi i(\Gamma_5^{(1)} - \mathbb{1}_{N_c L_1 L_2})/4]$, where $\widetilde{H} = U_5^{(1)} H U_5^{(1)\,-1}$ becomes an anti-self-dual Hermitian matrix ($\widetilde{H} = \widetilde{H}^\dagger = -\tau_2 \widetilde{H}^T \tau_2 = -\tau_2 \widetilde{H}^* \tau_2$). This is the class C of the tenfold classification [40] and $\widetilde{H}$ is an element in the Lie-algebra of the group USp $(N_c L_1 L_2)$. In this basis, the Dirac operator reads

$$D = \mathrm{diag}\,(U_5^{(1)}, U_5^{(1)\,-1}) \begin{pmatrix} 0 & \widetilde{H} \\ -\widetilde{H} & 0 \end{pmatrix} \mathrm{diag}\,(U_5^{(1)\,-1}, U_5^{(1)}). \tag{82}$$

Note that $\Gamma_5^{(1)} = U_5^{(1)\,2}$.

What does this imply for the spectrum of the Dirac operator? The anti-unitary symmetry leads to a pair of eigenvalues $\pm\lambda$ of the Hermitian matrix $\widetilde{H}$. Indeed, if λ is an eigenvalue of $\widetilde{H}$ with the eigenvector $|\phi\rangle$,

$$\widetilde{H}|\phi\rangle = \lambda|\phi\rangle, \tag{83}$$

then the state $\tau_2|\phi^*\rangle$ is an eigenvector with eigenvalue $-\lambda$,

$$\widetilde{H}\tau_2|\phi^*\rangle = -\lambda\tau_2|\phi^*\rangle. \tag{84}$$

Therefore the Dirac operator (82) has the eigenvalues $\pm i\lambda$ which are doubly degenerate. This leads to a doubling of the number of flavors and the spectrum of D is twice the spectrum of $i\widetilde{H}$. In addition, because of

$$\bar{\psi}^T \widetilde{H} \psi = \frac{1}{2}(\bar{\psi}^T \widetilde{H} \psi - (\tau_2 \psi)^T \widetilde{H} \tau_2 \bar{\psi}), \tag{85}$$

the flavor symmetry is enhanced to $\mathrm{USp}\,(4N_\mathrm{f})$, cf. Eq. (55). Because

$$\det(D+m\mathbb{1})=\det(\widetilde{H}^2+m^2\mathbb{1})=\det(\widetilde{H}+im\mathbb{1})\det(\widetilde{H}-im\mathbb{1}), \tag{86}$$

a nonzero eigenvalues density of $\widetilde{H}$ leads to a nonzero eigenvalue density of the Dirac operator, D. The symmetry $\mathrm{USp}\,(4N_\mathrm{f})$ is thus spontaneously broken by the formation of a condensate with m as source term. However this condensate is still invariant under a $\mathrm{U}\,(2N_\mathrm{f})$ subgroup of $\mathrm{USp}\,(4N_\mathrm{f})$

$$[\mathrm{diag}\,(U,U^*)]^T\begin{pmatrix}0 & m\mathbb{1}_{2N_\mathrm{f}}\\ -m\mathbb{1}_{2N_\mathrm{f}} & 0\end{pmatrix}\mathrm{diag}\,(U,U^*)=\begin{pmatrix}0 & m\mathbb{1}_{2N_\mathrm{f}}\\ -m\mathbb{1}_{2N_\mathrm{f}} & 0\end{pmatrix}. \tag{87}$$

Thus the symmetry breaking pattern is $\mathrm{USp}\,(4N_\mathrm{f})\to\mathrm{U}\,(2N_\mathrm{f})$ in agreement with the symmetry breaking pattern of the corresponding random matrix ensemble [40].

The joint probability distribution of the symmetry class C coincides with the distribution of the non-zero eigenvalues of chGUE for $\nu=1/2$. The microscopic level density is thus given by [42, 48, 49]

$$\rho(x)=\frac{1}{\pi}-\frac{\sin(2x)}{2\pi x}. \tag{88}$$

In Fig. 1b we compare this result to lattice simulations. We find only good agreement to about one eigenvalue spacing. The reason for the strong disagreement above the average position of the first eigenvalue is not clear. Nevertheless, the quadratic repulsion of the eigenvalues from the origin, the double degeneracy of the eigenvalues, and the fact that there are no generic zero modes are confirmed by the lattice simulations.

3. The Even-Even Case

Finally, we consider the case with both L_1 and L_2 even. Then the Dirac operator has the structure given in Eq. (75). After combining the chiral structure of W with the anti-unitary symmetry (80) the Dirac operator takes the form

$$D=\mathrm{diag}\,(U_5^{(1)},U_5^{(1)\,-1})\left(\begin{array}{cc|cc}\multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & \widetilde{W}\\ & & \widetilde{W}^\dagger & 0\\ \hline 0 & -\widetilde{W} & \multicolumn{2}{c}{\multirow{2}{*}{0}}\\ -\widetilde{W}^\dagger & 0 & & \end{array}\right)\mathrm{diag}\,(U_5^{(1)\,-1},U_5^{(1)}) \tag{89}$$

with $\widetilde{W}^*=\tau_2\widetilde{W}\tau_2$ a quaternion matrix without any further symmetries. The unitary transformation $\mathrm{diag}\,(U_5^{(1)\,-1},U_5^{(1)})$ is exactly the same as in the previous subsection and keeps the spectrum invariant such that the global symmetries of the lattice Dirac operator D essentially coincide with the continuum Dirac operator in four dimensions with the fermions in the adjoint representation. Therefore, the random matrix ensemble corresponding to this type of lattice theory is the chGSE with the chiral symmetry breaking pattern $\mathrm{U}\,(4N_\mathrm{f})\to\mathrm{O}\,(4N_\mathrm{f})$. The degeneracy of the eigenvalues is four because of Kramers degeneracy and the doubling of flavors. In table II we summarize the main properties of this ensemble.

The microscopic level density of the lattice QCD Dirac operator in this class is given by the $\nu=0$ result of chGSE [47, 48] (note that $\widetilde{W}$ is a square matrix),

$$\rho(x)=x\left[J_0^2(2x)+J_1^2(2x)\right]-\frac{1}{2}J_0(2x)\int_0^{2x}J_0(\widetilde{x})d\widetilde{x}. \tag{90}$$

There are no generic zero modes and the levels show a cubic repulsion from the origin.

In Fig. 1c we compare the result (90) to lattice simulations of the two-dimensional Dirac operator for QCD with two colors. There is an excellent agreement for the first few eigenvalues confirming our predictions.

C. $\mathrm{SU}(N_\mathrm{c})$ and fermions in the adjoint representation.

For the fermions in the adjoint representation of the gauge group $\mathrm{SU}\,(N_\mathrm{c}\geq 2)$ the translation matrices are real and, hence, satisfy the anti-unitary symmetry

$$[K,T_\mu]_-=T_\mu. \tag{91}$$

On a $L_1 \times L_2$ lattice, the translation matrices are represented by a subset of matrices in the orthogonal group $\mathrm{O}((N_\mathrm{c}^2-1)L_1L_2)$. The symmetry (91) carries over to the two-dimensional lattice Dirac operator

$$[iD, \sigma_2 K]_- = 0, \tag{92}$$

and its off-diagonal block matrix

$$W = -W^T. \tag{93}$$

We combine this symmetry with the symmetries (68), (70), and (73) along the same lines as shown in subsection III C. Thereby we discuss the odd-odd, even-odd, and even-even lattices in subsections III C 1, III C 2, and III C 3 , respectively.

1. The Odd-Odd Case

In the case where both the number of lattice sites L_1 and L_2 are odd, the Dirac operator has the same symmetries as in the continuum limit resulting in the same pattern of chiral symmetry breaking ($\mathrm{O}(2N_\mathrm{f}) \times \mathrm{O}(2N_\mathrm{f}) \to \mathrm{O}(2N_\mathrm{f})$) and the same microscopic spectral properties (see table II). Depending on the number of colors the off-diagonal block W of the lattice Dirac operator is either even or odd dimensional and the corresponding symmetry class is given by the second Bogolyubov-de Gennes ensemble DIII, see Ref. [40, 42], which can be also either even or odd, respectively. The microscopic level density was obtained in Ref. [48] and is given by

$$\rho(x) = \frac{x}{2}\left[2J_1^2(2x) + J_0^2(2x) - J_0(2x)J_2(2x)\right] + \frac{1}{2}J_1(2x) \tag{94}$$

for N_c odd and

$$\rho(x) = 2\delta(x) + \frac{x}{2}\left[2J_1^2(2x) + J_0^2(2x) - J_0(2x)J_2(2x)\right] - \frac{1}{2}J_1(2x) \tag{95}$$

for N_c even. Notice that the lattice Dirac operator has one additional pair of generic zero-modes if the number of colors is even otherwise there are no generic zero modes. Therefore the repulsion of the eigenvalues from the origin is stronger. However, the level repulsion is always quartic, see table II. Moreover, the full spectrum is Kramers degenerate. This is a characteristic for ensembles associated to the Dyson index $\beta_\mathrm{D} = 4$.

In Figs. 2a and 2b we compare the low lying lattice Dirac spectra and the analytical results of (94) and (95) for two and three colors, respectively. The agreement is good for the first few eigenvalues and becomes better when increasing the number of colors.

2. The Even-Odd Case

Next we consider the mixed situation where the lattice has an even L_1 and an odd L_2. The combination of the symmetries (70) and (93) can be again simplified via the same unitary transformation $\mathrm{diag}\,(U_5^{(1)\,-1}, U_5^{(1)})$ as introduced in subsection III B 2. Then the lattice Dirac operator can be written as

$$D = \mathrm{diag}\,(U_5^{(1)}, U_5^{(1)\,-1}) \begin{pmatrix} 0 & \widetilde{H} \\ -\widetilde{H} & 0 \end{pmatrix} \mathrm{diag}\,(U_5^{(1)\,-1}, U_5^{(1)}), \tag{96}$$

where $\widetilde{H}$ is purely imaginary and anti-symmetric. Thus the symmetry class is equivalent to a random matrix ensemble with the matrices in the Lie-algebra of the orthogonal group $\mathrm{O}\,(L_1L_2(N_\mathrm{c}^2-1))$ which is denoted by the Cartan symbol D [40]. Although for this ensemble one also has to distinguish between even and odd matrix size N because of an additional pair of generic zero modes, the lattice Dirac operator always yields an even sized matrix $\widetilde{H}$. The reason is that $\widetilde{H}$ is $L_1L_2(N_\mathrm{c}^2-1) \times L_1L_2(N_\mathrm{c}^2-1)$ dimensional where L_1 is even. Therefore we expect a quadratic level repulsion, no repulsion of the levels of D from the origin and no generic zero modes, cf. table II. The number of flavors is doubled because of the particular block structure (96).

The quark bilinear can be written as

$$\bar{\psi}^T \widetilde{H} \psi = \frac{1}{2}(\bar{\psi}\widetilde{H}\psi + \psi\widetilde{H}\bar{\psi}), \tag{97}$$

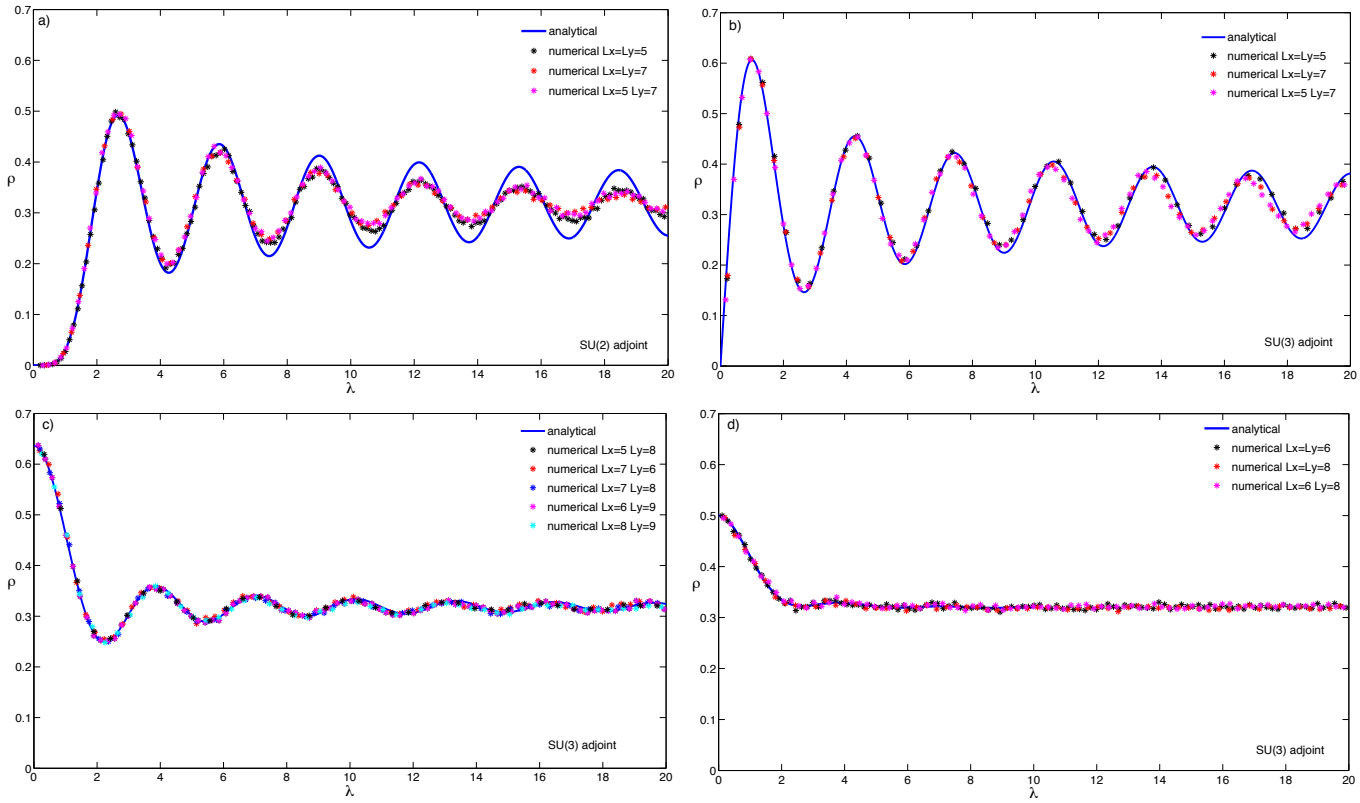

FIG. 2: Comparison of the microscopic level densities of the lattice QCD Dirac operator in the strong coupling limit at various lattice sizes (stars) and the analytical results derived from the corresponding random matrix theories (solid curves). Results are shown for the lattice theories: a) SU (2) adjoint and L_1, L_2 odd, b) SU (3) adjoint and L_1, L_2 odd, c) SU (3) adjoint and L_1 odd and L_2 even, and d) SU (3) adjoint and L_1, L_2 even.

so that the symmetry group is O $(4N_{\rm f})$. As was shown in the case $\beta_{\rm D} = 1$, see subsection III B 2, a nonzero eigenvalue density of $\widetilde{H}$ results in a nonzero eigenvalues density of the Dirac operator resulting in a chiral condensate with source term m. This condensate breaks the O $(4N_{\rm f})$ symmetry group to the subgroup satisfying

$$O^T \begin{pmatrix} 0 & m\mathbb{1}_{2N_{\rm f}} \\ -m\mathbb{1}_{2N_{\rm f}} & 0 \end{pmatrix} O = \begin{pmatrix} 0 & m\mathbb{1}_{2N_{\rm f}} \\ -m\mathbb{1}_{2N_{\rm f}} & 0 \end{pmatrix} \tag{98}$$

This equation enforces the matrix O to a block structure

$$O = \begin{pmatrix} O_1 & O_2 \\ -O_2 & O_1 \end{pmatrix}. \tag{99}$$

The orthogonality of O requires that

$$(O_1 + iO_2)^\dagger (O_1 + iO_2) = \mathbb{1} \tag{100}$$

so that O is equivalent to a unitary transformation. Moreover each unitary matrix $U \in$ U $(2N_{\rm f})$ can be decomposed into the real matrices $O_1 = \frac{1}{2}(U + U^*)$ and $O_2 = -i(U - U^*)$. Hence the remaining group invariance is equal to U $(2N_{\rm f})$ yielding the symmetry breaking pattern O $(4N_{\rm f}) \to$ U $(2N_{\rm f})$.

The microscopic level density can be calculated from the corresponding random matrix ensemble in class D and is given by [42, 48]

$$\rho(x) = \frac{1}{\pi} + \frac{\sin(2x)}{2\pi x}. \tag{101}$$

In Fig. 2c we compare this analytical result to strong coupling lattice simulations for naive quarks in the adjoint representation of SU (3). The lattice data show excellent agreement for the low-lying Dirac spectrum. Moreover the simulations confirm the double degeneracy of the Dirac operator (eigenvalues have also the degeneracy two) and the fact that there are no generic zero modes.

3. The Even-Even Case

Let L_1 and L_2 be even. This is the case related to the staggered Dirac operator. With help of the symmetries (73) and (93) the lattice Dirac operator can, by choosing a particular gauge field independent basis, be brought to the form

$$D = \mathrm{diag}\,(U_5^{(1)}, U_5^{(1)\,-1}) \left(\begin{array}{cc|cc} & 0 & 0 & \widetilde{W} \\ & & \widetilde{W}^\dagger & 0 \\ \hline 0 & -\widetilde{W} & & \\ -\widetilde{W}^\dagger & 0 & 0 & \end{array}\right) \mathrm{diag}\,(U_5^{(1)\,-1}, U_5^{(1)}), \tag{102}$$

where $\widetilde{W}$ is a real $L_1L_2(N_c^2-1)/2 \times L_1L_2(N_c^2-1)/2$ matrix without any additional restrictions. The additional chiral structure is related to the parity of the lattice sites.

The unitary transformation $\mathrm{diag}\,(U_5^{(1)\,-1}, U_5^{(1)})$ does not change the spectrum. Therefore the naive lattice Dirac operator (102) is in the class of chGOE with index $\nu = 0$ (because $\widetilde{W}$ is a square matrix). The Dirac spectrum is doubly degenerate which is taken care of when constructing the staggered Dirac operator. The symmetry breaking pattern is $\mathrm{U}\,(4N_\mathrm{f}) \to \mathrm{USp}\,(4N_\mathrm{f})$ [35] and the microscopic spectral density is given by the $\nu = 0$ result of the chGOE [50]

$$\rho(x) = \frac{x}{2}\left[J_0^2(x) - J_1^2(x)\right] + \frac{1}{2}J_0(x)\left[1 - \int_0^x J_0(\widetilde{x})d\widetilde{x}\right], \tag{103}$$

Therefore the level repulsion is linear, the levels have no repulsion from the origin and there are no generic zero modes. The analytical result (103) is compared with lattice data in Fig. 2d showing a perfect agreement.

D. QCD with more than Two Colors and Fermions in the Fundamental Representation

In this case there are no anti-unitary symmetries. The structure and the symmetry class of the Dirac operator are only related to the parity of the lattice. Hence, we have to take the structure of the naive lattice Dirac operator as shown in Eqs. (68), (70), and (73).

The odd-odd and even-even lattices are in the same universality class and are both discussed in subsection III D 1. The case of one even number of lattice sites and one odd number is considered in subsection III D 2.

1. The Odd-Odd and Even-Even Case

If the parity of both directions is odd, there are no additional symmetries and we are in the universality class of chGUE with the symmetry breaking pattern $\mathrm{U}\,(N_\mathrm{f}) \times \mathrm{U}\,(N_\mathrm{f}) \to \mathrm{U}\,(N_\mathrm{f})$. The Dirac operator has the form (69). The eigenvalues of D show no degeneracies and the microscopic spectral density is given by the $\nu = 0$ result of chGUE [41]

$$\rho(x) = \frac{x}{2}\left[J_0^2(x) + J_1^2(x)\right]. \tag{104}$$

Note that the two-dimensional Dirac operator has no zero modes. Therefore the level repulsion is quadratic and the repulsion from the origin is linear.

If both numbers of lattice sites, L_1 and L_2, are even, the off-diagonal block W becomes itself chiral and the Dirac-operator takes the form (75). Since we have no additional symmetries the symmetry class is again the one of chGUE. The only difference with the odd-odd case is a doubling of the number of flavors with the chiral symmetry breaking pattern $\mathrm{U}\,(2N_\mathrm{f}) \times \mathrm{U}\,(2N_\mathrm{f}) \to \mathrm{U}\,(2N_\mathrm{f})$. Apart from an additional degeneracy from the doubling of the flavors, the spectral properties remain the same. In particular, the microscopic spectral density has index $\nu = 0$ and is given by Eq. (104).

In Fig. 3a we show lattice data for the spectral density of the Dirac operator in the case that both L_1 and L_2 are either odd or even. There is an excellent agreement with the analytical random matrix result (104). Also the degree of degeneracy and the fact that there are no zero modes is confirmed by the lattice simulations.

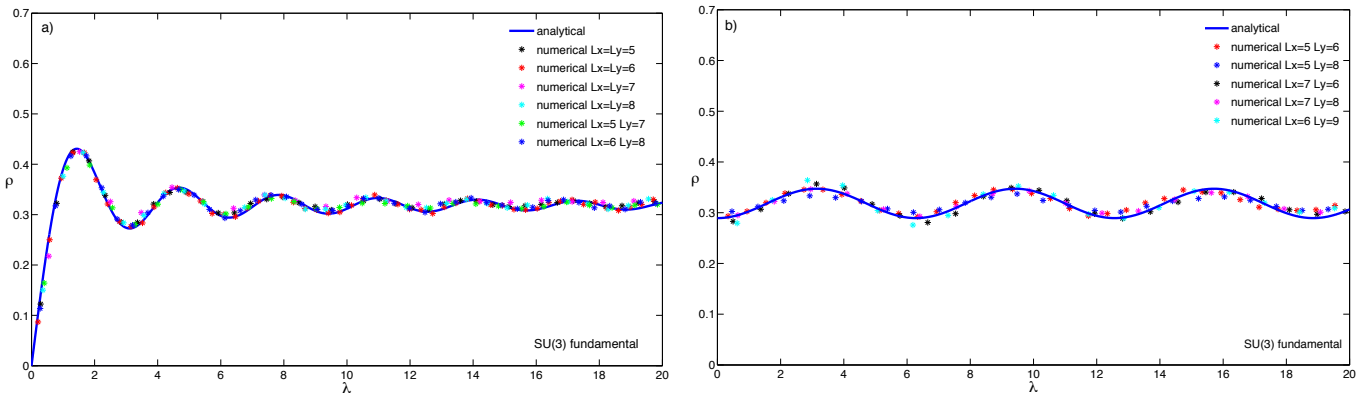

FIG. 3: Comparison of the microscopic level density of lattice QCD data in the strong coupling limit at various lattice sizes (stars) and the analytical results predicted by the corresponding random matrix theories (solid curves). The presented lattice gauge theories are: a) SU (3) fundamental with $L_1 + L_2 =$ even and b) SU (3) fundamental with $L_1 + L_2 =$ odd. Note that in figure b) we have a strong oscillation on top of the universal result which is a constant equal to $1/\pi$. Therefore we plotted the GUE result with its first correction in a $1/n$ expansion in its matrix size n. Astoundingly also this non-universal term seems to fit the lattice data quite well.

2. The Even-Odd Case

The situation changes if $L_1 + L_2$ is odd. Then the Dirac operator D follows the structure (72) where the $L_1 L_2 N_c \times L_1 L_2 N_c$ matrix H is Hermitian. The corresponding symmetry class is represented by the GUE and denoted by the Cartan symbol A [40]. Due to the structure (72) and the Hermiticity of H, the flavor symmetry is doubled to U $(2N_f)$. However the eigenvalues of D are not doubly degenerate but come in complex conjugate pairs $\pm i\lambda$ because H appears in the off-diagonal blocks.

The lattice Dirac operator is in the same universality class as the three dimensional continuum theory. Hence the symmetry breaking pattern for this case is already known from QCD in three dimensions [36]. A nonzero spectral density of H results in a nonzero spectral density of the Dirac operator (see the discussion in subsection III B 2) resulting in a chiral condensate with source term m. The chiral condensate is invariant under a transformation with the unitary matrix $U \in$ U $(2N_f)$ if it fulfills

$$U^\dagger \begin{pmatrix} 0 & m\mathbb{1}_{N_f} \\ -m\mathbb{1}_{N_f} & 0 \end{pmatrix} U = \begin{pmatrix} 0 & m\mathbb{1}_{N_f} \\ -m\mathbb{1}_{N_f} & 0 \end{pmatrix}. \tag{105}$$

This breaks chiral symmetry according to the pattern U $(2N_f) \to$ U $(N_f) \times$ U (N_f).

The microscopic level density including the $O(1/n)$ corrections of a $2n \times 2n$ GUE is given by

$$\rho(x) = \frac{1}{\pi}\left[1 + \frac{\cos 2x}{8n}\right]. \tag{106}$$

In order to obtain a better fit of the analytical result to the lattice data, we have included the correction term multiplied by a fitting parameter. In Fig. 3b we compare the microscopic level density of GUE and lattice results. The lattice data exhibit much larger oscillations than the ones given by the $O(1/n)$ correction in Eq. (106). One possible mechanism that may contribute to this enhancement is the condition that the off-diagonal block H of D is traceless, tr $H = 0$, since the translation matrices (64) have no diagonal elements. In appendix B we evaluate the spectral density for the random matrix ensemble that interpolates between the GUE and the traceless GUE. The result is given by

$$\rho_t(x) \;=\; \frac{1}{\pi}\left(1 + \frac{1}{8n}\exp\left[\frac{2t}{t+1}\right]\cos\left[2x\right]\right), \tag{107}$$

which shows oscillations that are enhanced by a factor of $e^2 \approx 7.4$ for a traceless random matrix ($t \to \infty$) in comparison to the original GUE ($t = 0$). Because the lattice Dirac operator is sparse the effective value of n is expected much less than the size of the matrix. Nevertheless we would also expect that n still increases with the lattice size. However when using n in Eq. (107) as a fitting parameter we find that $n \approx 7$ for almost all simulations. It is not clear why

the amplitude of the oscillations does not depend on the lattice size which should be analyzed in more detail. Also other conditions such as the fixed Euclidean norm of H, i.e. $\mathrm{tr}\, H^2 = 4N_c L_1 L_2$, may contribute to the amplitude of the oscillations.

IV. CONCLUSIONS

We have analyzed quenched two-dimensional lattice QCD Dirac spectra at strong coupling. The main differences with QCD in four dimensions are the absence of Goldstone bosons, the absence of topology corresponding to the Atiyah-Singer index theorem, and the non-commutativity of the anti-unitary symmetries and the axial symmetry. As is the case in four dimensions, the symmetries of the Dirac operator depend on the parity of the number of lattice points in each direction. However in two dimensions we find a much richer classification of symmetry breaking patterns. As is the case in four dimensions, the corresponding random matrix class is determined by the anti-unitary and the involutive symmetries. This is consistent with the maximum spontaneous breaking of chiral symmetry.

The simulations were performed with periodic boundary conditions in both directions even though we have also checked the effect of anti-periodicity in one direction. Our results remain unaffected in terms of the identifications of the universality class. Only a marginal increase of the Thouless energy was observed by this modification.

Notwithstanding the Mermin-Wagner-Coleman theorem, we find that the agreement with random matrix theory is qualitatively the same in two and four dimensions. The agreement is particularly good if the Goldstone manifold contains a $\mathrm{U}(1)$ or $\mathrm{O}(1) \simeq \mathbb{Z}_2$ group (i.e for the classes D, DII, BDI, CII and $AIII$). This raises the possibility that the long range correlations that give rise to random matrix statistics are related to the topological properties of the Goldstone manifold [51].

In this paper all numerical results are at nonzero lattice spacing. We did not attempt to perform an extrapolation to the continuum limit. Based on a bosonized form of the QCD partition function in terms of hadronic fields, one would expect a domain of low-lying eigenvalues that is dominated by the fluctuations of the zero momentum modes so that they are correlated according to random matrix theory. In the continuum limit the two dimensional theory is expected to renormalize to a theory without spontaneous symmetry breaking. What is disturbing is that we do not observe a qualitative different behavior between QCD in two and four dimensions.

Since quenched spectra are obtained by a supersymmetric extension of the partition function, our results seem to favor the suggestion by Niedermaier and Seiler that noncompact symmetries can be broken spontaneously in two dimensions. One of the signatures of this type of spontaneous symmetry breaking is an order parameter that wanders off to infinity. Indeed, in [16] it was found that the chiral condensate of the quenched Schwinger model seems to diverge in the thermodynamic limit. On the other hand, the Dirac spectrum of the $N_f = 1$ Schwinger model behaves as predicted by random matrix theory. It is clear that the chiral condensate is determined by the anomaly and does not involve any noncompact symmetries. Because of the absence of massless excitations the partition function of the one flavor Schwinger model must be smooth as a function of the quark mass. This implies that the condensate due to the nonzero Dirac eigenvalues must be the same as the condensate from the one-instanton configurations in the massless limit. This suggests that the eigenfunctions of the low-lying nonzero mode states must be delocalized and that the eigenvalue fluctuations are described by random matrix theory, so that the supersymmetric partition function that generates the Dirac spectrum looks like it has spontaneous symmetry breaking.

An alternative scenario arises because of the finiteness of the Thouless energy in units of the average level spacing. The fermion determinant due to massless quarks may push all eigenvalues beyond the Thouless energy into the localized domain resulting in a partition function with no spontaneous breaking of chiral symmetry. To find out if this is the case we would have to study two-dimensional lattice QCD with dynamical quarks. This scenario is not favored by simulations of the Schwinger model. Both the one- and two-flavor Schwinger model show excellent agreement with random matrix statistics and the agreement improves with increasing volumes which also excludes the possibility that the localization length is larger than the size of the box.

Our study raises many questions. The most fundamental issue is the reconciliation of the agreement with random matrix theory and the implied spontaneous breaking of chiral symmetry with the Mermin-Wagner-Coleman theorem. In particular, can the noncompact symmetry of the supersymmetric generating function for the Dirac spectrum of two-dimensional QCD-like theories be spontaneously broken? To address this we have to analyze the approach to the thermodynamic limit and the continuum limit. Such studies could also settle whether or not the localization length of the low-lying states exceeds the size of the box used in the present work. This is supported by Dirac spectra of the quenched Schwinger model which deviate more from random matrix theory with increasing volume [16], but there is no hint of this in our results. Another intriguing question is the possibility that all states become localized beyond a critical number of flavors. A final issue concerns the number of generic zero modes of the QCD Dirac operator for fermions in the adjoint representation. With chiral perturbation theory and random matrix theory we predict that the Dirac operator may have no or only two generic zero modes of opposite chirality. In future work we hope to

address the nature of these zero modes and the possible relation with the complexified zero modes found in Ref. [45].

V. ACKNOWLEDGMENTS

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Appendix A: Random matrix theories of two-dimensional continuum QCD

In this appendix we evaluate the joint probability density of the eigenvalues and the pattern of chiral symmetry breaking of random matrix theory corresponding to the continuum limit of two dimensional QCD. The case of two colors with fundamental fermions is worked out in the subsection A 1 and the case with two or more colors with fermions in the adjoint representation is discussed in the subsection A 2. The case with three or more colors with fermions in the fundamental representation follows the same pattern in two and four dimensions and is not discussed here. Although the results of this appendix are known, discussing them in the present framework will add to the readability of this paper.

1. Random Matrix Theory for Two-Dimensional QCD with Two Colors in the Fundamental Representation

For two colors with the quarks in the fundamental representation we can find a gauge field independent basis for which the Dirac operator becomes real. In two dimensions this transformation does not commute with the transformation to a block structure reflecting its chiral symmetry, see section II A. We choose to preserve the chiral structure of the Dirac operator. Then the consequence of the anti-unitary symmetry is that the off-diagonal block of the Dirac operator is complex anti-self-dual which is unitarily equivalent to a random matrix theory with an off-diagonal block that is complex symmetric. The corresponding chiral random matrix theory is given by

$$D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix}, \quad W = -\tau_2 W^T \tau_2 \in \mathbb{C}^{2n\times 2n}, \tag{A1}$$

or equivalently by

$$D' = \begin{bmatrix} 0 & W\tau_2 \\ -W^\dagger\tau_2 & 0 \end{bmatrix}, \quad (W\tau_2)^T = W\tau_2 \in \mathbb{C}^{2n\times 2n}. \tag{A2}$$

The probability distribution is taken to be Gaussian

$$P(W)d[W] \propto \exp\left[-n\mathrm{tr}\, WW^\dagger\right] \prod_{1\le i\le j\le 2n} d\,\mathrm{Re}\; W_{ij}\, d\,\mathrm{Im}\; W_{ij}. \tag{A3}$$

In the subsection A 1 a we calculate the joint eigenvalue probability density of this theory (see Ref. [42]). In the subsection A 1 b we rederive its partition function which was already summarized for all chiral ensembles in Ref. [40, 48].

a. Joint Probability Density

The joint probability density of the eigenvalues of the random matrix D denoted by $p(\Lambda)$ is defined by

$$\int_{\mathbb{C}^{2n\times 2n}} f(D)P(W)d[W] = \int_{\mathbb{R}_+^{2n}} f(\pm i\Lambda)p(\Lambda) \prod_{1\le j\le 2n} d\lambda_j \tag{A4}$$

for any function f invariant under

$$f(D) = f(VDV^\dagger) \tag{A5}$$

for all $V = \mathrm{diag}\,(\widetilde{V}, \tau_2\widetilde{V}^*\tau_2)$ or $W \to \widetilde{V}W\tau_2\widetilde{V}^T\tau_2$ with $\widetilde{V} \in \mathrm{U}\,(2n)$.

The characteristic polynomial of D can be rewritten as

$$\det(D - i\lambda\mathbb{1}_{4n}) = \det(WW^\dagger - \lambda^2\mathbb{1}_{2n}) = \det(W^\dagger W - \lambda^2\mathbb{1}_{2n}). \tag{A6}$$

Let $U \in \mathrm{U}\,(2n)/\mathrm{U}^{\,2n}(1)$ be the matrix diagonalizing $WW^\dagger$, i.e. $WW^\dagger = U\Lambda^2U^\dagger$ with the positive definite, diagonal matrix $\Lambda^2 \in \mathbb{R}_+^{2n}$. Then we can relate the eigenvectors of $WW^\dagger$ to those of $W^\dagger W$. Let

$$WW^\dagger U = (W\tau_2)(W\tau_2)^\dagger = U\Lambda^2, \tag{A7}$$

then complex conjugation results in

$$(W\tau_2)^*(W\tau_2)^T U^* = U^*\Lambda^2, \tag{A8}$$

and because of the symmetry of $W\tau_2$, we also have

$$(W\tau_2)^\dagger W\tau_2 U^* = U^*\Lambda^2. \tag{A9}$$

Hence the eigenvalue decomposition of $W^\dagger W$ reads

$$(W\tau_2)^\dagger(W\tau_2) = U^*\Lambda^2U^T. \tag{A10}$$

The combination of this decomposition with $WW^\dagger = U\Lambda^2U^\dagger$ yields a singular value decomposition of W,

$$W\tau_2 \;=\; UZU^T \tag{A11}$$

with the complex, diagonal matrix $Z \in \mathbb{C}^{2n}$ such that $|Z| = \Lambda$ and $U \in \mathrm{U}\,(2n)/\mathrm{U}^{\,2n}(1)$. The number of degrees of freedom is $2n(2n+1)$ on both sides of Eq. (A11). Hence, the right hand side of Eq. (A11) can be used as a parameterization of W. The phases of Z can be absorbed in U so that W can be parameterized as

$$W\tau_2 \;=\; U\Lambda U^T \tag{A12}$$

with the positive definite, diagonal matrix $\Lambda \in \mathbb{R}_+^{2n}$ and $U \in \mathrm{U}\,(2n)$.

In the next step we calculate the invariant length element which directly yields the Haar measure of W in the coordinates (A12),

$$\begin{aligned}
\mathrm{tr}\, dW dW^\dagger \;&=\; \mathrm{tr}\, d(W\tau_2)d(W\tau_2)^\dagger \qquad\qquad (A13)\\
&=\; \mathrm{tr}\, d\Lambda^2 + \mathrm{tr}\,\left(U^\dagger dU\Lambda + \Lambda(U^\dagger dU)^T\right)\left(U^\dagger dU\Lambda + \Lambda(U^\dagger dU)^T\right)^\dagger\\
&=\; \sum_{1\le i\le 2n} (d\lambda_i^2 + 4\lambda_i^2(U^\dagger dU)_{ii}^2)\\
&\quad + \sum_{1\le i<j\le 2n} \left[\, (U^\dagger dU)_{ij},\ (U^\dagger dU)_{ij}^* \,\right] \begin{bmatrix} \lambda_i\lambda_j & -\dfrac{\lambda_i^2+\lambda_j^2}{2} \\ -\dfrac{\lambda_i^2+\lambda_j^2}{2} & \lambda_i\lambda_j \end{bmatrix} \begin{bmatrix} (U^\dagger dU)_{ij}, \\ (U^\dagger dU)_{ij}^* \end{bmatrix}.
\end{aligned}$$

Note that the Pauli matrix τ_2 drops out. Moreover we have used the anti-Hermiticity of $U^\dagger dU$. From the invariant length (A13) we find the joint probability density

$$p(\Lambda)\prod_{1\le j\le 2n} d\lambda_j \propto |\Delta_{2n}(\Lambda^2)| \prod_{1\le j\le 2n} \exp\left[-n\lambda_j^2\right]\lambda_j d\lambda_j, \tag{A14}$$

cf. Ref. [42, 48]. This coincides with the joint probability density of the nonzero eigenvalues of the chiral GOE with $\nu = 1$, which has one zero mode while the present model has no zero modes at all. Its microscopic spectral density has a linear slope at the origin and the level repulsion is also linear at small distances, cf. Fig 1a.

b. Partition Function

The partition function with N_f flavors is defined by

$$Z(N_\mathrm{f}) = \int d[W] \prod_{k=1}^{N_\mathrm{f}} \det(D + m_k\mathbb{1}_{4n})P(W). \tag{A15}$$

Due to the decomposition (A12) we multiply D by the unitary matrix $\mathrm{diag}\,(\mathbb{1}_{2n},\tau_2)$ from the left and from the right which keeps the spectrum invariant. To evaluate the average (A15) we first rewrite the determinants as Gaussians over Grassmann variables

$$\begin{aligned} Z(M) \;\propto\; & \int d[W,V]\exp\left[-n\mathrm{tr}\,W\tau_2(W\tau_2)^\dagger\right] \\ & \times\exp\left[\mathrm{tr}\,V_R^\dagger W\tau_2 V_L-\mathrm{tr}\,V_L^\dagger(W\tau_2)^\dagger V_R+\mathrm{tr}\,M(V_R^\dagger V_R+V_L^\dagger V_L)\right] \end{aligned} \tag{A16}$$

with the mass matrix $M=\mathrm{diag}\,(m_1,\ldots,m_{N_\mathrm{f}})$. The matrices V_R and V_L are both $2n\times N_\mathrm{f}$ rectangular matrices comprising independent Grassmann variables as matrix elements. Because $W\tau_2$ is symmetric we have to symmetrize the matrices $V_LV_R^\dagger$ and $V_RV_L^\dagger$. After integrating over W we obtain

$$\begin{aligned} Z(M) \;\propto\; & \int d[V]\exp\left[-\frac{1}{4n}\mathrm{tr}\,(V_LV_R^\dagger-V_R^*V_L^T)(V_RV_L^\dagger-V_L^*V_R^T)+\mathrm{tr}\,M(V_R^\dagger V_R+V_L^\dagger V_L)\right] \\ \propto\; & \int d[V]\exp\left[\frac{1}{4n}\mathrm{tr}\,(\widetilde{\tau}_2\otimes\mathbb{1}_{N_\mathrm{f}})\sigma(\widetilde{\tau}_2\otimes\mathbb{1}_{N_\mathrm{f}})\sigma^T+\mathrm{tr}\,(\mathbb{1}_2\otimes M)\sigma\right], \end{aligned} \tag{A17}$$

where τ_2 completely drops out. The second Pauli matrix $\widetilde{\tau}_2$ acts on flavor space and should not be confused with τ_2 which acts on color space for QCD and its analogue in random matrix theory. The dyadic super matrix

$$\sigma=\begin{bmatrix} V_R^\dagger \\ -V_L^T \end{bmatrix}\begin{bmatrix} V_R, & V_L^* \end{bmatrix}. \tag{A18}$$

is nilpotent and can be replaced by a unitary matrix $U\in\mathrm{U}\,(2N_\mathrm{f})$ via the superbosonization formula [52–54]. By rescaling $U\to 2nU$ and introducing the rescaled mass matrix $\widehat{M}=2nM$, we arrive at

$$Z(\widehat{M}) \;\propto\; \int\limits_{\mathrm{U}\,(2N_\mathrm{f})}\exp\left[n\mathrm{tr}\,(\widetilde{\tau}_2\otimes\mathbb{1}_{N_\mathrm{f}})U(\widetilde{\tau}_2\otimes\mathbb{1}_{N_\mathrm{f}})U^T+\mathrm{tr}\,(\mathbb{1}_2\otimes\widehat{M})U\right]\det{}^{-2n}U d\mu(U), \tag{A19}$$

where $d\mu$ is the normalized Haar-measure.

In the microscopic limit ($n\to\infty$ and $\widehat{M}$ fixed) we can apply the saddlepoint approximation. The saddlepoint equation is given by

$$U^{-1}=(\widetilde{\tau}_2\otimes\mathbb{1}_{N_\mathrm{f}})U^T(\widetilde{\tau}_2\otimes\mathbb{1}_{N_\mathrm{f}}). \tag{A20}$$

Since $U\in\mathrm{U}\,(2N_\mathrm{f})$ Eq. (A20) implies $U\in\mathrm{USp}\,(2N_\mathrm{f})$. The final result is given by

$$\begin{aligned} Z(\widehat{M}) \;&=\; \int\limits_{\mathrm{USp}\,(2N_\mathrm{f})}\exp\left[\mathrm{tr}\,(\mathbb{1}_2\otimes\widehat{M})U\right]d\mu(U) \\ &=\; \int\limits_{\mathrm{USp}\,(2N_\mathrm{f})}\exp\left[\frac{1}{2}\mathrm{tr}\,(\mathbb{1}_2\otimes\widehat{M})(U+U^{-1})\right]d\mu(U). \end{aligned} \tag{A21}$$

Although the joint probability density of the eigenvalues coincides with chGOE, the chiral symmetry breaking pattern ($\mathrm{USp}\,(2N_\mathrm{f})\times\mathrm{USp}\,(2N_\mathrm{f})\to\mathrm{USp}\,(2N_\mathrm{f})$) turns out to be different and agrees with Ref. [31, 40]. Especially there are no zero modes such that the partition function does not vanish at $M=0$ which would be the case for chGOE with the index $\nu=1$, see Ref. [35].

2. Two Dimensional QCD in the Adjoint Representation

For two dimensional QCD with quarks in the adjoint representation the anti-unitary symmetry of the Dirac operator allows us to choose a gauge field independent basis for which the Dirac operator becomes quaternion real. However, when performing this transformation we will lose the chiral block structure. We choose to preserve this structure. Then the anti-unitary symmetry requires that the off-diagonal block of the Dirac operator becomes anti-symmetric. The corresponding random matrix theory is given by

$$D=\begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix},\quad W=-W^T\in\mathbb{C}^{(2n+\nu)\times(2n+\nu)}. \tag{A22}$$

with the probability distribution

$$P(W)d[W] \propto \exp\left[-n\mathrm{tr}\, WW^\dagger\right] \prod_{1\le i<j\le(2n+\nu)} d\,\mathrm{Re}\; W_{ij} d\,\mathrm{Im}\; W_{ij}. \tag{A23}$$

Because odd-dimensional anti-symmetric matrices have one generic zero eigenvalue we have to distinguish the even and odd dimensional case (denoted by $\nu = 0$ and $\nu = 1$ respectively).

In subsection A 2 a we evaluate the joint probability density of the eigenvalues and in subsection A 2 b we discuss the corresponding partition function for $\nu = 0, 1$. These results were obtained previously in Refs. [31, 40, 42, 48].

a. Joint Probability Distribution

The joint probability density $p(\Lambda)$ is defined as in Eq. (A4) while the arbitrary function f has the invariance

$$f(D) = f(VDV^\dagger), \quad \forall\, V = \mathrm{diag}\,(\widetilde{V}, \widetilde{V}^*) \text{ with } \widetilde{V} \in \mathrm{U}\,(2n+\nu). \tag{A24}$$

Let $\nu = 0$, i.e. W is even dimensional. Analogous to the discussion in subsection A 1 a we can quasi-diagonalize W, i.e.

$$W \;=\; U(\tau_2 \otimes \Lambda)U^T \tag{A25}$$

with a positive definite, diagonal matrix $\Lambda \in \mathbb{R}_+^n$ and unitary matrix $U \in \mathrm{U}\,(2n)/\mathrm{SU}^n(2)$. The division with the subgroup $\mathrm{SU}^n(2)$ is the result of the identity $\widetilde{U}\tau_2\widetilde{U}^T = \tau_2$ for all $\widetilde{U} \in \mathrm{SU}(2)$.

The matrix $\tau_2 \otimes \Lambda$ has $\pm\lambda_j$ as eigenvalues. We can use the result (A14) by replacing $\mathrm{diag}\,(\lambda_1, \ldots, \lambda_{2n}) \to \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)$ and taking care of the fact that some degrees of freedom of $\mathrm{U}\,(2n)$ are missing. We can apply the result (A14) because the invariant length element is calculated similar to the ($\beta_{\mathrm{D}} = 1,\ d = 2$)-case. Hence, we find the joint probability density

$$p(\Lambda) \prod_{1\le j\le 2n} d\lambda_j \propto \Delta_n^4(\Lambda^2) \prod_{1\le j\le n} \exp\left[-n\lambda_j^2\right] \lambda_j d\lambda_j, \tag{A26}$$

cf. Ref. [42, 48]. This density coincides with the chGSE for $\nu = -1/2$.

Let us consider the case with an odd dimension, $W = -W^T \in \mathbb{C}^{(2n+1)\times(2n+1)}$. Since an odd dimensional anti-symmetric matrix has one generic zero mode we have to modify the decomposition (A25) according to

$$W \;=\; U\mathrm{diag}\,(\tau_2 \otimes \Lambda, 0)U^T, \tag{A27}$$

where $\Lambda \in \mathbb{R}_+^n$ and $U \in \mathrm{U}\,(2n+1)/[\mathrm{SU}^n(2) \times \mathrm{U}\,(1)]$. Hence, the joint probability density (A26) becomes [42, 48]

$$p(\Lambda) \prod_{1\le j\le 2n} d\lambda_j \propto \Delta_n^4(\Lambda^2) \prod_{1\le j\le n} \exp\left[-n\lambda_j^2\right] \lambda_j^5 d\lambda_j \tag{A28}$$

by employing the result (A14) with the replacement $\mathrm{diag}\,(\lambda_1, \ldots, \lambda_{2n}) \to \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n, 0)$ and taking care of the subgroup $\mathrm{SU}^n(2)\times\mathrm{U}\,(1)$ that is divided out. This coincides with the joint probability density of the non-zero eigenvalues of chGSE with $\nu = 1/2$.

b. Partition Function

The partition function with N_{f} fermionic flavors (A15) can be again mapped to flavor space via the rectangular $(2n+\nu) \times N_{\mathrm{f}}$ matrices V_R and V_L comprising Grassmann variables only. The analogue of Eq. (A17) is given by

$$\begin{aligned} Z(M) \;&\propto\; \int d[V] \exp\left[-\frac{1}{4n}\mathrm{tr}\,(V_L V_R^\dagger + V_R^* V_L^T)(V_R V_L^\dagger + V_L^* V_R^T) + \mathrm{tr}\, M(V_R^\dagger V_R + V_L^\dagger V_L)\right] \\ &\propto\; \int \exp\left[\frac{1}{4n}\mathrm{tr}\,\sigma\sigma^T + \mathrm{tr}\,(\widetilde{\tau}_1 \otimes M)\sigma\right] d[V] \end{aligned} \tag{A29}$$

with the dyadic supermatrix

$$\sigma = \begin{bmatrix} -V_L^T \\ V_R^\dagger \end{bmatrix} \begin{bmatrix} V_R, & V_L^* \end{bmatrix}. \tag{A30}$$

The first Pauli matrix $\widetilde{\tau}_1$ acts on flavor space. The superbosonization formula [52–54] yields

$$Z(\widehat{M}) \propto \int\limits_{\mathrm{U}\,(2N_{\mathrm{f}})} \exp\left[n\mathrm{tr}\,UU^T + \mathrm{tr}\,(\widetilde{\tau}_1 \otimes \widehat{M})U\right] \det{}^{-2n-\nu} U d\mu(U). \tag{A31}$$

In the microscopic limit by taking n to infinity we have to solve the saddlepoint equation

$$U^{-1} = U^T. \tag{A32}$$

Therefore we end up with an integral over the group $\mathrm{O}(2N_{\mathrm{f}})$, i.e.

$$\begin{aligned} Z(\widehat{M}) &= \int\limits_{\mathrm{O}(2N_{\mathrm{f}})} \exp\left[\mathrm{tr}\,(\widetilde{\tau}_1 \otimes \widehat{M})U\right] \det{}^{\nu} U d\mu(U), \\ &= \int\limits_{\mathrm{O}(2N_{\mathrm{f}})} \exp\left[\mathrm{tr}\,(\mathbb{1}_2 \otimes \widehat{M})U\right] \det{}^{\nu} U d\mu(U), \\ &= \int\limits_{\mathrm{O}(2N_{\mathrm{f}})} \exp\left[\frac{1}{2}\mathrm{tr}\,(\mathbb{1}_2 \otimes \widehat{M})(U + U^{-1})\right] \det{}^{\nu} U d\mu(U) \end{aligned} \tag{A33}$$

with $\widehat{M} = 2nM$. For $\nu = 0$ the partition function is of order one for $\widehat{M} \ll 1$ while for $\nu = 1$, the sum over two disconnected components of $\mathrm{O}\,(2N_{\mathrm{f}})$, results in a partition function $Z(\widehat{M}) \propto \widehat{M}^2$ for $\widehat{M} \ll 1$. This property as well as the symmetry breaking pattern $\mathrm{O}(2N_{\mathrm{f}}) \times \mathrm{O}(2N_{\mathrm{f}}) \to \mathrm{O}(2N_{\mathrm{f}})$ underlines the difference of the random matrix ensemble (A22) with chGSE, see Refs. [35, 40]. The sum over $\nu = 0$ and $\nu = 1$ gives an integral over $\mathrm{SO}\,(2N_{\mathrm{f}})$ corresponding to the symmetry breaking pattern of the full partition function [31].

As was shown in Ref. [45] gauge fields with nonzero topology exist for two-dimensional QCD with adjoint fermions and both partition functions for $\nu = 0$ and $\nu = 1$ are realized. The argument of Ref. [45] predicting additional values of ν for $N_{\mathrm{c}} > 2$ seems to be in conflict with chiral perturbation theory [31] and random matrix theory, but we hope to address this puzzle in future work. In lattice QCD at strong coupling in the case of an odd-odd lattice only $\nu = 0$ and $\nu = 1$ are realized for N_{c} odd and N_{c} even, respectively. Our simulations confirm this prediction, see Fig. 2.

Appendix B: Corrections to the Traceless Ensemble

In this appendix we calculate the eigenvalue density including $1/n$ corrections for an even-dimensional GUE with the additional condition that the trace of the matrices may vanish. This condition is implemented via a Lagrange multiplier. The level density is thus given by the random matrix integral,

$$\rho_t^{(n)}(x) = \frac{\int_{\mathrm{Herm}\,(2n)} d[H] \exp\left[-\mathrm{tr}\,H^2/(4n) - t\mathrm{tr}^{\,2}H/(8n^2)\right] \mathrm{tr}\,\delta(H - x\mathbb{1}_{2n})}{\int_{\mathrm{Herm}\,(2n)} d[H] \exp\left[-\mathrm{tr}\,H^2/(4n) - t\mathrm{tr}^{\,2}H/(8n^2)\right]}. \tag{B1}$$

The parameter t interpolates between the traceless condition ($t \to \infty$) and the ordinary GUE ($t \to 0$). The square of the trace in H can be linearized by a Gaussian integral over an auxiliary scalar variable λ, meaning that we can trace back the whole problem to ordinary GUE

$$\rho_t^{(n)}(x) = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1+t}{2t\pi}} d\lambda \exp\left[-\frac{1+t}{2t}\lambda^2\right] \rho_0^{(n)}(x + i\lambda). \tag{B2}$$

The level density of GUE is given in terms of Hermite polynomials, $H_j(x) = x^j + \ldots$, in the following formula [43]

$$\begin{aligned} \rho_0^{(n)}(x) &= \frac{(2n)!}{\sqrt{4\pi n}} \exp\left[-\frac{x^2}{4n}\right] \\ &\times \left(\frac{(2n)^{2n-1}}{((2n-1)!)^2} H_{2n-1}^2\left(\frac{x}{\sqrt{2n}}\right) - \frac{(2n)^{2n-1}}{(2n)!(2n-2)!} H_{2n}\left(\frac{x}{\sqrt{2n}}\right) H_{2n-2}\left(\frac{x}{\sqrt{2n}}\right)\right). \end{aligned} \tag{B3}$$

The large n asymptotics of $H_{2n}(x/\sqrt{2n})$ where x is fixed can be obtained by the relation between Hermite polynomials with an even order and the associated Laguerre polynomials, $L_n^{(-1/2)}(x) = x^n + \ldots$,

$$H_{2n}\left(\frac{x}{\sqrt{2n}}\right) \;=\; 2^n L_n^{(-1/2)}\left(\frac{x^2}{2n}\right). \tag{B4}$$

Note that we employ for both polynomials the monic normalization. The associated Laguerre polynomials $L_n^{(\nu)}$ with a positive integer index ν have a simple representation as a contour integral,

$$L_n^{(\nu)}\left(\frac{x^2}{2n}\right) \;=\; \frac{n!}{(2n)^n}\int_0^{2\pi}\frac{d\varphi}{2\pi}e^{i\nu\varphi}\left(1-\frac{e^{-i\varphi}}{2n}\right)^{n+\nu}\exp[x^2e^{i\varphi}], \tag{B5}$$

which can be expanded asymptotically

$$\begin{aligned} L_n^{(\nu)}\left(\frac{x^2}{2n}\right) &\overset{n\gg 1}{\approx} \frac{1}{(2n)^n}\int_0^{2\pi}\frac{d\varphi}{2\pi}e^{i\nu\varphi}\exp\left[x^2e^{i\varphi}-(n+\nu)\left(\frac{e^{-i\varphi}}{2n}+\frac{e^{-2i\varphi}}{8n^2}\right)\right] \\ &\approx \frac{1}{(2n)^n}\left(\frac{J_{-\nu}(x)}{x^\nu}-\frac{\nu J_{1-\nu}(x)}{2nx^{\nu-1}}-\frac{J_{2-\nu}(x)}{8nx^{\nu-2}}\right). \end{aligned} \tag{B6}$$

The functions J_j are the Bessel functions of the first kind and can be analytically continued in their index j. For $\nu = -1/2$ the expansion for the Hermite polynomials reads

$$\begin{aligned} \frac{1}{n!}H_{2n}\left(\frac{x}{\sqrt{2n}}\right) &\overset{n\gg 1}{\approx} \frac{1}{n^n}\left(\sqrt{x}J_{1/2}(x)+\frac{x^{3/2}J_{3/2}(x)}{4n}-\frac{x^{5/2}J_{5/2}(x)}{8n}\right) \\ &= \frac{1}{n^n}\sqrt{\frac{2}{\pi}}\left(\sin x+\frac{1}{8n}\left((x^2-1)\sin x+x\cos x\right)\right). \end{aligned} \tag{B7}$$

From this asymptotic expansion it also follows

$$\begin{aligned} \frac{1}{(n-1)!}H_{2n-1}\left(\frac{x}{\sqrt{2n}}\right) &= \sqrt{\frac{n}{2}}\frac{\partial}{\partial x}\frac{1}{n!}H_{2n}\left(\frac{x}{\sqrt{2n}}\right) \\ &\overset{n\gg 1}{\approx} \frac{1}{n^n}\sqrt{\frac{n}{\pi}}\left(\cos x+\frac{1}{8n}\left(x\sin x+x^2\cos x\right)\right). \end{aligned} \tag{B8}$$

and

$$\begin{aligned} \frac{1}{(n-2)!}H_{2n-2}\left(\frac{x}{\sqrt{2n}}\right) &= \frac{\sqrt{2n}(n-1)}{2n-1}\frac{\partial}{\partial x}\frac{1}{(n-1)!}H_{2n-1}\left(\frac{x}{\sqrt{2n}}\right) \\ &\overset{n\gg 1}{\approx} \frac{n-1}{(2n-1)n^{n-1}}\sqrt{\frac{2}{\pi}}\left(-\sin x+\frac{1}{8n}\left((1-x^2)\sin x+3x\cos x\right)\right), \end{aligned} \tag{B9}$$

with help of the recurrence relation of the Hermite polynomials and the Stirling formula including subleading corrections

$$n! \overset{n\gg 1}{\approx} \sqrt{2\pi n}n^n e^{-n}\left(1+\frac{1}{12n}\right). \tag{B10}$$

Summarizing all these asymptotic expansions and plugging everything into the level density (B3) we find the first correction to the GUE asymptotics

$$\begin{aligned} \rho_0^{(n)}(x) &\overset{n\gg 1}{\approx} \frac{1}{\pi}\left(1-\frac{x^2}{4n}\right)\left(1+\frac{1}{8n}\right)\left(\cos^2 x+\frac{1}{4n}\cos x\left(x\sin x+x^2\cos x\right)\right. \\ &\quad \left.-\left(-\sin^2 x+\frac{1}{4n}\sin x\left((1-x^2)\sin x+x\cos x\right)\right)\right) \\ &\approx \frac{1}{\pi}\left(1+\frac{\cos 2x}{8n}\right). \end{aligned} \tag{B11}$$

One can now perform the integral (B2) which yields the result (107).

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B Articles summarized in Subsection 3.2

On the Eigenvalue Density of Real and Complex Wishart Correlation Matrices

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Wishart correlation matrices are the standard model for the statistical analysis of time series. The ensemble averaged eigenvalue density is of considerable practical and theoretical interest. For complex time series and correlation matrices, the eigenvalue density is known exactly. In the real case, however, a fundamental mathematical obstacle made it forbidingly complicated to obtain exact results. We use the supersymmetry method to fully circumvent this problem. We present an exact formula for the eigenvalue density in the real case in terms of twofold integrals and finite sums.

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Time series analysis is an indispensable tool in the study of complex systems with numerous applications in physics, climate research, medicine, signal transmission, finance and many other fields [1–3]. A time series is an observable such as the water level of a river, the temperature, the intensity of transmitted radiation, the neuron activity in electroencephalography (EEG), the price of a stock, etc., measured at usually equidistant times $k = 1, \ldots, n$. Suppose we measure p such time series $M_j,\ j = 1, \ldots, p$, for example, in the case of EEG, at p electrodes placed on the scalp or, in the case of temperatures, at p different locations. Our data set then consists of the $p \times n$ rectangular matrix M with entries M_{jk}. The time series M_j are usually real (labeled $\beta = 1$), but in some applications they can be complex ($\beta = 2$). Often, one is interested in the correlations between the time series. To estimate them, one normalizes the time series M_j to zero mean and unit variance. The correlation coefficient between the time series M_j and M_l is then given as the sample average

$$C_{jl} = \frac{1}{n}\sum_{k=1}^{n} M_{jk}M_{lk}^{*} \qquad \text{and} \qquad C = \frac{1}{n}MM^{\dagger} \tag{1}$$

is the correlation matrix. For real time series ($\beta = 1$), the complex conjugation is not needed and the adjoint is simply the transpose. We notice that C is a $p \times p$ real–symmetric ($\beta = 1$) or Hermitean ($\beta = 2$) matrix.

The eigenvalues of C provide important information, see recent examples in Refs. [4, 5]. As the empirical information is limited, it is desirable to compare the measured eigenvalue density with a "null hypothesis" that results from a statistical ensemble. The ensemble is defined [6] by synthetic real or complex time series $W_j,\ j = 1, \ldots, p$ which yield the *empirical* correlation matrix C upon averaging over the probability density function

$$P_\beta(W, C) \sim \exp\left(-\frac{\beta}{2}\mathrm{tr}W^{\dagger}C^{-1}W\right)\ , \tag{2}$$

that is, we have by construction

$$\int d[W]P_\beta(W, C)\frac{1}{n}WW^{\dagger} = C\ , \tag{3}$$

where the measure $d[W]$ is the product of the differentials of all independent elements in W. To ensure that C is invertible, we always assume $n \geq p$. When going to higher order statistics, the Gaussian assumption (2) is not necessarily justified, but it often is a good approximation. This multivariate statistical approach is closely related to Random Matrix Theory [7], and the matrices $WW^{\dagger}$ are referred to as Wishart correlation matrices. One is interested in the ensemble averaged eigenvalue density of these matrices. In terms of the resolvent, it reads

$$S_\beta(x) = -\frac{1}{p\pi}\mathrm{Im}\int d[W]P_\beta(W, C)\mathrm{tr}\frac{\mathbb{1}_p}{x^{+}\mathbb{1}_p - WW^{\dagger}}, \tag{4}$$

where $\mathbb{1}_p$ is the $p \times p$ unit matrix. The argument x carries a small positive imaginary increment $\varepsilon > 0$, indicated by the notation $x^{+} = x + i\varepsilon$. The limit $\varepsilon \to 0$ is implicit in our notation. Due to the invariance of the trace and the measure, the ensemble averaged eigenvalue density $S_\beta(x)$ only depends on the eigenvalues $\Lambda_j,\ j = 1, \ldots, p$ of C. Hence we may replace C in Eq. (4) by the $p \times p$ diagonal matrix $\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_p)$. We notice that the eigenvalues are positive definite, $\Lambda_j > 0$.

A large body of literature is devoted to the eigenvalue density (4). Its asymptotic form for large n and p has been studied in great detail, see Refs. [8, 9]. However, an exact closed–form result for finite n and p is only available in the complex case [10, 11]. Unfortunately, a deep, structural mathematical reason made it up to now impossible to derive such a closed–form result in the real case which is the more relevant one for applications. We have three goals: We, first, introduce the powerful supersymmetry method [12–14] to Wishart correlation matrices for arbitrary C. This has, to the best of our knowledge, not been done before. We, second, use the thereby achieved unique structural clearness to derive a new and exact closed–form result for the eigenvalue density in the real case for finite n and p. We, third, show that our results are easily numerically tractable and compare them with Monte Carlo simulations.

Why does the real case pose such a substantial problem? — This is best seen by going to the polar decom-

position $W = UwV$, where $U \in \mathrm{O}(p)$, $V \in \mathrm{O}(n)$ for $\beta = 1$ and $U \in \mathrm{U}(p)$, $V \in \mathrm{U}(n)$ for $\beta = 2$ and where w is the $p \times n$ matrix containing the singular values or radial coordinates w_j, $j = 1, \ldots, p$. In particular, one has $WW^\dagger = Uw^2U^\dagger$, with $w^2 = ww^\dagger$. When inserting into (4), one sees that the non–trival group integral

$$\Phi_\beta(\Lambda, w^2) = \int \exp\left(-\frac{\beta}{2}\mathrm{tr}U^\dagger\Lambda^{-1}Uw^2\right) d\mu(U) \qquad (5)$$

has to be done to obtain the joint probability density function of the radial coordinates w_j. Here, $d\mu(U)$ is the invariant Haar measure. For $\beta = 2$, this integral is the celebrated Harish-Chandra–Itzykson–Zuber integral and known explicitly [15, 16]. For $\beta = 1$, however, $\Phi_1(\Lambda, w^2)$, is not a Harish-Chandra spherical function, it rather belongs to the Gelfand class [17] and a closed–form expression is lacking. The only explicit form known is a cumbersome, multiple infinite series expansion in terms of zonal or Jack polynomials [6, 18]. This inconvenient feature then carries over to the eigenvalue density (4), but we will arrive at a finite series over twofold integrals.

The supersymmetry method is based on writing

$$S_\beta(x) = -\frac{1}{\pi p}\mathrm{Im}\frac{\partial Z_\beta(J)}{\partial J}\bigg|_{J=0} \qquad (6)$$

as the derivative of the generating function

$$Z_\beta(J) = \int d[W]P_\beta(W,\Lambda)\frac{\det(x^+\mathbf{1}_p + J\mathbf{1}_p - WW^\dagger)}{\det(x^+\mathbf{1}_p - WW^\dagger)} \qquad (7)$$

with respect to the source variable J at $J = 0$. One has the normaliziation $Z_\beta(0) = 1$ at $J = 0$. We consider the real and the complex case and use the latter as test. We map $Z_\beta(J)$ onto superspace using steps which are by now standard, see Refs. [13, 14]. A particularly handy approach for applications such as the present one is given in Ref. [19], we use the same conventions and find

$$\begin{aligned} Z_\beta(J) \;=\; & \int d[\rho] I_\beta(\rho) \\ & \times \prod_{j=1}^{p} \mathrm{sdet}^{-\beta/2}\left(x^+\mathbf{1}_{4/\beta} - J\gamma - \frac{\beta}{2}\Lambda_j\rho\right). \end{aligned} \qquad (8)$$

The merit of this transformation is the drastic reduction in the number of degrees of freedom, because the variables to be intergrated over form the $4/\beta \times 4/\beta$ Hermitean supermatrix

$$\rho \;=\; \begin{bmatrix} \rho_{11} & \rho_{12}^\dagger \\ \rho_{12} & i\rho_{22} \end{bmatrix} . \qquad (9)$$

For $\beta = 2$, ρ_{11} and ρ_{22} are scalar, real commuting variables and ρ_{12} is a complex anticommuting scalar variable. For $\beta = 1$, ρ_{11} is a 2×2 real symmetric matrix, ρ_{22} has to be multiplied with $\mathbf{1}_2$ and we have

$$\rho_{12} = \begin{bmatrix} \chi & \chi^* \\ \xi & \xi^* \end{bmatrix} , \qquad (10)$$

where χ, ξ and χ^*, ξ^* denote anticommuting variables and their complex conjugates, respectively. We also introduced the matrix $\gamma = \mathrm{diag}(0_{2/\beta}, -\mathbf{1}_{2/\beta})$ and the supersymmetric Ingham–Siegel integral

$$I_\beta(\rho) = \int d[\sigma]\mathrm{sdet}^{-n\beta/2}(\mathbf{1}_{4/\beta} + i\sigma)\exp(i\mathrm{str}\sigma\rho) , \qquad (11)$$

where σ has the same form as ρ. The supertrace and superdeterminant [20] are denoted by str and sdet.

Starting from the generating function (8) we first consider the complex case $\beta = 2$. By introducing eigenvalue–angle coordinates for the supermatrix ρ, we rederive in a straightforward calculation the eigenvalue density $S_2(x)$ as found in Ref. [10]. In the real case $\beta = 1$, the analogous approach leads to inconvenient Efetov–Wegner terms [14], and we thus proceed differently. Since the generating function remains invariant under rotations of the matrix ρ_{11}, we introduce its eigenvalues $R_1 = \mathrm{diag}(r_1, r_2)$ and the diagonalizing angle as new coordiantes. This yields the Jacobian $|\Delta_2(R_1)| = |r_1 - r_2|$. The next step is to evaluate the Ingham–Siegel integral $I_1(\rho)$. The supermatrix σ in Eq. (11) has the same form as ρ in Eq. (9). Doing the integral over σ followed by an expansion in the anticommuting variables of ρ according to Eq. (10) gives

$$\begin{aligned} I_1(\rho) \sim\; & \det{}^{(n-1)/2}R_1 \exp\left(-\mathrm{str}\rho\right)\Theta(R_1) \\ & \left(\left(\frac{\partial}{i\partial\rho_{22}}\right)^{n-2} - \left(\frac{\chi\chi^*}{r_1} + \frac{\xi\xi^*}{r_2}\right)\left(\frac{\partial}{i\partial\rho_{22}}\right)^{n-1}\right. \\ & \left. + \frac{1}{r_1 r_2}\chi\chi^*\xi\xi^*\left(\frac{\partial}{i\partial\rho_{22}}\right)^{n}\right)\delta(\rho_{22}) . \end{aligned} \qquad (12)$$

In a simple, direct calculation, we also expand the product of the superdeterminats in Eq. (8) in the anticommuting variables of ρ. We collect everything and do the integration over the anticommuting variables. With the notation $Q_j = x^+ - 2\Lambda_j i\rho_{22}$, we obtain

$$\begin{aligned} Z_1(J) \sim\; & \int d[R_1]\int d\rho_{22}|\Delta_2(R_1)|\det{}^{(n-1)/2}R_1\Theta(R_1) \\ & \exp(-(r_1 + r_2 - 2i\rho_{22}))\prod_{j=1}^{p}\frac{(J+Q_j)}{\det{}^{1/2}(x^+\mathbf{1}_2 - 2\Lambda_j r_1)} \\ & \left(\det{}^{-1}R_1\left(\frac{\partial}{i\partial\rho_{22}}\right)^n + \sum_{j=1}^{p}\frac{(2\Lambda_j)^2}{(J+Q_j)}\left(\frac{1}{(x^+ - 2\Lambda_j r_1)r_2}\right.\right. \\ & \left. + \frac{1}{(x^+ - 2\Lambda_j r_2)r_1}\right)\left(\frac{\partial}{i\partial\rho_{22}}\right)^{n-1} \\ & + \sum_{j\neq k}^{p}\left(\frac{(2\Lambda_j)^2}{(J+Q_j)(x^+ - 2\Lambda_j r_1)}\right. \\ & \left.\left. \times\frac{(2\Lambda_k)^2}{(J+Q_k)(x^+ - 2\Lambda_k r_2)}\left(\frac{\partial}{i\partial\rho_{22}}\right)^{n-2}\right)\right)\delta(\rho_{22}) . \end{aligned} \qquad (13)$$

According to Eq. (6) we have to take the derivative with

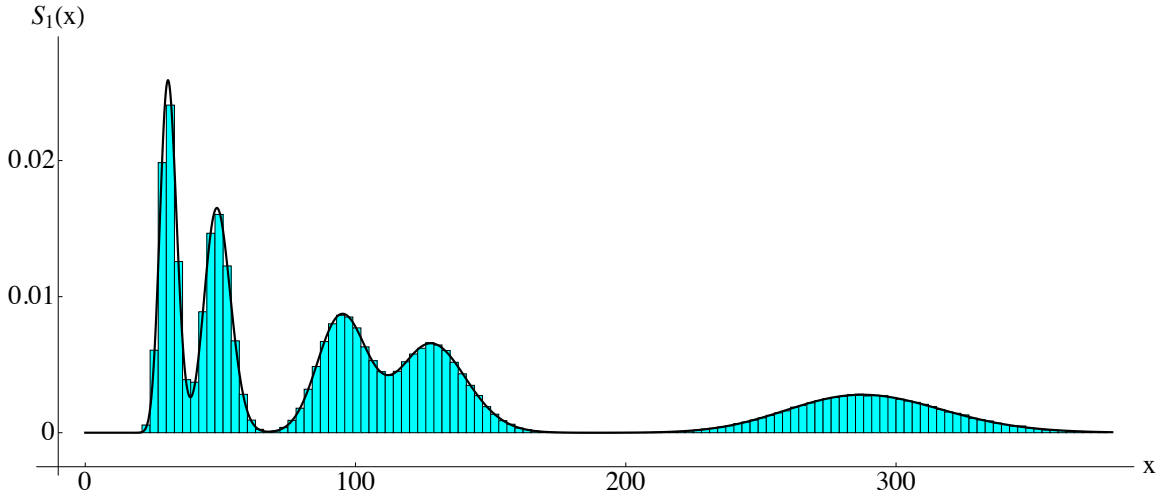

FIG. 1: Eigenvalue density for $p = 5$ and $n = 200$: analytical formula (solid lines) and Monte Carlo simulations (histogram with bin width 3).

respect to J. This leads to the three relations

$$\frac{\partial}{\partial J}\prod_{l=1}^{p}(J+Q_l)\bigg|_{J=0} = E_{p-1}(Q) \tag{14}$$

$$\frac{\partial}{\partial J}\frac{1}{J+Q_j}\prod_{l=1}^{p}(J+Q_l)\bigg|_{J=0} = E_{p-2;j}(Q)$$

$$\frac{\partial}{\partial J}\frac{1}{J+Q_j}\frac{1}{J+Q_k}\prod_{l=1}^{p}(J+Q_l)\bigg|_{J=0} = E_{p-3;j,k}(Q)\ ,$$

where $E_{m;i,k}(Q)$ denotes the elementary symmetric polynomial of order m in the variables $Q_j,\ j = 1,\ldots,p \neq i,k$ with Q_i and Q_k omitted,

$$E_{m;i,k}(Q) = \sum_{\substack{1\le i_1<\cdots<i_m\le p\\ \neq i\neq k}} Q_{i_1}\ldots Q_{i_m}\ . \tag{15}$$

We finally arrive at

$$\begin{aligned} S_1(x) = c\,\mathrm{Im}\int d[R_1]\int d\rho_{22}\frac{|\Delta_2(R_1)|\det{}^{(n-1)/2}R_1}{\prod_{j=1}^{p}\det{}^{1/2}(x^+\mathbb{1}_2-2\Lambda_jR_1)}\\ \Theta(R_1)e^{-(r_1+r_2-2i\rho_{22})}\Bigg(\det{}^{-1}R_1E_{p-1}(Q)\left(\frac{\partial}{i\partial\rho_{22}}\right)^n\\ +\sum_{j=1}^{p}(2\Lambda_j)^2\left(\frac{1}{(x^+-2\Lambda_jr_2)r_1}\right.\\ \left.+\frac{1}{(x^+-2\Lambda_jr_1)r_2}\right)E_{p;j}(Q)\left(\frac{\partial}{i\partial\rho_{22}}\right)^{n-1}\\ +\sum_{j\neq k}^{p}\frac{(2\Lambda_j)^2(2\Lambda_k)^2E_{p-3,j,k}(Q)}{(x^+-2\Lambda_jr_1)(x^+-2\Lambda_kr_2)}\left(\frac{\partial}{i\partial\rho_{22}}\right)^{n-2}\Bigg)\delta(\rho_{22})\ , \end{aligned} \tag{16}$$

where the constant reads $c = (-1)^{n+1}/(4\pi p(n-2)!)$. Due to the δ–distribution, the integral over ρ_{22} are elementary. Hence we end up with an expression for the eigenvalue density $S_1(x)$ which essentially is a twofold integral.

The integrals in Eq. (16) can be numerically evaluated by using a regularisation technique of the type

$$\begin{aligned} \mathrm{Im}\int_0^\infty dr_1\int_0^\infty dr_2\frac{f(r_1,r_2)}{\prod_{l=1}^{p}\sqrt{r_1-\Lambda_l^{-1}}\sqrt{r_2-\Lambda_l^{-1}}} =\\ =\sum_{\substack{0\le i,j\le p\\ (i+j)\in 2\mathbb{N}+1}}\int_{\Lambda_{i+1}^{-1}}^{\Lambda_i^{-1}}dr_1\int_{\Lambda_{j+1}^{-1}}^{\Lambda_j^{-1}}dr_2 f(r_1,r_2)\\ \frac{1}{\prod_{l=1}^{p}\sqrt{r_1-\Lambda_l^{-1}}\sqrt{r_2-\Lambda_l^{-1}}}\ . \end{aligned} \tag{17}$$

Here we assume an ordering of the eigenvalues such that $\Lambda_0 > \Lambda_1 > \ldots > \Lambda_p > \Lambda_{p+1}$ with $\Lambda_0^{-1} = 0$ and $\Lambda_{p+1}^{-1} = \infty$. The real function $f(r_1,r_2)$ is independent of ε and has no singularities. The singularties at the boundaries of the domain are integrable. Using the commercial software MATHEMATICA® [21], we evaluate our formula (17) numerically. For independent comparison, we also carry out Monte Carlo simulations with ensembles of 10^5 matrices. In Figs. 1 and 2, we show the

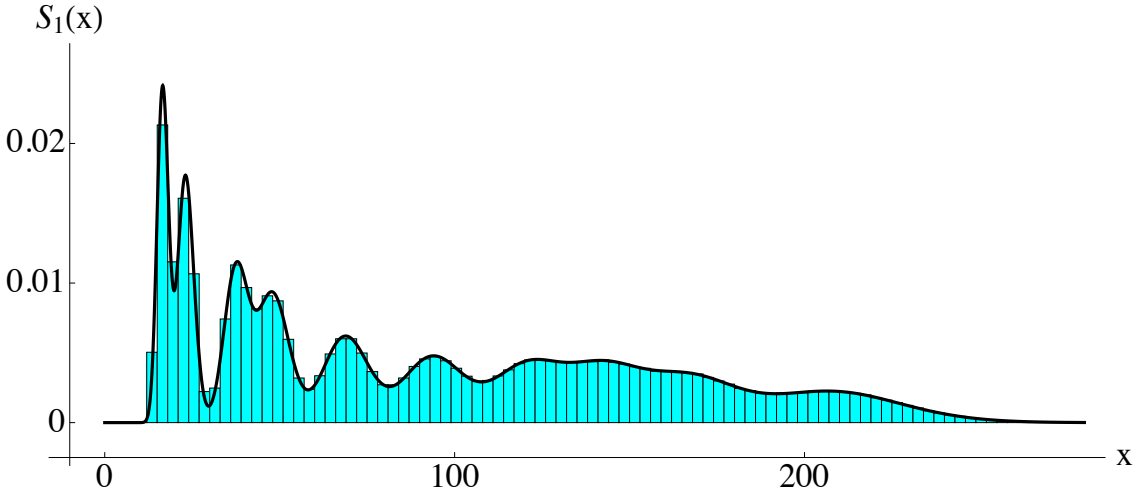

FIG. 2: Eigenvalue density for $p = 10$ and $n = 200$: analytical formula (solid lines) and Monte Carlo simulations (histogram with bin width 3).

results for $p = 5$ and $p = 10$ and $n = 200$ with the chosen empirical eigenvalues Λ_j, $j = 1, \ldots, 5$ of $\{1.44, 0.64, 0.49, 0.25, 0.16\}$ and Λ_j, $j = 1, \ldots, 10$ of $\{1, 0.81, 0.7225, 0.64, 0.45, 0.36, 0.25, 0.2025, 0.1225, 0.03\}$, respectively. The agreement is perfect.

In conclusion, we introduced the supersymmetry method for the first time to Wishart correlation matrices. We thereby derived exact expressions for the eigenvalue density in terms of low–dimensional integrals. This is a drastic reduction, as the original order of integrals is np. Our approach solves a serious mathematical obstacle in the real case. A presentation for a mathematics audience will be given elsewhere [22]. Here, we derived and discussed the formulae needed for applications. In the real case ($\beta = 1$), we obtained the previously unknown exact solution in terms of a finite sum of twofold integrals. We evaluated our formula numerically and confirmed it by comparing to Monte Carlo simulations.

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Supersymmetry approach to Wishart correlation matrices: Exact results

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Abstract We calculate the 'one-point function', meaning the marginal probability density function for any single eigenvalue, of real and complex Wishart correlation matrices. No explicit expression had been obtained for the real case so far. We succeed in doing so by using supersymmetry techniques to express the one-point function of real Wishart correlation matrices as a twofold integral. The result can be viewed as a resummation of a series of Jack polynomials in a non-trivial case. We illustrate our formula by numerical simulations. We also rederive a known expression for the one-point function of complex Wishart correlation matrices.

Keywords Wishart correlation matrices

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1 Introduction

Complex systems of many different kinds are in the focus of modern research [1,2,3,4]. Correlation matrices obtained from data sampling are a key tool to study such systems [5,6]. A standard approach in multivariate statistics is to use random matrix theory (RMT) to model the correlation matrices [7]. Adopting the framework of RMT, we will calculate the marginal probability density function for any single eigenvalue to take a given value x (referred to as the one-point function $S_\beta(x)$ for short) for the case of real ($\beta = 1$) and complex ($\beta = 2$) correlation matrices. In the complex case, an explicit expression is known [8]. Real correlation matrices are encountered more frequently, but unfortunately, closed expressions for their one-point functions and related quantities have not been obtained so far. This is because a certain integral over the orthogonal group is not available in explicit form. Sophisticated power series techniques have been developed in order to tackle this problem [7]. However, the resulting expressions suffer from the drawback that a resummation of the infinite series has not been possible so far. For correlation matrices of large dimension, asymptotic results were derived in [9]. Recently some new results for the one-point function and the two-point correlation function in the asymptotic regime have been found [10].

Here we provide exact results for the one-point functions of real and complex correlation matrices. We use an alternative approach to circumvent the problems mentioned above. Our approach relies on the supersymmetry method [11] – nowadays a standard tool for RMT applications in physics [12]. We derive an exact expression for the one-point function of real correlation matrices as a twofold integral. We also rederive the known result for the one-point function of complex correlation matrices. In Ref. [13], we presented our main results to make them available for applications. Here, we give the full derivation of our results in a form that addresses not only physicists and practitioners, but also the mathematics and statistics community.

The article is organized as follows. We formulate the problem and introduce our notation in Sec. 2. In Sec. 3 we apply the supersymmetry technique, pursuing in parallel two different approaches put forward in the literature, namely the generalized Hubbard-Stratonovich transformation [14,15] and the superbosonization formula [16,17]. These two approaches are equivalent [18], but they yield different forms of the final expressions, each having their own advantages and disadvantages. We explicitly calculate the one-point function in Sec. 4. In Sec. 5 we numerically integrate our formula to compare with Monte-Carlo simulations. We summarize and conclude in Sec. 6.

2 Formulating the Problem

In Sec. 2.1 we define the ensemble of random matrices to be considered in this paper. In Sec. 2.2 we introduce a generating function for the one-point function. This generating function will serve as the starting point for the supersymmetry method.

2.1 Ensemble of Wishart correlation matrices and one-point function

We briefly sketch the RMT approach to correlation matrices as set up in Ref. [7]. We consider real and complex Wishart correlation matrices. The building block for these are rectangular $p \times n$ matrices which we denote by $W = [W_{jk}]$, with $j = 1, \ldots, p$ and $k = 1, \ldots, n$. The p rows can be viewed as the model time series of length n. The $p \times p$ matrix $WW^\dagger$ is the model correlation matrix, also referred to as the Wishart correlation matrix. We always assume $p \leq n$. The entries of W are either real or complex random variables. These two cases are labeled by the Dyson index β taking the value $\beta = 1$ for real entries ($W_{jk} \in \mathbb{R}$) and $\beta = 2$ for complex entries ($W_{jk} \in \mathbb{C}$). For the joint probability distribution of the entries of W one chooses the multivariate Gaussian distribution

$$P_\beta(W, C) = D_\beta \exp\left(-\frac{\beta}{2} \operatorname{tr} W^\dagger C^{-1} W\right), \tag{1}$$

where C is the empirical, *i.e.* given correlation matrix. By construction, the ensemble-averaged Wishart correlation matrix $WW^\dagger/n$ equals C, the empirical one. The Gaussian assumption (1) is justified in most if not all situations of interest. The full measure is $P_\beta(W, C)d[W]$ where

$$d[W] = \begin{cases} \displaystyle\prod_{j=1}^{p} \prod_{k=1}^{n} dW_{jk} & \text{for } \beta = 1, \\ \displaystyle\prod_{j=1}^{p} \prod_{k=1}^{n} d\mathrm{Re}W_{jk} d\mathrm{Im}W_{jk} & \text{for } \beta = 2, \end{cases} \tag{2}$$

is the corresponding volume element. This measure fulfills the invariance condition

$$P_\beta(W, C)d[W] = P_\beta(UW, UCU^\dagger)d[UW] \tag{3}$$

for an arbitrary orthogonal ($\beta = 1$) or unitary ($\beta = 2$) $p \times p$ matrix U. Since the domain of W ($\mathbb{R}^{p\times n}$ for $\beta = 1$, $\mathbb{C}^{p\times n}$ for $\beta = 2$) is invariant under the transformation $W \mapsto UW$, we may replace C by the diagonal matrix of its eigenvalues as long as invariant quantities such as the one-point function (see below) are studied. Thus, we set $C \equiv \Lambda$ where $\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_p)$ is the diagonal matrix containing the eigenvalues. By the definition of C as a correlation matrix we have $\Lambda_j > 0$. The constant D_β in Eq. (1) ensures the normalization of $P_\beta(W, C)d[W]$ to unity and is given by

$$D_\beta = \left((2\pi/\beta)^p \det\Lambda\right)^{-n\beta/2}. \tag{4}$$

The set of random matrices $WW^\dagger$ with the entries of W distributed according to Eq. (1) is referred to as the ensemble of Wishart correlation matrices (sometimes also as the correlated Wishart ensemble). We mention in passing that for the choice $\Lambda = \mathbb{1}_p$, where $\mathbb{1}_p$ denotes the $p \times p$ unit matrix, the ensemble defined by Eq. (1) is equivalent to the so-called Gaussian chiral random matrix ensemble, which serves as a model for the universal eigenvalue statistics of the Dirac operator in Quantum Chromodynamics [19].

The one-point function for the eigenvalues λ_j of $WW^\dagger$ is defined by

$$S_\beta(x) := \int d[W] P_\beta(W,\Lambda) \frac{1}{p} \sum_{j=1}^{p} \delta(x-\lambda_j)$$
$$= \frac{1}{\pi p} \lim_{\varepsilon \to 0+} \mathrm{Im} \int d[W] P_\beta(W,\Lambda) \,\mathrm{tr}\, \frac{\mathbb{1}_p}{(x-\mathrm{i}\varepsilon)\mathbb{1}_p - WW^\dagger}, \tag{5}$$

where in the second line we have passed to an expression involving the resolvent. By the definition (5), the one-point function is a function of x which parametrically depends on the empirical eigenvalues $\Lambda_1, \ldots, \Lambda_p$. We drop the dependence on Λ in writing $S_\beta(x)$.

Having defined the object of interest, we briefly comment on why it is difficult to handle the real case $\beta = 1$ by the traditional techniques of multivariate analysis. In the standard approach to calculating the one-point function (5) one makes a singular-value decomposition

$$W = UwV, \tag{6}$$

where $U \in \mathrm{O}(p)$, $V \in \mathrm{O}(n)$ for $\beta = 1$ and $U \in \mathrm{U}(p)$, $V \in \mathrm{U}(n)$ for $\beta = 2$. The $p \times n$ matrix w contains the singular values $w_j \in \mathbb{R}$ $(j = 1, \ldots, p)$ of W. For the matrix $WW^\dagger$ this decomposition yields

$$WW^\dagger = Uw^2U^\dagger \quad \text{with} \quad w^2 = \mathrm{diag}(w_1^2, \ldots, w_p^2). \tag{7}$$

While the substitution $W \mapsto UW$ leaves the resolvent in Eq. (5) invariant, the diagonalizing matrix U does not drop out of the probability distribution function $P_\beta(W,\Lambda)$ containing the matrix $\Lambda \neq \mathbb{1}_p$. The decomposition thus leads to the group integral

$$\Phi_\beta(\Lambda, w^2) = \int \exp\Big(-\frac{\beta}{2} \,\mathrm{tr}\, U^\dagger \Lambda^{-1} U w^2 \Big) d\mu(U). \tag{8}$$

For $\beta = 2$ this is the celebrated Harish–Chandra-Itzykson-Zuber integral [20, 21] for which there exists an explicit expression. On the other hand, for $\beta = 1$ no simple expression is known. The only expression available [7] is an infinite series in terms of zonal polynomials or equivalently, Jack polynomials, which in turn are only known recursively. A resummation of the infinite series has not been possible so far. In the present work we circumvent this problem by using a supersymmetry approach. This allows us to derive an expression for the one-point function as a twofold integral.

2.2 Generating function

The starting point for the supersymmetry approach is the generating function

$$Z_\beta(x_0, x_1) = \int d[W] P_\beta(W,\Lambda) \frac{\det(x_1\mathbb{1}_p - WW^\dagger)}{\det(x_0\mathbb{1}_p - WW^\dagger)}, \tag{9}$$

where x_0, x_1 are complex variables, $x_0 \notin \mathbb{R}_+$. The one-point function can be computed from it by taking a derivative:

$$S_\beta(x) = (2\pi \mathrm{i} p)^{-1} \frac{\partial}{\partial x_1}\bigg|_{x_1=x} \lim_{\varepsilon\to 0+} \big(Z_\beta(x-\mathrm{i}\varepsilon, x_1) - Z_\beta(x+\mathrm{i}\varepsilon, x_1)\big). \quad (10)$$

Note that the generating function $Z_\beta(x_0, x_1)$ equals unity at $x_0 = x_1$. In the following we derive simple and computationally useful expressions for it by applying the supersymmetry technique.

3 Passing to superspace

There are several ways to express the generating function as an integral over a suitable superspace. Of particular prominence are the generalized Hubbard-Stratonovich transformation put forward in Ref. [14,15] and the superbosonization formula derived in Ref. [17]. Superbosonization [22] was first proposed in a field theoretical context. It was then explored how the supersymmetry method can be extended to arbitrary invariant random matrix ensembles. In the unitary case, this problem was solved [14] by introducing the generalized Hubbard-Stratonovich transformation. In Ref. [17], rigorous superbosonization was developed for all classical Lie symmetries (unitary, orthogonal, symplectic). The approach of Ref. [14] was then completed by transcribing it to the orthogonal and symplectic cases [15]. The equivalence of the superbosonization of Ref. [17] and the generalized Hubbard-Stratonovich transformation of Refs. [14,15] was demonstrated in [18].

In Sec. 3.1, we write the ratio of determinants in the generating function as a Gaussian integral over a rectangular supermatrix. Then we carry out the ensemble average. To the reader not experienced with anticommuting variables, we recommend the introductory parts of Refs. [11,23,24] and the book by Berezin [25]. In Sec. 3.2, we use a duality between dyadic ordinary matrices and dyadic supermatrices to express the result of the ensemble average as a supermatrix integral. After analyzing certain symmetries of this dyadic supermatrix we replace it by a supermatrix of the same symmetries but independent matrix elements by means of the generalized Hubbard-Stratonovich transformation (Sec. 3.3), and alternatively with the help of the superbosonization formula (Sec. 3.4).

3.1 Ensemble average

The determinant in the denominator of Eq. (9) can be expressed as a Gaussian integral over a vector comprising ordinary commuting variables. The determinant in the numerator can be expressed as a Gaussian integral over a vector with anticommuting entries [23,11]. By combining both expressions we obtain a representation for the ratio of determinants in Eq. (9) as a Gaussian integral over a rectangular supermatrix A comprising both vectors:

$$\frac{\det\left(x_1 \mathbb{1}_p - WW^\dagger\right)}{\det\left(x_0 \mathbb{1}_p - WW^\dagger\right)} = \int d[A] \exp\left(\frac{\mathrm{i}\beta}{2} \operatorname{str}\left(XA^\dagger A - A^\dagger WW^\dagger A\right)\right), \quad (11)$$

where the diagonal matrix X is given by $X = \mathrm{diag}(x_0, x_0, x_1, x_1)$ for $\beta = 1$ and $X = \mathrm{diag}(x_0, x_1)$ for $\beta = 2$. We take $\mathrm{Im}\, x_0 > 0$ in order for the Gaussian integral to converge. The rectangular supermatrix

$$\begin{aligned} A &= \left[u_a,\, v_a,\, \zeta_a^*,\, \zeta_a\right]_{1\le a\le p}, \qquad A^\dagger = \begin{bmatrix} u_b \\ v_b \\ \zeta_b \\ -\zeta_b^* \end{bmatrix}_{1\le b\le p} \quad \text{for } \beta = 1, \\ A &= \left[z_a^*\ \zeta_a^*\right]_{1\le a\le p}, \qquad\qquad A^\dagger = \begin{bmatrix} z_b \\ \zeta_b \end{bmatrix}_{1\le b\le p} \quad \text{for } \beta = 2, \end{aligned} \tag{12}$$

is $p \times (2/\beta|2/\beta)$ dimensional. Here $u_j, v_j \in \mathbb{R}$ and $z_j \in \mathbb{C}$ are ordinary real or complex variables while ζ_j, ζ_j^* are anticommuting variables, also referred to as Grassmann variables. We denote by z_j^* the complex conjugate of z_j. In Eq. (11), $d[A]$ denotes the following product:

$$d[A] = \begin{cases} (2\pi)^{-p} \prod_{j=1}^{p} du_j\, dv_j\, \partial_{\zeta_j^*}\partial_{\zeta_j} & \text{for } \beta = 1\,, \\ \pi^{-p} \prod_{j=1}^{p} d\mathrm{Re}\, z_j\, d\mathrm{Im}\, z_j\, \partial_{\zeta_j^*}\partial_{\zeta_j} & \text{for } \beta = 2\,. \end{cases} \tag{13}$$

It should be mentioned that, in this context, one often writes $d\zeta \equiv \partial_\zeta$ for a Grassmann variable ζ. However, the transformation law $d(t\zeta) = t^{-1}d\zeta$ for $t \in \mathbb{C}$ shows that $d\zeta = \partial/\partial\zeta$ really is a partial derivative, *not* a differential!

By inserting the representation (11) into the generating function (9) and changing the order of doing the integrals we find

$$\begin{aligned} Z_\beta(x_0, x_1) &= \int d[A] \exp\left(\frac{\mathrm{i}\beta}{2}\,\mathrm{str}\, X A^\dagger A\right) \\ &\quad \times D_\beta \int d[W] \exp\left(-\frac{\beta}{2}\,\mathrm{tr}\, W^\dagger(\Lambda^{-1} + \mathrm{i}AA^\dagger)W\right) \\ &= \int d[A] \exp\left(\frac{\mathrm{i}\beta}{2}\,\mathrm{str}\, X A^\dagger A\right) \mathrm{det}^{-n\beta/2}\left(\mathbb{1}_p + \mathrm{i}AA^\dagger\Lambda\right). \end{aligned} \tag{14}$$

In the last step, we performed the Gaussian integral over W.

3.2 Duality between ordinary and superspace

We now rewrite the determinant in Eq. (14) as a superdeterminant. This is possible due to duality relations between ordinary spaces and superspaces, see Refs. [26,14,15]. In the present context the duality amounts to the relation [14,15]

$$\mathrm{det}\left(\mathbb{1}_p + \mathrm{i}AA^\dagger\Lambda\right) = \mathrm{sdet}\left(\mathbb{1}_{4/\beta} + \mathrm{i}A^\dagger\Lambda A\right). \tag{15}$$

We notice that the determinant is a polynomial while the superdeterminant is in principle a rational function. The relation originates from the identity $\mathrm{tr}\,(AA^\dagger\Lambda) = \mathrm{str}\,(A^\dagger\Lambda A)$ and a Taylor expansion in the Grassmann variables,

which is always a finite sum. The supermatrix $A^\dagger \Lambda A$ has dimension 4×4 and 2×2 for $\beta = 1$ and $\beta = 2$, respectively. On the other hand, the original matrix $AA^\dagger$ is $p \times p$ dimensional. This dimensional reduction is the crucial advantage of the supersymmetry method.

For present use, we take a look at the symmetry properties of the supermatrix $A^\dagger \Lambda A$. We see that in both cases ($\beta = 1, 2$) the left upper block (a.k.a. the boson-boson block) of $A^\dagger \Lambda A$ is a Hermitian matrix. This observation will constrain some of the matrix blocks appearing below. What about the complex linear symmetries (*i.e.* those not involving complex conjugation)? For $\beta = 2$ there are no such symmetries, but for $\beta = 1$ we have

$$(A^\dagger \Lambda A)^T = S^T A^\dagger \Lambda A S \quad \text{with} \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad , \tag{16}$$

reflecting the fact that the related $p \times p$ matrix $\Lambda^{1/2} A A^\dagger \Lambda^{1/2}$ is symmetric.

Our aim now is to replace the supermatrix $A^\dagger \Lambda A$ by a supermatrix σ with independent matrix elements. We have two approaches at our disposal: the generalized Hubbard-Stratonovich transformation [14,15], and the superbosonization formula derived in Refs. [16,17].

3.3 Generalized Hubbard-Stratonovich transformation

We proceed by introducing a (super-)Fourier representation of the required power of the superdeterminant function on the right-hand side of Eq. (15):

$$\operatorname{sdet}^{-n\beta/2}(\mathbb{1}_{4/\beta} + \mathrm{i} A^\dagger \Lambda A) = \int d[\varrho] I_\beta(\varrho) \exp\left(-\frac{\mathrm{i}\beta}{2} \operatorname{str} A^\dagger \Lambda A \varrho\right). \tag{17}$$

The Fourier transform $I_\beta(\varrho)$ is

$$I_\beta(\varrho) = \int d[\sigma] \operatorname{sdet}^{-n\beta/2}(\mathbb{1}_{4/\beta} + \mathrm{i}\sigma) \exp\left(\frac{\mathrm{i}\beta}{2} \operatorname{str} \sigma \varrho\right). \tag{18}$$

In order for this integral representation to be formally consistent, the two supermatrices σ and ϱ have to share the complex linear symmetries of $A^\dagger \Lambda A$. Hence σ and ϱ are supermatrices of dimension $4{\times}4$ for $\beta = 1$ and $2{\times}2$ for $\beta = 2$, and for $\beta = 1$ the complex linear constraint (16) is imposed.

We write the supermatrix σ as

$$\sigma = \begin{bmatrix} \sigma_0 & \chi \\ \tilde{\chi} & \mathrm{i}\sigma_1 \mathbb{1}_{2/\beta} \end{bmatrix}, \tag{19}$$

where the entries are $2/\beta$ dimensional square matrices. The matrix σ_0 (akin to the boson-boson block of $A^\dagger \Lambda A$) is Hermitian, and σ_1 is a real scalar. (The reason for putting the imaginary unit in front of σ_1 will become clear presently.) The off-diagonal blocks χ and $\tilde{\chi}$ contain all anticommuting variables of σ. For $\beta = 2$ the diagonal block σ_0 is simply a real number whereas

χ and $\tilde{\chi} \equiv \chi^*$ are two Grassmann variables. For $\beta = 1$ the diagonal blocks σ_0 and $\sigma_1 \mathbb{1}_2$ are real symmetric 2×2 matrices. The off-diagonal blocks for $\beta = 1$ have the following structure:

$$\chi = \begin{bmatrix} \eta & \eta^* \\ \xi & \xi^* \end{bmatrix}, \qquad \tilde{\chi} = \begin{bmatrix} \eta^* & \xi^* \\ -\eta & -\xi \end{bmatrix}, \tag{20}$$

where η and ξ denote Grassmann variables. With this choice σ satisfies the constraint (16). The supermatrix ϱ, similar to σ, is chosen as

$$\varrho = \begin{bmatrix} \varrho_0 & \omega \\ \tilde{\omega} & \mathrm{i}\varrho_1 \mathbb{1}_{2/\beta} \end{bmatrix}. \tag{21}$$

In a self-evident way, ϱ is divided into blocks having the same symmetries as the blocks of σ in Eq. (19).

The super-integration measure for σ in Eq. (17) is flat and reads

$$d[\sigma] = \begin{cases} (2\pi)^{-2} d\sigma_{0aa}\, d\sigma_{0ab}\, d\sigma_{0bb}\, d\sigma_1\, \partial_\eta \partial_{\eta^*} \partial_\xi \partial_{\xi^*} & \text{for} \quad \beta = 1, \\ (2\pi)^{-1} d\sigma_0\, d\sigma_1\, \partial_\chi \partial_{\chi^*} & \text{for} \quad \beta = 2, \end{cases} \tag{22}$$

where σ_{0aa} and σ_{0bb} are the diagonal elements and σ_{0ab} is the off-diagonal element of the real symmetric matrix σ_0 for $\beta = 1$. The measure $d[\varrho]$ is defined by an identical expression.

We now insert the representation (17) into Eq. (14). Our ensemble-averaged generating function then becomes a supermatrix integral:

$$\begin{aligned} Z_\beta(x_0, x_1) &= \int d[\varrho] I_\beta(\rho) \int d[A] \exp\left(\frac{\mathrm{i}\beta}{2} \operatorname{str} \left(X A^\dagger A - A^\dagger \Lambda A \varrho \right) \right) \\ &= \int d[\varrho] I_\beta(\varrho) \prod_{j=1}^{p} \operatorname{sdet}^{-\beta/2} \left(X - \varrho \Lambda_j \right), \end{aligned} \tag{23}$$

where in the last step we performed the integrals over A. Eq. (23) is the desired superspace representation of the generating function. Originally, the generating function was an integral over ordinary $p \times n$ matrices W. The representation (23) is an integral over supermatrices ϱ of dimension 4×4 for $\beta = 1$ and 2×2 for $\beta = 2$. This drastically reduces the number of integrals to be calculated.

To complete the description of our result (23), we must discuss the function $I_\beta(\varrho)$. This is a super-version of what has come to be called the Ingham-Siegel integral [14,15]. (The name is due to Fyodorov [27] who introduced such an integral in a related, non-super context.) An important point to appreciate is that the superdeterminant under the integral sign of (18) depends *polynomially* on the variable σ_1. Therefore the Fourier transform (18) does not converge in that variable and the supersymmetric Ingham-Siegel integral $I_\beta(\varrho)$ cannot exist as a regular function of ϱ. Nevertheless, we can make sense of $I_\beta(\varrho)$ as a distribution in ϱ, as follows. (In the next subsection, we will reproduce the same result (23) by employing convergent integrals only.)

The Ingham-Siegel integral is invariant under conjugation of σ by elements of the supergroups UOSp(2|2) or U(1|1) for $\beta = 1, 2$, respectively.

This is *not* automatic, but does hold true once the integral is properly regularized by an *invariant* cutoff function. Thus, choosing an invariant cutoff of Gaussian form, we define the supersymmetric Ingham-Siegel integral by

$$I_\beta(\varrho) := \lim_{\varepsilon\to 0+} \int d[\sigma]\, \mathrm{sdet}^{-n\beta/2}(\mathbb{1}_{4/\beta} + \mathrm{i}\sigma) \exp\left(\frac{\mathrm{i}\beta}{2}\,\mathrm{str}\,\sigma\varrho\right) \mathrm{e}^{-\varepsilon\,\mathrm{str}\,\sigma^2}. \quad (24)$$

Here we see the reason why the block $\sigma_1\mathbb{1}_2$ is multiplied by $\mathrm{i} = \sqrt{-1}$: this factor cancels the minus sign from the supertrace, thereby making the Gaussian cutoff function decrease with increasing real integration variable σ_1 .

By construction, the invariantly regularized Fourier transform (24) is invariant and thus the distribution $I_\beta(\varrho)$ depends only on the eigenvalues of $\varrho = URU^{-1} = U\mathrm{diag}(R_0, R_1\mathbb{1}_{2/\beta})U^{-1}$. We note that the diagonal matrix R_0 has the dimension $2/\beta$ while R_1 is a scalar. An explicit formula for the Ingham-Siegel distribution $I_\beta(\varrho)$ was derived in Refs. [14,15]. The result is

$$\begin{aligned} I_\beta(\varrho) &= K_\beta\, \Theta(R_0)\, \mathrm{det}^{(n+1)\beta/2-1}(R_0) \\ &\quad \times \exp\left(-\frac{\beta}{2}\,\mathrm{str}\,R\right) \left(\mathrm{i}\frac{\partial}{\partial R_1}\right)^{n-2/\beta} \delta(R_1), \end{aligned} \quad (25)$$

with the constants

$$K_1 = \frac{\pi}{(n-2)!}, \qquad K_2 = \frac{2\pi}{(n-1)!}, \quad (26)$$

and the Heaviside distribution

$$\Theta(R_0) = \begin{cases} 1 & \text{if } R_0 \text{ is a positive definite matrix,} \\ 0 & \text{else.} \end{cases} \quad (27)$$

This completes our description of the result (23).

3.4 Approach using superbosonization

We now rederive the result (23) (or, more precisely, an equivalent formula) by a different approach, avoiding the use of super-distributions. This will be mathematically clean in every respect. The price to pay is that we rely on a "black box", namely the superbosonization formula proved in [17].

We start over from the very beginning, Eq. (11). Motivated by the fact that our probability measure $P_\beta(W,\Lambda)d[W]$ for $W \in \mathbb{C}^{p\times n}$ (resp. $W \in \mathbb{R}^{p\times n}$) is right-invariant but not left-invariant, we first pass from determinants on the left space $\mathbb{C}^p$ ($\mathbb{R}^p$) to determinants on the right space $\mathbb{C}^n$ ($\mathbb{R}^n$), and only afterwards introduce the standard Gaussian integral representation:

$$\begin{aligned} &\left(\frac{x_0}{x_1}\right)^{p-n} \frac{\det\left(x_1\mathbb{1}_p - WW^\dagger\right)}{\det\left(x_0\mathbb{1}_p - WW^\dagger\right)} = \frac{\det\left(x_1\mathbb{1}_n - W^\dagger W\right)}{\det\left(x_0\mathbb{1}_n - W^\dagger W\right)} \\ &= \mathrm{sdet}^{-\beta/2}\left(X\otimes\mathbb{1}_n - \mathbb{1}_{4/\beta}\otimes W^\dagger W\right) \\ &= \int d[A] \exp\left(\frac{\mathrm{i}\beta}{2}\,\mathrm{str}\left(XA^\dagger A - A^\dagger W^\dagger W A\right)\right). \end{aligned} \quad (28)$$

X is the same diagonal matrix as before, and we still take $\operatorname{Im} x_0 > 0$ in order for the Gaussian integral to converge. The rectangular supermatrix A is the same as before except for the change in dimension $p \to n$.

Next we take the expectation with respect to the probability measure $P_\beta(W,\Lambda)d[W]$ of the W-dependent factor under the integral sign:

$$\begin{aligned}&\int d[W]P_\beta(W,\Lambda)\exp\left(-\frac{\mathrm{i}\beta}{2}\operatorname{tr} WAA^\dagger W^\dagger\right)\\&=\det{}^{-\beta/2}\left(\mathbb{1}_n\otimes\mathbb{1}_p+\mathrm{i}AA^\dagger\otimes\Lambda\right)\\&=\operatorname{sdet}{}^{-\beta/2}\left(\mathbb{1}_{4/\beta}\otimes\mathbb{1}_p+\mathrm{i}A^\dagger A\otimes\Lambda\right).\end{aligned}\tag{29}$$

In the last step we invoked the duality of Sec. 3.2. For the generating function (9) we then get the formula

$$\begin{aligned}Z_\beta(x_0,x_1)=(x_0/x_1)^{n-p}\int d[A]\exp\left(\frac{\mathrm{i}\beta}{2}\operatorname{str} XA^\dagger A\right)\\\times\prod_{j=1}^{p}\operatorname{sdet}{}^{-\beta/2}\left(\mathbb{1}_{4/\beta}+\mathrm{i}A^\dagger A\Lambda_j\right).\end{aligned}\tag{30}$$

Now a beautiful feature of the integrand in (30) is that it depends only on the product $A^\dagger A$ which is invariant under left translations $A\mapsto UA$ by $U\in\mathrm{O}(n)$ (resp. $\mathrm{U}(n)$) for $\beta=1$ ($\beta=2$). This means that we are exactly in the situation where the superbosonization formula of [16,17] applies.

In a nutshell, superbosonization lets us switch from our A-integral to the same integral over a supermatrix Q replacing $A^\dagger A$. The result is

$$Z_\beta(x_0,x_1)=(x_0/x_1)^{n-p}\int DQ\,F_\beta(Q),\tag{31}$$

$$F_\beta(Q)=\operatorname{sdet}{}^{n\beta/2}(Q)\exp\left(\frac{\mathrm{i}\beta}{2}\operatorname{str} XQ\right)\prod_{j=1}^{p}\operatorname{sdet}{}^{-\beta/2}\left(\mathbb{1}_{4/\beta}+\mathrm{i}Q\Lambda_j\right).$$

Q is formally identical to the supermatrix ϱ of Sec. 3.3. The only difference is in the domain of integration: the present Q-integral is over a Riemannian symmetric superspace (of Cartan type A) with invariant (or Berezin-Haar) measure DQ. This is to say that Q runs over the positive matrices in the left upper (or boson-boson) block and the unitary matrices in the right lower (or fermion-fermion) block. For $\beta=2$ the measure DQ is the flat measure $d[Q]\equiv d[\sigma]$ of (22) (up to a normalization factor), for $\beta=1$ it is the flat one times an extra factor of $\operatorname{sdet}{}^{-1/2}(Q)$.

Our two expressions (23) and (31) become identical upon making the substitution $Q\to\mathrm{i}Q/X$. In fact, the Heaviside distribution in R_0 of (25) restricts the integration over all Hermitian matrices ϱ_0 to the positive ones, and the derivatives of the δ-distribution in R_1 of (25) have exactly the same effect as doing the integral over the unitary variable in the fermion-fermion block of Q by the residue theorem. Thus the ϱ-integral in (23) is effectively over the Riemannian symmetric superspace parameterized by Q; see [18] for more discussion of how the two approaches are related.

4 Explicit expressions for the one-point function

We consider in Sec. 4.1 the complex case ($\beta = 2$) and rederive the result found in Ref. [8]. In Sec. 4.2 we study the real case ($\beta = 1$) and derive an expression as a twofold integral. In both cases we have a choice between the methods of superbosonization and generalized Hubbard-Stratonovich transformation.

4.1 Complex case

We show how to reproduce in a few more steps the known result [8] for the one-point function for $\beta = 2$. The supermatrix Q in this case is of size 2×2. We begin by introducing its eigen-representation:

$$Q = URU^{-1}, \quad U = \begin{bmatrix} \sqrt{1-\alpha\alpha^*} & -\alpha \\ \alpha^* & \sqrt{1-\alpha^*\alpha} \end{bmatrix}, \quad R = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}, \tag{32}$$

with eigenvalues r, s and two Grassmann variables α, α^*. This parametrization is singular (more precisely, degenerate in the Grassmann variables) at $r = s$. To suppress any effects of the singularity, we are going to utilize the scale invariance of DQ to change the integration radius $|s| = 1$ to $|s| = q$ with $q \to \infty$. By the change-of-variables formula for superintegrals, our invariant integral (31) for the choice of parametrization (32) then takes the form

$$\int DQ\, F_2(Q) = (2\pi\mathrm{i})^{-1} \lim_{q\to\infty} \int_0^q dr \oint_{|s|=q} ds\, (r-s)^{-2} \partial_\alpha \partial_{\alpha^*} F_2\,, \tag{33}$$

provided that $F_2(Q)$ vanishes on the locus of the coordinate singularity $r = s = q \to \infty$. (Otherwise, so-called Efetov-Wegner boundary terms appear.) To arrange for this vanishing property to hold, notice that for large values of $r = s = q$ the function $F_2(q\mathbb{1}_2)$ behaves as $\mathrm{e}^{\mathrm{i}q(x_0-x_1)}$. If $\mathrm{Im}(x_0 - x_1) > 0$ then this exponential factor makes $F_2(q\mathbb{1}_2)$ vanish on the singular locus for $q \to +\infty$. We therefore assume the inequality $\mathrm{Im}\, x_0 > \mathrm{Im}\, x_1$.

Since the two superdeterminants in $F_2(Q)$ are functions of the eigenvalues r and s only, all of the dependence on α, α^* resides in the factor $\mathrm{e}^{\mathrm{i}\,\mathrm{str}\,XQ}$ and the process of integrating (actually, differentiating) w.r.t. the Grassmann variables is simply done by

$$\partial_\alpha \partial_{\alpha^*}\, \mathrm{e}^{\mathrm{i}\,\mathrm{str}\,XQ} = \mathrm{e}^{\mathrm{i}(x_0 r - x_1 s)} \mathrm{i}(x_0 - x_1)(r - s). \tag{34}$$

By using (33) and (34) in (31) we obtain

$$Z_2(x_0, x_1) = (x_0/x_1)^{n-p} \lim_{q\to\infty} \int_0^q dr \oint_{|s|=q} \frac{ds}{2\pi}\, \frac{r^n (x_0 - x_1)\, g_\Lambda(x_1; s)}{s^n (r-s)\, g_\Lambda(x_0; r)}, \tag{35}$$

where $g_\Lambda(x; s)$ is the function

$$g_\Lambda(x; s) = \mathrm{e}^{-\mathrm{i}xs} \prod_{j=1}^{p} (1 + \mathrm{i}s\Lambda_j). \tag{36}$$

Our next step is to perform the inner integral over the circle variable s. This is done by invoking Cauchy's integral theorem to show that for any complex-analytic function $f(s)$ one has

$$\frac{1}{2\pi\mathrm{i}}\oint\limits_{|s|=q}\frac{f(s)\,ds}{s^n(r-s)}=\begin{cases}-r^{-n}f^{[\geq n]}(r), & q>r,\\ +r^{-n}f^{[<n]}(r), & q<r,\end{cases} \tag{37}$$

where $f^{[\geq n]}(r)$ stands for the Taylor series of $f(r)$ in r with all terms up to order $n-1$ deleted, and $f^{[<n]}(r)=f(r)-f^{[\geq n]}(r)$. Application of this formula to (35) yields the result

$$Z_2(x_0,x_1)=-\mathrm{i}\,(x_0-x_1)(x_0/x_1)^{n-p}\int\limits_0^\infty dr\,\frac{g_\Lambda^{[\geq n]}(x_1;r)}{g_\Lambda(x_0;r)}. \tag{38}$$

Recall that $\mathrm{Im}\,x_0>\mathrm{Im}\,x_1$ is required in order for this to hold. The correct expression for the opposite case of $\mathrm{Im}\,x_0<\mathrm{Im}\,x_1$ is obtained by replacing in (38) every occurrence of i by $-\mathrm{i}$ or, equivalently, by sending $r\to -r$.

We now turn to the calculation of the one-point function $S_2(x)$. We recall the expression (10) involving $Z_2(x\pm\mathrm{i}\varepsilon,x_1)$ and apply our result (38) to the case of $x_0=x\pm\mathrm{i}\varepsilon$ with real parameters x,x_1 and $\varepsilon>0$. Our integral representations of $Z_2(x\pm\mathrm{i}\varepsilon,x_1)$ then combine to a *single* integral:

$$\begin{aligned}&\lim_{\varepsilon\to 0+}\big(Z_2(x-\mathrm{i}\varepsilon,x_1)-Z_2(x+\mathrm{i}\varepsilon,x_1)\big)\\ &=\mathrm{i}\,(x-x_1)(x/x_1)^{n-p}\lim_{\varepsilon\to 0+}\int\limits_{-\infty}^{\infty}dr\,\mathrm{e}^{-\varepsilon|r|}\,\frac{g_\Lambda^{[\geq n]}(x_1;r)}{g_\Lambda(x;r)}.\end{aligned} \tag{39}$$

If we re-express the numerator of the integrand as $g_\Lambda^{[\geq n]}\equiv g_\Lambda-g_\Lambda^{[<n]}$ then the term g_Λ contributes $(x-x_1)\delta(x-x_1)=0$. Hence we replace this numerator by $-g_\Lambda^{[<n]}(x_1;r)$, which is a polynomial in r of degree $n-1$.

Now for $x<0$ we may close the integration contour around the lower half of the complex r-plane. From (36) one sees that the integrand is holomorphic in r for $\mathrm{Im}\,r<0$. We therefore get $S_2(x)=0$ for $x<0$. On the other hand, for $x>0$ the contour has to be closed around the upper half-plane. Again, recall Eq. (10). Because the r-integral is now manifestly finite for all values of x_1, we may safely take the x_1-derivative at $x_1=x$ by simply removing the prefactor x_1-x and replacing x_1 by x under the integral sign. (Please be warned that in (38) the r-integral has to diverge at $x_1=x_0$ to arrange for $Z_2(x_0,x_0)=1$ in spite of the factor x_0-x_1.) The result for $x>0$ is

$$S_2(x)=\frac{1}{2\pi p}\oint dr\,\frac{g_\Lambda^{[<n]}(x;r)}{g_\Lambda(x;r)}=-\frac{1}{p}\sum_{j=1}^{p}(\mathrm{i}r+1/\Lambda_j)\frac{g_\Lambda^{[\geq n]}(x;r)}{g_\Lambda(x;r)}\bigg|_{r\to\mathrm{i}/\Lambda_j}, \tag{40}$$

where the integration contour in the first expression encloses the poles of $1/g_\Lambda(x;r)$ at $r=\mathrm{i}/\Lambda_j$ $(j=1,\ldots,p)$. To get the second expression, we switched from $g_\Lambda^{[<n]}$ back to $-g_\Lambda^{[\geq n]}$ and applied the residue theorem.

The residues are best computed by re-inserting the integral representation (37) for $g_\Lambda^{[\geq n]}$, which gives

$$-g_\Lambda^{[\geq n]}(x;\mathrm{i}/\Lambda_j) = (\mathrm{i}/\Lambda_j)^{n-1} \oint \frac{ds}{2\pi\mathrm{i}}\, s^{-n}\mathrm{e}^{-\mathrm{i}xs} \prod_{l(\neq j)} (1+\mathrm{i}s\Lambda_l)\,. \tag{41}$$

We now use the expansion

$$\prod_{l(\neq j)} (1+\mathrm{i}s\Lambda_l) = \sum_{k=1}^{p} (\mathrm{i}s)^{k-1} E_{k-1}(\Lambda^{\widehat{j}}) \tag{42}$$

where E_k are the elementary symmetric functions

$$E_k(\Lambda) := \sum_{1\leq j_1<j_2<\ldots<j_k\leq p} \Lambda_{j_1}\Lambda_{j_2}\cdots\Lambda_{j_k}\,. \tag{43}$$

The notation $\Lambda^{\widehat{j}}$ means that the eigenvalue Λ_j is to be dropped from the diagonal matrix Λ. Now, inserting the expansion (42) into (41) we carry out the s-integral. Our final result for the one-point function then reads

$$S_2(x) = \frac{\Theta(x)}{p} \sum_{j=1}^{p} \frac{\mathrm{e}^{-x/\Lambda_j}(1/\Lambda_j)^n}{\prod_{l(\neq j)}(1-\Lambda_l/\Lambda_j)} \sum_{k=1}^{p} \frac{x^{n-k}}{(n-k)!} E_{k-1}(-\Lambda^{\widehat{j}}). \tag{44}$$

To conclude this subsection, we present an alternative expression for the result (44) as a ratio of determinants:

$$S_2(x) = \frac{\Theta(x)}{p} \det \begin{bmatrix} 0 & B_\Lambda(x) \\ C(x) & D_\Lambda \end{bmatrix} / \det D_\Lambda\,, \tag{45}$$

where D_Λ is the $p\times p$ matrix with matrix elements $(D_\Lambda)_{k,j} = \Lambda_j^{-k+1}$, and $B_\Lambda(x)$ and $C(x)$ are row and column vectors with entries

$$B_\Lambda(x)_j = \mathrm{e}^{-x/\Lambda_j}(1/\Lambda_j)^n, \qquad C(x)_k = -\frac{x^{n-k}}{(n-k)!}. \tag{46}$$

The denominator $\det D_\Lambda = \prod_{j>j'}(\Lambda_j^{-1}-\Lambda_{j'}^{-1})$ is essentially the Vandermonde determinant associated with the numbers $\Lambda_1,\ldots,\Lambda_p$.

To verify the expression (45) one expands the $(p+1)\times(p+1)$ determinant with respect to the first row and first column and then uses a standard identity [29] for the elementary symmetric functions:

$$\frac{\det D_\Lambda^{(kj)}}{\det D_\Lambda} = \frac{(-1)^{j-1}E_{k-1}(\Lambda^{\widehat{j}})}{\prod_{l(\neq j)}(1-\Lambda_l/\Lambda_j)}, \tag{47}$$

where $D_\lambda^{(kj)}$ is D_Λ with the k^{th} row and j^{th} column removed. In this way one immediately retrieves (44) from (45).

The expression (45) may be useful for certain applications. Also, it is easily seen to be directly equivalent to the expression given in [8].

4.2 Real case

4.2.1 Approach using the eigenvalues of Q

The main trick in deriving the explicit result for $\beta = 2$ was to use the eigenrepresentation (32) for the supermatrix Q. We are now going to carry out an analogous derivation for $\beta = 1$. The outcome will be somewhat different in that the integrand is no longer meromorphic but has square-root singularities in the radial variables of the boson-boson block.

We use the parametrization

$$Q = k \begin{bmatrix} Q_0 & 0 \\ 0 & s\mathbb{1}_2 \end{bmatrix} k^{-1}, \quad k = \begin{bmatrix} k_0 & 0 \\ 0 & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} \sqrt{1-\alpha\tilde{\alpha}} & -\alpha \\ \tilde{\alpha} & \sqrt{1-\tilde{\alpha}\alpha} \end{bmatrix}, \tag{48}$$
$$k_0 = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \quad \alpha = \begin{bmatrix} \eta & \eta^* \\ \xi & \xi^* \end{bmatrix}, \quad \tilde{\alpha} = \begin{bmatrix} \eta^* & \xi^* \\ -\eta & -\xi \end{bmatrix}, \quad Q_0 = \begin{bmatrix} r_a & 0 \\ 0 & r_b \end{bmatrix},$$

which has coordinate singularities at $r_a = s$ and $r_b = s$. As before, we will suppress their effects by using the scale invariance of DQ to place the singular locus of the coordinate system on the zero locus of the integrand.

In the coordinates given by (48) our invariant integral (31) takes the form

$$\int DQ\, F_1(Q) = \lim_{q\to\infty} \iint\limits_{[0,q)^2} dr_a dr_b \oint\limits_{|s|=q} \frac{ds}{16\pi\mathrm{i}} \frac{|r_a - r_b|}{(r_a - s)^2 (r_b - s)^2} \tag{49}$$
$$\times \frac{s}{\sqrt{r_a r_b}} \int_0^{2\pi} \frac{d\phi}{2\pi}\, \partial_\xi \partial_{\xi^*} \partial_\eta \partial_{\eta^*} (1 + \xi^*\xi + \eta^*\eta) F_1,$$

where F_1 is the function

$$F_1(Q) = \mathrm{sdet}^{n/2}(Q) \exp\left(\frac{\mathrm{i}}{2} \operatorname{str} XQ\right) \prod_{j=1}^{p} \mathrm{sdet}^{-1/2} \left(\mathbb{1}_4 + \mathrm{i}Q\Lambda_j\right). \tag{50}$$

The expression (49) holds if $F_1(Q)$ and its derivatives vanish on the singular locus $r_a = s = q \to \infty$ and $r_b = s = q \to \infty$. Observing that our integrand $F_1(Q)$ for $r_{a/b} = s = q$ contains the exponential $\mathrm{e}^{\mathrm{i}q(\frac{1}{2}x_0 - x_1)}$, we see that F_1 has the required property if the parameter range is restricted by $\frac{1}{2}\mathrm{Im}\, x_0 > \mathrm{Im}\, x_1$. We thus impose this restriction.

It is clear that the integrand (50) does not depend on the angle ϕ, and all of the dependence on the Grassmann variables is in the factor $\mathrm{e}^{\mathrm{i}\operatorname{str} XQ/2}$. Hence the result of doing the Grassmann integral is given by

$$\partial_\xi \partial_{\xi^*} \partial_\eta \partial_{\eta^*} (1 + \xi^*\xi + \eta^*\eta)\, \mathrm{e}^{\mathrm{i}\operatorname{str} XQ/2} = \mathrm{e}^{\mathrm{i}x_0(r_a + r_b)/2 - \mathrm{i}x_1 s}$$
$$\times \left(\mathrm{i}(x_0 - x_1)(r_a + r_b - 2s) - (x_0 - x_1)^2 (r_a - s)(r_b - s)\right). \tag{51}$$

A slight reorganization of the integrand then leads to the following expression for the generating function:

$$Z_1(x_0,x_1) = \left(\frac{x_0}{x_1}\right)^{n-p} \lim_{q\to\infty} \iint\limits_{[0,q)^2} dr_a dr_b\, |r_a - r_b| \oint\limits_{|s|=q} \frac{ds}{16\pi \mathrm{i}} \left(\frac{\sqrt{r_a r_b}}{s}\right)^{n-1} \tag{52}$$
$$\times \left(\frac{\mathrm{i}(x_0-x_1)(r_a+r_b-2s)}{(r_a-s)^2(r_b-s)^2} - \frac{(x_0-x_1)^2}{(r_a-s)(r_b-s)}\right) \frac{g_\Lambda(x_1;s)}{\sqrt{g_\Lambda(x_0;r_a)g_\Lambda(x_0;r_b)}},$$

where g_Λ was defined in (36).

As before, it is possible to carry out the complex contour integral over the variable s. By making partial fraction decompositions to express the integrand in (52) by isolated poles $(r_a-s)^{-\ell}$ and $(r_b-s)^{-\ell}$ of degrees $\ell = 1, 2$, and then applying the identity (37), we obtain

$$Z_1(x_0,x_1) = \frac{1}{8}\left(\frac{x_0}{x_1}\right)^{n-p} \iint\limits_{\mathbb{R}_+^2} dr_a dr_b\, \frac{|r_a-r_b|\sqrt{r_a r_b}^{\,n-1}}{\sqrt{g_\Lambda(x_0;r_a)g_\Lambda(x_0;r_b)}}$$
$$\times \sum_{\nu=1}^{2} (x_0-x_1)^\nu F_\Lambda^{(\nu)}(x_1;r_a,r_b), \tag{53}$$

where we have taken the upper limits of the r-integrals to infinity, and

$$F_\Lambda^{(\nu)}(x;r_a,r_b) = \frac{G_\Lambda^{(\nu)}(x;r_a) - G_\Lambda^{(\nu)}(x;r_b)}{r_a - r_b}, \tag{54}$$

$$G_\Lambda^{(1)}(x;r) = \mathrm{i}\frac{\partial}{\partial r} G_\Lambda^{(2)}(x;r), \quad G_\Lambda^{(2)}(x;r) = -\frac{g_\Lambda^{[\geq n-1]}(x;r)}{r^{n-1}}. \tag{55}$$

The result (53) is the analog for $\beta = 1$ of the earlier formula (38) for $\beta = 2$.

By recalling the formula (10) we can again deduce an expression for the one-point function, $S_1(x)$. (This requires a process of analytic continuation to remove the restriction $\operatorname{Im} x_0 > 2 \operatorname{Im} x_1$.) The result involves only the $\nu = 1$ term from (53) and holds for all $n > p+3$. Unfortunately, we do not know how to do the final two integrals over r_a and r_b in closed form, and an attempt to compute these integrals numerically met with the following complications.

The first complication is that we do not know how to pass from the expression (53–55) involving $g^>$ to an analog of Eq. (40) involving $g^<$. Working directly with $g^>$ is numerically expensive because many terms in the Taylor series have to be summed when r_a or r_b are large. On the other hand, if we use the identity $g^> = g - g^<$ then we incur cancelations, with ensuing rounding errors, due to taking the difference of two large numbers.

The second difficulty is that the exponential part of the integrand oscillates. For the most part, these oscillations can be cured by rotating the integration contours for r_a, r_b to the positive imaginary axis. Ultimately, however, these variables still have to go to infinity in the direction of the real axis to retain convergence of the integral. The contribution from this ultimate part of the integration contour converges rather slowly.

These problems notwithstanding, we are still able to verify our formula for $S_1(x)$. However, by numerical integration using Wolfram-Mathematical we are not able to produce stable results in a wide range of parameters. We therefore refrain from even writing down that formula, and abandon now the eigen-representation (48). Instead, we pursue another approach, exploiting the original coordinate system for Q described in Secs. 3.3 and 3.4.

4.2.2 Direct approach

Below, we give two further expressions for the generating function. Their logical order of presentation depends on which of our two formalisms is used.

Adopting the generalized Hubbard-Stratonovich transformation, we start from the integral representation (23) and evaluate the delta-distributions featured in (24–25). We then carry out the Grassmann integrals (or rather, derivatives) according to the flat measure specified in (22) to obtain

$$Z_1(x_0,x_1) = \frac{1}{8}\iint\limits_{\mathbb{R}_+^2} dr_a dr_b\, \frac{|r_a-r_b|\,\sqrt{r_a r_b}^{\,n-3}\mathrm{e}^{-(r_a+r_b)/2}}{\prod_{i=1}^p \sqrt{(x_0-\Lambda_i r_a)(x_0-\Lambda_i r_b)}}$$
$$\times \sum_{k=0}^{p} \frac{(-1)^k x_1^{p-k}}{(n-k)!}\Bigg(n(n-1)E_k(\Lambda) + \sum_{j\neq l}\frac{\Lambda_j^2\Lambda_l^2 E_{k-2}(\Lambda^{\widehat{jl}})\, r_a r_b}{(x_0-\Lambda_j r_a)(x_0-\Lambda_l r_b)} \tag{56}$$
$$+ (n-1)\sum_{j=1}^{p}\Lambda_j^2 E_{k-1}(\Lambda^{\widehat{j}})\left(\frac{r_a}{x_0-\Lambda_j r_a}+\frac{r_b}{x_0-\Lambda_j r_b}\right)\Bigg),$$

where E_k, defined in (43), is understood to vanish when $k<0$. The notation $\Lambda^{\widehat{jl}}$ means that both Λ_j and Λ_l are removed from the set $\{\Lambda_i\}_{i=1,\dots,p}$.

Alternatively, we can take the formula (31) from superbosonization and make the substitution $Q \mapsto \mathrm{i}Q/X$. Then, by carrying out the Grassmann integrals in the original parametrization specified in Sec. 3.4 we obtain

$$Z_1(x_0,x_1) = \frac{(x_1/x_0)^p}{16\pi\mathrm{i}}\iint\limits_{\mathbb{R}_+^2} dr_a dr_b\, \frac{|r_a-r_b|\,\sqrt{r_a r_b}^{\,n-3}}{\sqrt{g_\Lambda(x_0;\mathrm{i}r_a/x_0)\,g_\Lambda(x_0;\mathrm{i}r_b/x_0)}}$$
$$\times \oint\limits_{|s|=1} ds\, \frac{g_\Lambda(x_1;\mathrm{i}s/x_1)}{s^{n+1}}\Bigg(\sum_{j\neq l}\frac{(\Lambda_j r_a)(\Lambda_j s)(\Lambda_l r_b)(\Lambda_l s)}{(x_0-\Lambda_j r_a)(x_1-\Lambda_j s)(x_0-\Lambda_l r_b)(x_1-\Lambda_l s)}$$
$$-(n-1)\sum_{j=1}^{p}\frac{\Lambda_j s}{x_1-\Lambda_j s}\left(\frac{\Lambda_j r_a}{x_0-\Lambda_j r_a}+\frac{\Lambda_j r_b}{x_0-\Lambda_j r_b}\right)+n(n-1)\Bigg). \tag{57}$$

If we go on to carry out the s-integral, we again arrive at (56) above. Note that (56) and (57) make immediate sense as convergent integrals for all $x_1\in\mathbb{C}$ and $x_0\in\mathbb{C}\setminus\mathbb{R}_+$. A non-trivial check is $Z_1(x_0,x_0)=1$.

From the expression (57) it is easy to verify the large-n behavior predicted by the central limit theorem (CLT). Indeed, for large n our integrand

develops a saddle point at $r_a = r_b = s \simeq n$ due to the presence of the factor $(\sqrt{r_a r_b}/s)^n \mathrm{e}^{-(r_a+r_b)/2+s}$. By steepest descent evaluation of the integral around this saddle point one finds

$$\lim_{n\to\infty} Z_1(nx_0, nx_1) = \prod_{l=1}^{p} \frac{x_1 - \Lambda_l}{x_0 - \Lambda_l}, \tag{58}$$

which is the result expected from CLT for the random matrix $(WW^T)_{ij} = \sum_{k=1}^{n} W_{ik}W_{jk}$ with independent identically distributed matrix elements W_{jk}.

5 Numerical computation of $S_1(x)$

On recalling the basic identity (10), one immediately produces an expression for the one-point function $S_1(x)$; see below. The result looks more involved than the one obtained from the eigen-representation of Q, and the reciprocal square roots visible in Eqs. (56, 57) remain serious obstacles for further analytical calculations. From a numerical perspective, however, the new formula has two clear advantages over the previous one: its integrand is free of oscillatory exponentials, and high powers of the integration variables r_a and r_b occur only through the factor $\mathrm{e}^{-(r_a+r_b)/2}$, which is numerically inexpensive to compute. Thus numerical evaluation of the integral (61) for $S_1(x)$ is straightforward, provided that the singularities at $r_a, r_b = x/\Lambda_j$ $(j = 1, \ldots, p)$ are treated with care. We now discuss how to organize this computation.

To prepare our final formula for $S_1(x)$, we observe that the discontinuity $Z(x - \mathrm{i}\varepsilon, x_1) - Z(x + \mathrm{i}\varepsilon, x_1)$ for $\varepsilon \to 0$ arises solely from the product of factors

$$1/\sqrt{(x - \Lambda_i r_a)(x - \Lambda_i r_b)}, \tag{59}$$

since all other terms in (56) are single-valued across the real x-axis. By the derivation of the result (56), the reciprocal square root (59) has the real value $(x - \Lambda_i r)^{-1}$ for $r_a = r_b = r$. Its values away from the diagonal $r_a = r_b$ are defined by the process of analytic continuation.

For definiteness, let the Λ-values be ordered by

$$0 < \Lambda_1 < \Lambda_2 < \ldots < \Lambda_p\,.$$

Fixing any $x > 0$, we partition (a dense open subset of) the domain $\mathbb{R}_+$ for r_a and r_b into $p+1$ intervals $U_0(x) := (0, x/\Lambda_p)$, $U_p(x) := (x/\Lambda_1, \infty)$, and

$$U_{p-l}(x) := (x/\Lambda_{l+1}, x/\Lambda_l) \qquad (l = 1, \ldots, p-1). \tag{60}$$

The product over $i = 1, \ldots, p$ of the square roots in (59) is then discontinuous across the x-axis only for $(r_a, r_b) \in U_l(x) \times U_{l'}(x)$ with $l + l'$ an odd number. Thus we arrive at an integral representation for $S_1(x)$ of the form

$$S_1(x) = \sum_{l+l' \,\mathrm{odd}} \int_{U_l(x)} dr_a \int_{U_{l'}(x)} dr_b \, \frac{(-1)^{(l+l'-1)/2}}{\prod_{i=1}^{p} \sqrt{|(x - \Lambda_i r_a)(x - \Lambda_i r_b)|}} \tag{61}$$

$$\times \left(f_{x,\Lambda}(r_a, r_b) + \sum_{j=1}^{p} \frac{f^{(j)}_{x,\Lambda}(r_a, r_b)}{x - \Lambda_j r_a} + \sum_{j \neq j'} \frac{f^{(jj')}_{x,\Lambda}(r_a, r_b)}{(x - \Lambda_j r_a)(x - \Lambda_{j'} r_b)} \right).$$

The outer sum is over the pairs $(l, l') \in \{0, 1, \ldots, p\}^2$ with odd sum $l + l'$, and the $f^{(..)}_{x,\Lambda}(r_a, r_b)$ are certain real-valued analytic functions, which do not need to be written down as they are easy to read off from (56) using (10).

What we do need is some further discussion of the precise meaning of the integral (61). There is no problem with the first summand (containing $f_{x,\Lambda}$), as the square root singularities from (59) are integrable. However, in the second and third summand of (61) the order of the singularity is enhanced from $(x - \Lambda_j r)^{-1/2}$ to $(x - \Lambda_j r)^{-3/2}$ for one (j) or two ($j \neq j'$) factors of the product over $i = 1, \ldots, p$. These terms have to be properly understood as principal-value integrals by their definition as a limit $\varepsilon \to 0$. To render them suitable for numerical evaluation, we use a partial integration identity of the sort

$$\frac{1}{2} \lim_{\varepsilon \to 0} \int_{-\delta_-}^{\delta_+} \frac{g(r)\, dr}{(r - \mathrm{i}\varepsilon)^{3/2}} = -\frac{g(r)}{r^{1/2}} \Bigg|_{r=-\delta_-}^{r=\delta_+} + \int_{-\delta_-}^{\delta_+} \frac{\partial_r g(r)\, dr}{r^{1/2}}, \tag{62}$$

which holds for any smooth function $g(r)$. We apply this identity to the present situation with $r \equiv r_a - \Lambda_j/x$ and/or $r \equiv r_b - \Lambda_{j'}/x$ (and suitable values for $\delta_\pm$), and thereby regularize the integral around the 3/2 singularities. In order for this regularization to be well-defined, we need the diagonal $r_a = r_b$ to lie (as it does) outside the domain of integration, so that our integrand $g(r)$ containing the factor $|r_a - r_b|$ is in fact smooth.

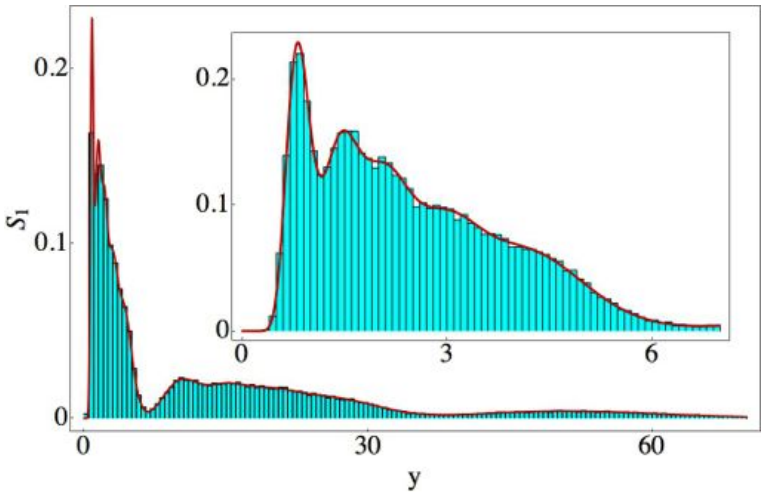

Fig. 1 One-point function $S_1(x)$ for the parameter values $p = 10$ and $n = 50$. The set of $\{\Lambda_j\}_{j=1,\ldots,10}$ is $\{1, 0.49, 0.4225, 0.36, 0.25, 0.09, 0.0729, 0.0529, 0.04, 0.0225\}$. The solid line is the result obtained by numerical integration of the analytical formula (61). The bin size of the histogram is 0.7 for the large figure and 0.1 for the inset. The inset magnifies $S_1(x)$ near the origin.

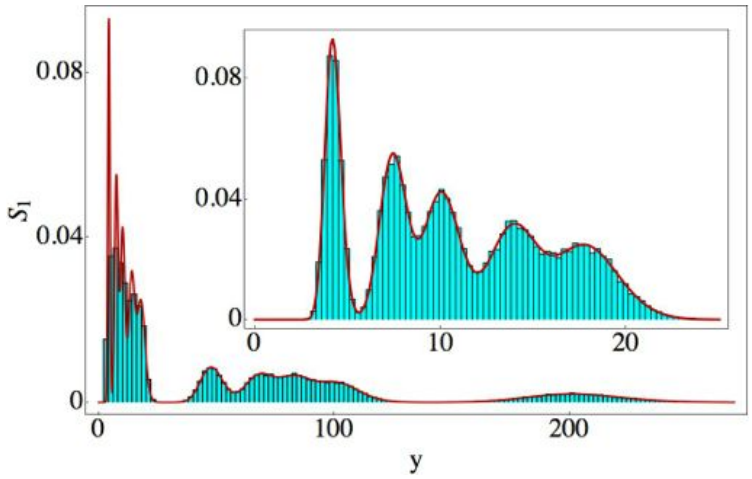

Fig. 2 $S_1(x)$ for the same values of p and $\{\Lambda_j\}$ but a larger parameter $n = 200$. The solid line is again the result obtained from Eq. (61). The bin size is 2 for the large figure and 0.3 for the inset. Note that because of the bigger value of n the peaks are more pronounced (than those in Fig. 1) around the CLT-values $n\Lambda_j$.

By using the commercial software MATHEMATICA® [28] we now numerically evaluate the integrals in Eq. (61). In Figs. 1 and 2 we show examples for $(p, n) = (10, 50)$ and $(10, 200)$, respectively. Our result (solid line) is compared with a Monte-Carlo simulation (histogram) using an ensemble of 10^5 random matrices. We see that the agreement between the Monte-Carlo simulation and our result from (61) is perfect.

We observe that the last peak, centered around $x = 50$ in Fig. 1 and $x = 200$ in Fig. 2, lies near $n\Lambda_p = n$, in agreement with what is expected by the central limit theorem (CLT). The other peaks are slightly shifted from their asymptotic CLT-positions $n\Lambda_j$. With increasing n these shifts become smaller and the peaks become more pronounced, as it should be.

6 Summary

We have developed a supersymmetry approach to derive exact expressions for the one-point function of real and complex Wishart correlation matrices. Supersymmetry got us around a difficult group integral which arises in the traditional approach for the real case. The crucial advantage of the supersymmetry approach is the drastic reduction of the number of integrals. For both cases, complex and real, we showed how to express the one-point function as an eigenvalue integral. In the complex case the supermatrix has dimension 2×2 and thus two eigenvalues. We carried out both integrals and demonstrated the equivalence of our result to that of Ref. [8].

In the more demanding real case ($\beta = 1$), the supermatrix has dimension 4×4 and 3 distinct eigenvalues. While the Efetov-Wegner boundary terms due to diagonalization for $\beta = 1$ have never been given in explicit form,

we showed how to suppress them by a variable substitution that pushes the coordinate singularities outside the domain of integration. One of the three eigenvalue integrals is easily done by residue calculus. We do not know how to calculate the remaining twofold integral by analytical means, and an attempt to compute it numerically met with some complications. We therefore abandoned the coordinate system given by the eigenvalues and turned to a direct approach using standard coordinates. Thus we produced a second formula for $S_1(x)$, still as a twofold integral, which we were able to compute numerically in a stable and efficient way. We also illustrated our result by comparison with a Monte-Carlo simulation.

Previous approaches to the real case had resulted in slowly converging series of Jack or zonal polynomials. We believe that our formula represents a considerable improvement over these previous results. From a conceptual viewpoint, one might say that our result re-sums a multiple infinite series of Jack polynomials (and integrals thereof) in a non-trivial real case.

Thus we hope that we have demonstrated that the supersymmetry method is a powerful tool to tackle problems in multivariate statistics.

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Completing the picture for the smallest eigenvalue of real Wishart matrices

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Rectangular real $N \times (N+\nu)$ matrices W with a Gaussian distribution appear very frequently in data analysis, condensed matter physics and quantum field theory. A central question concerns the correlations encoded in the spectral statistics of WW^T. The extreme eigenvalues of WW^T are of particular interest. We explicitly compute the distribution and the gap probability of the smallest non-zero eigenvalue in this ensemble, both for arbitrary fixed N and ν, and in the universal large N limit with ν fixed. We uncover an integrable Pfaffian structure valid for all even values of $\nu \geq 0$. This extends previous results for odd ν at infinite N and recursive results for finite N and for all ν. Our mathematical results include the computation of expectation values of half integer powers of characteristic polynomials.

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Introduction. To study generic statistical features of spectra, various kinds of random matrices are used. Following Wigner and Dyson [1], Hamiltonians of dynamical systems are modelled by real–symmetric, Hermitian or self-dual matrices in quantum chaos, many-body and mesoscopic physics. Due to universality, cf. [2, 3] and references therein, Gaussian probability densities suffice, leading to the Gaussian Orthogonal, Unitary and Symplectic Ensemble (GOE, GUE, GSE) [4]. This concept was extended to Dirac spectra [5] by imposing chiral symmetry as an additional constraint, resulting in the chiral ensembles chGOE, chGUE, chGSE [6]. Wishart [7] put forward random matrices to model spectra of correlation matrices in a quite different context. There are many applications in time series analysis [8–10] (including chaotic dynamics [11]), in a wide range of fields in physics [2, 3], biology [12], wireless communication [13] and finance [14]. In the most relevant case, $N \times (N+\nu)$ real matrices W model time series such that WW^T is the random correlation matrix. If it fluctuates around a given average correlation matrix C, the distribution reads

$$\mathbb{P}_{N,\nu}(W|C) \sim \exp\left[-\mathrm{Tr} WW^T C^{-1}/2\right] \ . \qquad (1)$$

For $C = \mathbb{1}_N$, this happens to coincide with the chGOE, where W and W^T model the non–zero blocks of the Dirac operator. Closing the circle, one can also extend Wishart's model by using non-Gaussian weights. Here and in the sequel, we focus on Eq. (1) with $C = \mathbb{1}_N$. Since WW^T has positive eigenvalues, the spectrum is bounded from below. Naturally, the distribution of the smallest (non–zero) eigenvalue is of particular importance.

Much interest in the chGOE was sparked by the observation [15] that in the limit $N \to \infty$ its spectral correlators describe the Dirac spectrum in quantum field theories with real Fermions and broken chiral symmetry, see [16] for a review. Based on earlier works for finite N [17, 18], the spectral density [15] and all higher density correlation functions [19] were computed in terms of a Pfaffian determinant of a matrix kernel for all ν. These quantities were shown later to be universal [20] for non-Gaussian potentials, and most recently for fixed trace ensembles in the context of quantum entanglement, see [21] and references therein. Further applications of the chGOE can be found in the recent review [22] on Majorana Fermions and topological superconductors.

In an influential paper [23] the condition number of a Wishart random matrix WW^T was investigated, that is the root of the ratio of the largest over the smallest non-zero eigenvalue of WW^T. This quantity is important for a generic matrix as it quantifies the difficulty of computing its inverse. In [24] the distribution of the smallest eigenvalue was calculated recursively in N for arbitrary rectangular chGOE matrices. Closed expressions were given for quadratic matrices $\nu = 0$ [23] (cf. [25]) and for $\nu = 1, 2, 3$ [24]. Later Pfaffian expressions were found in [26] for arbitrary odd ν valid for fixed and asymptotically large N. A more general consideration, including correlations with $C \neq \mathbb{1}_N$, of the smallest eigenvalue for ν odd was given in [27]. The limiting distributions of the k-th smallest eigenvalue were computed in [28], again for ν odd. These quantities are an efficient tool to test algorithms with exact chiral symmetry in lattice gauge theories [29], distinguishing clearly between different topologies labelled by ν. In [30] the distributions for higher even $\nu > 0$ were obtained from numerical chGOE simulations. Most recently efficient numerical algorithms have been applied, see e.g. [31], in order to compute smallest eigenvalue distributions for arbitrary ν using known analytic Fredholm determinant expressions [32].

It is our goal to complete the picture for the smallest chGOE eigenvalue distribution and its integral by finding explicit Pfaffian expressions for finite and infinite N valid for all even ν. Together with previous results this completes the integrability of this classical ensemble. A presentation with further results and more mathematical details will be given elsewhere [33].

Smallest eigenvalue and gap probability. First we define the quantities of interest and state the problem. In the analytic calculations below we set $C = \mathbb{1}_N$ in Eq. (1), and later we compare our universal large N results to numerical simulations with $C \neq \mathbb{1}_N$. Because we are only interested in correlations of the positive eigenvalues of $WW^T = OXO^T$ contained in $X = \mathrm{diag}(x_1, \ldots, x_N)$, we drop all normalisation constants depending on the orthogonal matrix O. Integrating the distribution (1) over all independent matrix elements with respect to flat Lebesgue measure we obtain the partition function expressed in terms of the eigenvalues as

$$\mathcal{Z}_{N,\gamma} = \prod_{i=1}^{N} \int_0^\infty dx_i w_\gamma(x_i) \; |\Delta_N(X)| \; , \tag{2}$$

up to a known constant. Here we introduce the weight function $w_\gamma(x)$ and Vandermonde determinant $\Delta_N(X)$ stemming from the Jacobian of the diagonalisation,

$$w_\gamma(x) \equiv x^\gamma \exp[-x/2] \; , \; \gamma \equiv (\nu - 1)/2 \; , \tag{3}$$

$$\Delta_N(X) \equiv \prod_{1 \leq i < j \leq N} (x_j - x_i) = \det_{1 \leq i,j \leq N} \left[x_i^{j-1} \right] . \tag{4}$$

We note that γ alternates between integer and half-integer values. The expectation value of an observable f only depending on X is defined as

$$\langle f(X) \rangle_{N,\gamma} \equiv \frac{\prod_{i=1}^N \int_0^\infty dx_i w_\gamma(x_i) \; f(X) |\Delta_N(X)|}{\mathcal{Z}_{N,\gamma}} . \tag{5}$$

Thus the gap probability that no eigenvalue occupies the interval $[0, t]$ is given by

$$\begin{aligned} E_{N,\gamma}(t) &\equiv \frac{1}{\mathcal{Z}_{N,\gamma}} \prod_{i=1}^{N} \int_t^\infty dx_i w_\gamma(x_i) \; |\Delta_N(X)| \\ &= \mathrm{e}^{-Nt/2} \frac{\mathcal{Z}_{N,0}}{\mathcal{Z}_{N,\gamma}} \left\langle \det{}^\gamma [X + t\mathbb{1}_N] \right\rangle_{N,0} \; . \end{aligned} \tag{6}$$

It is expressed as an expectation value of a characteristic polynomial to the power γ with respect to the weight function (3) without the pre-exponential factor, $w_0(x)$. This crucial identity follows from the translation invariance of the Vandermonde determinant (4).

The normalised distribution of the smallest non-zero eigenvalue, $P_{N,\gamma}(t)$, is obtained by differentiating Eq. (6)

$$\begin{aligned} P_{N,\gamma}(t) &\equiv -\frac{\partial E_{N,\gamma}(t)}{\partial t} \\ &= t^\gamma \mathrm{e}^{-Nt/2} \frac{N \mathcal{Z}_{N-1,1}}{\mathcal{Z}_{N,\gamma}} \left\langle \det{}^\gamma [X + t\mathbb{1}_{N-1}] \right\rangle_{N-1,1} , \end{aligned} \tag{7}$$

where the second line follows along the same steps as in Eq. (6). This relation is well known [26, 28], with the difficulty to compute the average (also called massive partition function) for γ half-integer, which is our main task.

To compute Eqs. (6) and (7) we need to know the normalising partition functions, which are given for arbitrary real $\nu > -1$ in terms of the Selberg integral, see also [34, 35], and the expectation values. For integer $\gamma = k$ corresponding to odd $\nu = 2k+1$ closed expressions of (7) exist [26], given in terms of Laguerre polynomials skew-orthogonal with respect to the weight (3). Therefore we concentrate on the case $\nu = 2k$ even.

Pfaffian structure and finite N results. To show that the gap probability (6) has a Pfaffian structure when γ is half-integer let us define the following parameter dependent weight function

$$w(x; t) \equiv \exp[-\eta x/2]/\sqrt{x + t} \; . \tag{8}$$

It absorbs the half-integer part in the expectation value (6) when $\nu = 2k$ is even. We set $\eta = 1$ unless otherwise stated. The monic polynomials $R_k(x; t) = x^k + \ldots$ are defined to be skew-orthogonal with respect to the following skew-symmetric scalar product

$$\langle f, g \rangle_t \equiv \int_0^\infty dy \int_0^y dx \, w(x; t) w(y; t) [f(x) g(y) - f(y) g(x)] \tag{9}$$

by satisfying for all $i, j = 0, 1, \ldots$ [36] the conditions

$$\begin{aligned} \langle R_{2j}, R_{2i} \rangle_t &= 0 = \langle R_{2j+1}, R_{2i+1} \rangle_t \\ \langle R_{2j+1}, R_{2i} \rangle_t &= r_j(t) \; \delta_{ij} \; . \end{aligned} \tag{10}$$

Their normalisations $r_j(t)$ depend on t. The partition function $\mathcal{Z}_N(t)$ of this new weight (8) is defined by

$$\mathcal{Z}_N(t) \equiv \prod_{i=1}^{N} \int_0^\infty dx_i w(x_i; t) |\Delta_N(X)| = N! \prod_{i=0}^{\frac{N}{2}-1} r_i(t). \tag{11}$$

The last step holds for N even [4]. Likewise we define expectation values $\langle f(X) \rangle_N^t$, following Eq. (5). Thus for even $\nu = 2k$, $k \in \mathbb{N}$, Eq. (6) reduces to

$$E_{N,k-\frac{1}{2}}(t) = \mathrm{e}^{-Nt/2} \frac{\mathcal{Z}_N(t)}{\mathcal{Z}_{N,k-\frac{1}{2}}} \left\langle \det{}^k [X + t\mathbb{1}_N] \right\rangle_N^t , \tag{12}$$

given in terms of an integer power of a characteristic polynomial. While the skew-orthogonal polynomials with respect to the weight (3) are know in terms of Laguerre polynomials [26], the difficulty here is to determine the t-dependent polynomials and normalisation constants for the non-standard weight (8). They can be computed following the observation [37]

$$R_{2j}(y, t) = \langle \det[X - y\mathbb{1}_{2j}] \rangle_{2j}^t \; , \tag{13}$$

$$\begin{aligned} R_{2j+1}(y, t) &= \langle (y + c + \mathrm{Tr} X) \det[X - y\mathbb{1}_{2j}] \rangle_{2j}^t \\ &= (y + c) R_{2j}(y, t) - 2 \, \frac{\partial}{\partial \eta} R_{2j}(y, t) \Big|_{\eta=1} . \end{aligned} \tag{14}$$

The odd polynomials are obtained by differentiation of the weight (8), generating $\mathrm{Tr} X$ in the average. Note that

the $R_{2j+1}(y,t)$ are not unique [37], we set $c=0$ in the following. The even polynomials (13) can be calculated by mapping them back to a proper matrix integral over an auxiliary $2j\times(2j+1)$ matrix $\overline{W}$ (corresponding to $\gamma=0$),

$$R_{2j}(y,t)=C_{2j}(t)\int d\overline{W}\,\frac{\det[\overline{W}\overline{W}^T-y\mathbf{1}_{2j}]}{\det{}^{\frac{1}{2}}[\overline{W}\overline{W}^T+t\mathbf{1}_{2j}]}\mathrm{e}^{-\frac{\eta}{2}\mathrm{Tr}\overline{W}\overline{W}^T}, \tag{15}$$

cf. [27]. The known normalisation constant $C_{2j}(t)$ follows from the fact that the polynomial is monic. Without giving details Eq. (15) can be computed exactly, representing the determinants by Gaussian integrals over commuting and anti-commuting variables and by using standard bosonisation techniques [38]. We arrive at

$$R^a_{2j}(y,t)=\frac{(2j)!\left(\mathrm{U}_j(t)L^{(a+1)}_{2j-a}(y)-\mathrm{U}'_j(t)L^{(a)}_{2j-a}(y)\right)}{(2j-a)!\mathrm{U}\left(\frac{2j+1}{2},\frac{3}{2},\frac{t}{2}\right)} \tag{16}$$

for the a-th derivatives of the polynomials, $\frac{\partial^a}{\partial y^a}R_j(y,t)\equiv R^a_j(y,t)$, $a=0,1,\ldots$ needed later. Here $\mathrm{U}_j(t)\equiv\mathrm{U}\left(\frac{2j+1}{2},\frac{1}{2},\frac{t}{2}\right)$ denotes the Tricomi confluent hypergeometric function, satisfying $\mathrm{U}'(a,b,t)=-a\mathrm{U}(a+1,b+1,t)$ [39]. The derivative in Eq. (16) acts only on the generalised Laguerre polynomials used in *monic normalisation* $L^{(a)}_j(y)=y^j+\ldots$ They satisfy $\frac{\partial^a}{\partial y^a}L^{(b)}_n(y)=\frac{n!}{(n-a)!}L^{(b+a)}_{n-a}(y)$, where we set $L^{(b)}_n(y)\equiv 0$ for $n<0$. For the odd polynomials we obtain

$$\begin{aligned}R^a_{2j+1}(y,t)&=\left(4j^2+4j+y\right)R^a_{2j}(y,t)+aR^{a-1}_{2j}(y,t)\\&+\frac{(2j)!/(2j-a)!}{\mathrm{U}\left(\frac{2N+1}{2},\frac{3}{2},\frac{t}{2}\right)}\Big\{t\mathrm{U}''_j(t)L^{(a)}_{2j-a}(y)+2\mathrm{U}'_j(t)\\&\times\left[aL^{(a)}_{2j-a}(y)+(2j-a)yL^{(a+1)}_{2j-a-1}(y)+\tfrac{t}{2}L^{(a+1)}_{2j-a}(y)\right]\\&-2\mathrm{U}_j(t)\left[aL^{(a+1)}_{2j-a}(y)+(2j-a)yL^{(a+2)}_{2j-a-1}(y)\right]\Big\},\end{aligned} \tag{17}$$

and the normalisation constants in Eq. (10) read

$$r_j(t)=2(2j)!(2j+1)!\frac{\mathrm{U}\left(\frac{2j+3}{2},\frac{3}{2},\frac{t}{2}\right)}{\mathrm{U}\left(\frac{2j+1}{2},\frac{3}{2},\frac{t}{2}\right)}\,. \tag{18}$$

Following [26] with their Laguerre weight $w_0(x)$ in Eq. (3) replaced by our weight (8), we express the gap probability (12) as a Pfaffian determinant with our kernel consisting of the skew-orthogonal polynomials (16) and (17). In a more general setting averages of characteristic polynomials such as Eq. (12) were considered in Refs. [40, 41] for arbitrary but unspecified weights. For finite even N and $\nu=2k$ with $k=2m$ even we obtain

$$\begin{aligned}E_{N,k-\frac{1}{2}}(t)&=\mathcal{C}_{N,\nu}\sqrt{t}\;\mathrm{e}^{-Nt/2}\mathrm{U}\left(\frac{N+2m+1}{2},\frac{3}{2},\frac{t}{2}\right)\\&\times\mathrm{Pf}\left[\sum_{j=0}^{\frac{N}{2}+m-1}\frac{R^a_{2j+1}(-t,t)\,R^b_{2j}(-t,t)-(a\leftrightarrow b)}{r_j(t)}\right]^{k-1}_{a,b=0}\end{aligned} \tag{19}$$

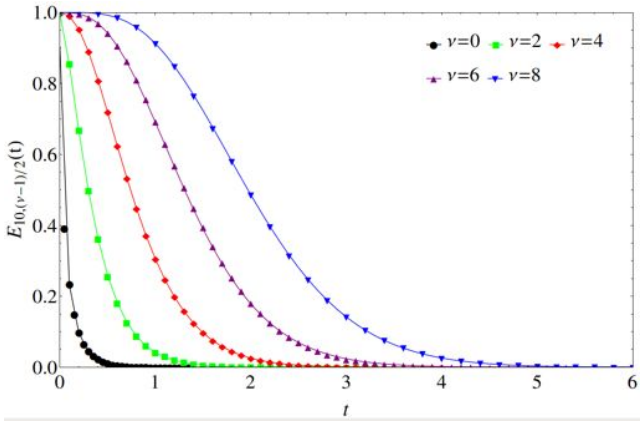

FIG. 1. The gap probability $E_{N,(\nu-1)/2}(t)$ (straight lines) for finite $N=10$ and $\nu=0,2,4,6,8$ (from left to right) vs. numerical simulations (symbols) of 40000 realisations of Wishart matrices, with $C=\mathbf{1}_N$.

For $k=2m-1$ odd the last row (and column) inside the Pfaffian is replaced by $(-)R^{b(a)}_{N+k-2}(-t,t)/r_{N/2+m-1}(t)$, respectively (for N odd cf. [33]). The known t-independent constant $\mathcal{C}_{N,\nu}$ is suppressed for simplicity, it ensures $E_{N,k-\frac{1}{2}}(t=0)=1$.

Eq. (19) is our first main result. A similar answer can be obtained for $P_{N,\gamma}(t)$ for even ν, given in terms of skew-orthogonal polynomials with respect to the weight $x\,w(x;t)$. This provides an explicit integrable Pfaffian structure for both $E_{N,\gamma}(t)$ and $P_{N,\gamma}(t)$. It extends the odd ν result for $P_{N,\gamma}(t)$ in [26] which is given by a Pfaffian determinant as well, but with a different kernel.

For illustration we give two examples. For $\nu=0$ the Pfaffian in Eq. (19) is absent,

$$E_{N,-\frac{1}{2}}(t)=\frac{(N-1)!\sqrt{t}\;\mathrm{e}^{-Nt/2}}{2^{N-1/2}\Gamma(N/2)}\mathrm{U}\left(\frac{N+1}{2},\frac{3}{2},\frac{t}{2}\right), \tag{20}$$

whereas for $\nu=2$ the kernel is absent, and only the polynomial (16) with $a=0$ contributes,

$$\begin{aligned}E_{N,+\frac{1}{2}}(t)&=\frac{\Gamma\left(\frac{N+1}{2}\right)\sqrt{t}\;\mathrm{e}^{-Nt/2}}{(-1)^N\sqrt{2\pi}N!}\\&\times\left[\mathrm{U}_N(t)L^{(1)}_N(-t)-\mathrm{U}'_N(t)L^{(0)}_N(-t)\right].\end{aligned} \tag{21}$$

Eqs. (20) and (21) are compared to numerical simulations in Fig. 1. They can be matched with the finite N results of [24] for $\nu=0,2$, after differentiating them and using identities for the Tricomi function [39].

Microscopic large N limit. We turn to the large N limit keeping ν fixed, referred to as hard edge limit. It is particularly important as the limiting density correlation functions are universal for non-Gaussian weight functions for any integer ν [20]. Because the gap probability can be expressed in terms of the limiting universal kernel [32] (see Eq. (30) for the corresponding density), its universal-

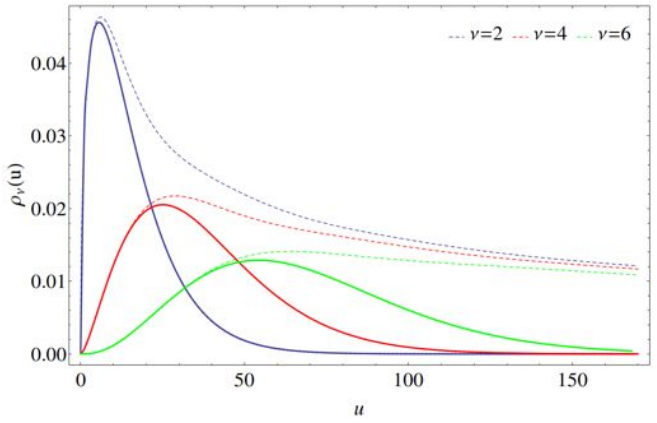

FIG. 2. The microscopic density $\rho(u)$ (30) (dashed lines) vs. the corresponding smallest eigenvalue distribution $\mathcal{P}_{(\nu-1)/2}(u)$ (straight lines) for $\nu = 2, 4, 6$ (from left to right). The smallest eigenvalue nicely follows the density for all ν.

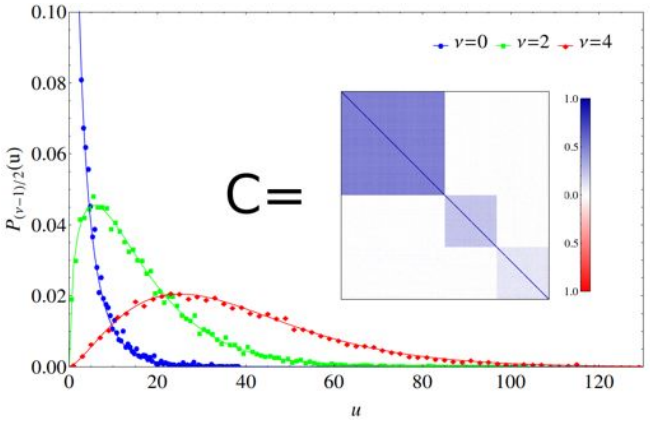

FIG. 3. The microscopic smallest eigenvalue distribution $\mathcal{P}_{(\nu-1)/2}(u)$ (straight lines) for $\nu = 0, 2, 4$ (from left to right) vs. numerical simulations (symbols) of 10000 realisations of matrices with $N = 200$ and correlation matrix $C \neq \mathbf{1}_N$ as indicated in the inset.

ity carries over to the distribution of the smallest eigenvalue. Moreover, in [27] it was shown for *both* ν even and odd, without explicitly calculating the distributions, that the presence of a nontrivial correlation matrix in Eq. (1) does not change the limiting smallest eigenvalue distribution when the spectrum of C has a finite distance to the origin.

The limiting gap probability and smallest eigenvalue distribution are defined as

$$\mathcal{E}_\gamma(u) \equiv \lim_{N\to\infty} E_{N,\gamma}\left(t = \frac{u}{4N}\right), \ \frac{\partial}{\partial u}\mathcal{E}_\gamma(u) = -\mathcal{P}_\gamma(u) \ . \tag{22}$$

In view of Eq. (19) we need the following asymptotic limit of the hypergeometric function,

$$\mathrm{U}\left(aN + c, b, \frac{u}{8N}\right) \approx \frac{2\left(N^2 8a/u\right)^{(b-1)/2}}{\Gamma\left(aN + c\right)} \mathrm{K}_{b-1}\left(\sqrt{\frac{au}{4}}\right). \tag{23}$$

For half integer index the modified Bessel function of second kind simplifies, *e.g.* for $b = 1/2, 3/2, 5/2$

$$K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\ \mathrm{e}^{-z}, \ K_{\frac{3}{2}}(z) = (1 - z^{-1})K_{\frac{1}{2}}(z) \ . \tag{24}$$

Inside the Pfaffian (19) the sum is replaced by an integral, $\sum_j \to \frac{N}{2}\int_0^1 dx$, with $j = Nx/2$. The limiting skew-orthogonal polynomials follow from Eq. (23) together with the standard Laguerre asymptotic in terms of modified Bessel functions of the first kind, see e.g. [39]. This leads to the following limiting kernel inside the Pfaffian (19), independently of N being even or odd,

$$\kappa_{ab}(u) \equiv \int_0^u \frac{dz}{u} z^{(a+b)/2}\left[2(b-a)\mathrm{I}_a(\sqrt{z})\mathrm{I}_b(\sqrt{z})\right. \tag{25}$$
$$\left.+(2b+1)\mathrm{I}_{a+1}(\sqrt{z})\mathrm{I}_b(\sqrt{z}) - (2a+1)\mathrm{I}_{b+1}(\sqrt{z})\mathrm{I}_a(\sqrt{z})\right].$$

The final answer for the limiting gap probability reads

$$\mathcal{E}_{k-1/2}(u) = C_e\ \mathrm{e}^{-\sqrt{u}/2-u/8}\mathrm{Pf}\left[\kappa_{ab}(u)\right]_{a,b=0}^{k-1} \tag{26}$$

for $\nu = 2k$ with $k = 2m$ even and

$$\mathcal{E}_{k-1/2}(u) = C_o\ \mathrm{e}^{-\sqrt{u}/2-u/8} \tag{27}$$
$$\times\mathrm{Pf}\left[\begin{array}{cc} \kappa_{ab}(u) & -u^{a/2}\left[\mathrm{I}_{a+1}\left(\sqrt{u}\right) + \mathrm{I}_a\left(\sqrt{u}\right)\right] \\ u^{b/2}\left[\mathrm{I}_{b+1}\left(\sqrt{u}\right) + \mathrm{I}_b\left(\sqrt{u}\right)\right] & 0 \end{array}\right]_{a,b=0}^{k-1}$$

for $k = 2m-1$ odd. We suppress the known u-independent normalisation constants $C_{e/o}$. The corresponding limiting result for the smallest eigenvalue distribution is

$$\mathcal{P}_{k-1/2}(u) = \widehat{C}_e u^k(1 + 2/\sqrt{u})\,\mathrm{e}^{-\sqrt{u}/2-u/8}\mathrm{Pf}\left[\widehat{\kappa}_{ab}(u)\right]_{a,b=0}^{k-1} \tag{28}$$

for $\nu = 2k$ with $k = 2m$ even, and

$$\mathcal{P}_{k-1/2}(u) = \widehat{C}_o u^k(1 + 2/\sqrt{u})\,\mathrm{e}^{-\sqrt{u}/2-u/8} \tag{29}$$
$$\times\mathrm{Pf}\left[\begin{array}{cc} \widehat{\kappa}_{ab}(u) & -\frac{\mathrm{I}_{a+2}\left(\sqrt{u}\right)+\frac{\sqrt{u}}{2+\sqrt{u}}\mathrm{I}_{a+3}\left(\sqrt{u}\right)}{u^{(a+2)/2}} \\ \frac{\mathrm{I}_{b+2}\left(\sqrt{u}\right)+\frac{\sqrt{u}}{2+\sqrt{u}}\mathrm{I}_{b+3}\left(\sqrt{u}\right)}{u^{(b+2)/2}} & 0 \end{array}\right]_{a,b=0}^{k-1}$$

for $k = 2m-1$ odd, suppressing again the u-independent normalisation constants $\widehat{C}_{e/o}$. Here $\widehat{\kappa}_{ab}(u)$ is the limiting kernel for the skew-orthogonal polynomials with respect to $x\,w(x;t)$ which is of a similar structure as Eq. (25). For $\nu = 0, 2$ the results (28) and (29) were known from [25], [21], respectively.

Eqs. (26) - (29) constitute our second main result and are universal. In Fig. 2 they are compared to the universal microscopic density [15, 42] valid for all ν-values

$$\rho_\nu(u) = \frac{1}{4}\left(J_\nu(\sqrt{u})^2 - J_{\nu-1}(\sqrt{u})J_{\nu+1}(\sqrt{u})\right)$$
$$+\frac{1}{4\sqrt{u}}J_\nu(\sqrt{u})\left(1 - \int_0^{\sqrt{u}} ds J_\nu(s)\right) \ . \tag{30}$$

We further illustrate the universality of our results by comparing to numerical simulations with a nontrivial correlation matrix C for large N, see Fig. 3.

Conclusions and outlook. We have computed closed expressions for the distribution of the smallest non-zero eigenvalue and its integral, the gap probability, for rectangular $N \times (N + \nu)$ real Wishart matrices with ν even, both for finite N and in the universal microscopic large N limit. They only depend on a single kernel instead of three different ones for the density correlation functions and are thus much simpler than these known results. We confirm our findings by numerical simulations even including a nontrivial correlation matrix C. This completes the calculation of all eigenvalue correlation functions in this classical ensemble of random matrices and shows its integrable structure. Furthermore, our finite N results allow to analyse deviations from the universal large N limit, as was very recently proposed in [43] for the chGUE.

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Limiting Statistics of the Largest and Smallest Eigenvalues in the Correlated Wishart Model

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The correlated Wishart model provides a standard tool for the analysis of correlations in a rich variety of systems. Although much is known for complex correlation matrices, the empirically much more important real case still poses substantial challenges. We put forward a new approach, which maps arbitrary statistical quantities, depending on invariants only, to invariant Hermitian matrix models. For completeness we also include the quaternion case and deal with all three cases in a unified way. As an important application, we study the statistics of the largest eigenvalue and its limiting distributions in the correlated Wishart model, because they help to estimate the behavior of large complex systems. We show that even for fully correlated Wishart ensembles, the Tracy-Widom distribution can be the limiting distribution of the largest as well as the smallest eigenvalue, provided that a certain scaling of the empirical eigenvalues holds.

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Time series analysis yields rich information about the dynamics but also about the correlations in numerous systems in physics, climate research, biology, medicine, wireless communication, finance and many other fields [1–12]. Suppose we have a set of p time series $M_j,\ j = 1, \ldots, p$ of n ($n \geq p$) time steps each, which are normalized to zero mean and unit variance. The entries are real, complex, quaternion, *i.e.* $M_j(t) \in \mathbb{R}, \mathbb{C}, \mathbb{H}$, for $\beta = 1, 2, 4$ and $t = 1, \ldots, n$. We arrange the time series as rows into a rectangular data matrix M of size $p \times n$. The empirical correlation matrix of these data,

$$C = \frac{1}{n} M M^{\dagger} \ , \tag{1}$$

with † the Hermitian conjugation, is positive definite and either real symmetric, Hermitian, or Hermitian self-dual for $\beta = 1, 2, 4$ and measures the linear correlations between the time series.

The largest and the smallest eigenvalue of a correlation matrix are highly relevant in many fields. In a simple, interacting dynamical system [13, 14], occurring in physics [15], biology [16], chemistry [17], ecology [18], *etc.*, the cumulative distribution function of the largest eigenvalue estimates the probability to find the system in a stable regime [15]. In high dimensional statistical inference, linear principal component analysis is a method to reduce the dimension of the observations to "significant directions" [19]. Especially, the largest eigenvalue corresponds to the most "significant" component [19–22]. Another example is factor analysis, where the largest eigenvalue can be used to study common properties [19]. The ratio of largest and smallest eigenvalue is important for the statistics of the condition number [23, 24], in numerical analysis including large random matrices. In wireless communication, eigenvalue based detection [25–27] is a promising technique for spectrum sensing in cognitive radio. It utilize the statistics of the ratio of largest and smallest eigenvalue to estimate certain statistical tests [28–30]. The smallest eigenvalue is important for estimates of the *error of a received signal* [31–33] in wireless communication, for estimates in linear discriminant [34] as well as in principal component analysis [4], it is most sensitive to *noise* in the data [4] and crucial for the identification of *single statistical outliers* [5]. In finance, it is related to the *optimal portfolio* [35].

These examples show the considerable theoretical and practical relevance to study the distributions $\mathcal{P}^{(\beta)}_{\max}(t)$, $\mathcal{P}^{(\beta)}_{\min}(s)$ of the largest, respectively, smallest eigenvalue. Both quantities can be traced back to gap probabilities, namely

$$\mathcal{P}^{(\beta)}_{\max}(t) = \frac{\mathrm{d}}{\mathrm{d}t} E^{(\beta)}_p([0,t];p) \ , \tag{2}$$

$$\mathcal{P}^{(\beta)}_{\min}(s) = -\frac{\mathrm{d}}{\mathrm{d}s} E^{(\beta)}_p([0,s];0) \ , \tag{3}$$

where $E^{(\beta)}_p([a,b];m)$ is the probability to find m out of p eigenvalues in the interval $[a,b]$.

This article has three major goals: First, we provide for the first time a framework to map a large class of invariant observables in correlated Wishart ensembles to invariant matrix models. Second, we explicitly apply this framework to the cumulative distribution function (2) of the largest eigenvalue and find an invariant matrix model. Third, we show that for a certain class of C's, p/n fixed and n, p tending to infinity the largest, respectively, smallest eigenvalue are Tracy-Widom distributed.

The ensemble of random Wishart correlation matrices $WW^{\dagger}/n$ [19, 20] consists of $p \times n$ model data matrices $W \in \mathrm{Mat}_{p\times n}(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ for $\beta = 1, 2, 4$, such that upon average $\langle WW^{\dagger}/n \rangle = C$. Data analysis strongly corroborate, see *e.g.* Refs. [3, 7–11, 36], the

Gaussian Wishart model [19, 20],

$$P(W|C) \sim \exp\left(-\frac{\beta}{2}\mathrm{tr}\; WW^\dagger C^{-1}\right) \ . \tag{4}$$

The matrix $WW^\dagger/n$ is known as Wishart correlation matrix. The corresponding measure $\mathrm{d}[W]$ and all other measures $\mathrm{d}[\cdot]$ occurring later on are flat, *i.e.*, the products of the independent differentials. Due to the invariance of $\mathrm{d}[W]$, invariant observables depend on average solely on the distinct, always non–negative eigenvalues Λ_j, $j = 1, \ldots, p$ of C which are referred to as the empirical ones. We arrange them in the diagonal matrix $\hat{\Lambda} = \Lambda \otimes \mathbb{1}_{\gamma_2}$ and introduce $\gamma_2 = 1$ if $\beta = 1, 2$ and $\gamma_2 = 2$ if $\beta = 4$ and for later purpose $\gamma_1 = 2\gamma_2/\beta$, where $\mathbb{1}_N$ is the unity matrix in N dimensions.

We consider an observable $\mathcal{O}(WW^\dagger)$ which is invariant under an arbitrary change of basis and $\mathcal{O}(WW^\dagger) = \mathcal{O}(W^\dagger W)$. This is a very weak assumption when studying the eigenvalue statistics of $WW^\dagger$. We are interested in the average

$$\left\langle \mathcal{O}\left(WW^\dagger\right)\right\rangle = K\int \mathrm{d}[W]\mathcal{O}\left(WW^\dagger\right)P(W|\hat{\Lambda}) \ , \tag{5}$$

where the integration domain is $\mathrm{Mat}_{p\times n}(\mathbb{K})$ and K is a normalization constant. The non-triviality of the integral (5) is due to a group integral of the form

$$\begin{aligned}&\Phi_\beta(X, \hat{\Lambda}^{-1}) = \\ &\int \mathrm{d}\mu(V)\exp\left(-\frac{\beta}{2}\mathrm{tr}V\left(X\otimes\mathbb{1}_{\gamma_2}\right)V^\dagger\hat{\Lambda}^{-1}\right),\end{aligned} \tag{6}$$

where $\mathbb{1}_N$ is N dimensional unit matrix, the integration domain is $\mathrm{O}(p), \mathrm{U}(p)$ or $\mathrm{USp}(2p)$ for $\beta = 1, 2, 4$, respectively, and $X = \mathrm{diag}(x_1, \ldots, x_p)$ are the distinct eigenvalues of $WW^\dagger = V\left(X\otimes\mathbb{1}_{\gamma_2}\right)V^\dagger$. It is known as the orthogonal, unitary or unitary-symplectic Itzykson-Zuber integral [37]. For the unitary case only, it can be computed analytically and is given in a closed form [37–39].

We replace the invariants of $WW^\dagger$ in Eq. (5) by those of the $n \times n$ matrix $W^\dagger W$. Thus, after introducing a δ-function and replacing $W^\dagger W$ by a matrix in the same symmetry class, say Q, we find

$$\begin{aligned}\left\langle \mathcal{O}\left(WW^\dagger\right)\right\rangle = K&\int \mathrm{d}[H,Q]\mathcal{O}\left(Q\right)\exp\left(\imath\mathrm{tr}HQ\right)\\ &\times\int \mathrm{d}[W]\exp\left(-\imath\mathrm{tr}HW^\dagger W\right)P(W|\hat{\Lambda}) \ ,\end{aligned} \tag{7}$$

where the integral of Q and H is over the set of real symmetric, Hermitian, Hermitian self-dual matrices of dimension $n \times n$ for $\beta = 1, 2, 4$, respectively. The H integral is the Fourier representation of the delta function. A detailed mathematical discussion will be given elsewhere [40]. The advantage of this approach is that H couples to $W^\dagger W$ while Λ^{-1} to $WW^\dagger$, see Eq. (4). Hence the integral over H is invariant under $H \to UHU^{-1}$ with U orthogonal, unitary, and unitary symplectic for $\beta = 1, 2, 4$ respectively. The remaining W integral becomes a Gaussian integral over an np-dimensional vector with entries in $\mathbb{K}$, yielding

$$\left\langle \mathcal{O}\left(WW^\dagger\right)\right\rangle = \int \mathrm{d}[H]\frac{\mathcal{F}_n\left[\mathcal{O}\right](H)}{\det{}^{1/\gamma_1}\left(\mathbb{1}_{np\gamma_2} + \imath H\otimes\frac{2}{\beta}\Lambda\right)} \ . \tag{8}$$

In the expression (8), we introduce the Fourier transform of the observable $\mathcal{O}$

$$\mathcal{F}_n\left[\mathcal{O}\right](H) = \frac{1}{(2\pi)^\mu}\int \mathrm{d}[Q]\mathcal{O}\left(Q\right)\exp\left(\imath\mathrm{tr}HQ\right) \tag{9}$$

where $\mu = n(n+1)/2, n^2, n(2n-1)$ is the number of real degrees of freedom of Q for $\beta = 1, 2, 4$, respectively. If we know $\mathcal{F}_n\left[\mathcal{O}\right](H)$, we can express the average (5) as an invariant matrix integral. Thereby we completely outmaneuver the Itzykson-Zuber integral (6).

We exploit this general observation to the statistics of the extreme eigenvalues. The gap probabilities in Eqs. (2) and (3) can be written as an ensemble averaged observable. We carry it out for Eq. (2) only, since for Eq. (3) it works analogously. The joint eigenvalue distribution function derived from Eq. (4) is

$$P(X|\Lambda) = K_{p\times n}\left|\Delta_p(X)\right|^\beta \det{}^\upsilon X\ \Phi_\beta(X, \hat{\Lambda}^{-1}) \ , \tag{10}$$

with normalization constant $K_{p\times n}$, Vandermonde determinant $\Delta_p(X) = \prod_{i<j}(x_j - x_i)$ and $\upsilon = \beta(n - p + 1 - 2/\beta)/2$, see Ref. [41, 42]. As Φ_2 is known, in the complex case the joint probability distribution function provides a representation that can be handled analytically [39]. The highly non–trivial part is the group integral Eq. (6). The gap probability to find all eigenvalues below t can then be cast into the form

$$\begin{aligned}E_p^{(\beta)}([0,t];p) = K_{p\times n}&\int \mathrm{d}[X]\left|\Delta_p(X)\right|^\beta\det{}^\upsilon X\\ &\times\prod_{i=1}^{p}\Theta\left(t\mathbb{1}_p - x_i\right)\ \Phi_\beta(X, \hat{\Lambda}^{-1}) \ ,\end{aligned} \tag{11}$$

where $\Theta(x_i)$ is the Heaviside Θ-function of scalar argument. The Heaviside function of matrix argument is known in terms of an Ingham-Siegel integral, see Ref. [43] and references therein. It is unity if its argument is positive definite and vanishes otherwise. Positive definiteness is an invariant property implying that the Θ-function depends on the eigenvalues a_i of A only,

$$\Theta\left(A\right) = \Theta\left(a\right) = \prod_{i=1}^{N}\Theta\left(a_i\right) \ . \tag{12}$$

Since the integral (11) is over the whole spectrum of $WW^\dagger$, we express the gap probability as averaged Θ-

function,

$$E_p^{(\beta)}([0,t];p) = K_{p\times n} t^{np\beta/2} \int \mathrm{d}[W] P(W|\sqrt{t}\hat{\Lambda}) \times \Theta\left(\mathbb{1}_{\gamma_2 p} - WW^\dagger\right) , \tag{13}$$

where $W \in \mathrm{Mat}_{p\times n}(\mathbb{K})$. Analogously, the gap probability (3) is given by

$$E_p^{(\beta)}([0,s];0) = K_{p\times n} s^{np\beta/2} \int \mathrm{d}[\widehat{W}] P(\widehat{W}|\sqrt{s}\hat{\Lambda}) \times\Theta\left(\widehat{W}\widehat{W}^\dagger - \mathbb{1}_{\gamma_2 p}\right) \det{}^{(n-p)/\gamma_1} \widehat{W}\widehat{W}^\dagger , \tag{14}$$

with $\widehat{W}$ is a square $p \times p$ matrix .

To employ our approach to the gap probability (13), we choose the observable to be $\mathcal{O}(WW^\dagger) = \Theta(\mathbb{1}_{\gamma_2 p} - WW^\dagger)$. The matrix $\mathbb{1}_{\gamma_2 n} - W^\dagger W$ has p distinct eigenvalues that coincide with those of $\mathbb{1}_{\gamma_2 p} - WW^\dagger$ and $n-p$ distinct eigenvalues that are exactly one. Hence, using Eq. (12) it is evident that $\mathcal{O}(WW^\dagger) = \mathcal{O}(W^\dagger W)$. To exchange the Q, H and the W integral, we shift the contour of H by $-\imath\mathbb{1}_{\gamma_2 n}$ and find the inverse Fourier-Laplace transform in Eq. (8)

$$\mathcal{F}_n\left[\Theta\right](H - \imath\mathbb{1}_{\gamma_2 n}) \sim \frac{\exp\left(\mathrm{tr}\left(\imath H + \mathbb{1}_{\gamma_2 n}\right)\right)}{\det{}^{\alpha/\gamma_1}\left(\imath H + \mathbb{1}_{\gamma_2 n}\right)} , \tag{15}$$

where $\alpha = n - 1 + 2/\beta$. If we diagonalize $H = U\left(Y \otimes \mathbb{1}_{\gamma_2}\right) U^\dagger$, where U is in one of the three groups and $Y = \mathrm{diag}(y_1, \ldots, y_n)$ is the matrix of distinct eigenvalues of H, we arrive at a remarkable, new expression for the gap probability (11)

$$E_p^{(\beta)}([0,t];p) = K_{p\times n} \int_{\mathbb{R}^n} \frac{\mathrm{d}[Y]\,|\Delta_n(Y)|^\beta}{\det{}^{\alpha_\beta \beta/2}\left(\imath Y + \mathbb{1}_n\right)} \times \frac{\exp\left(\gamma_2 \mathrm{tr}\left(\imath Y + \mathbb{1}_n\right)\right)}{\prod_{k=1}^p \det{}^{\beta/2}\left(\mathbb{1}_n + \left(\imath Y + \mathbb{1}_n\right) 2\Lambda_k/t\beta\right)} . \tag{16}$$

Likewise, we derive an invariant matrix model for the gap probability (13)

$$E_p^{(\beta)}([0,s];0) = K_{p\times n} \int_{\mathbb{R}^p} \mathrm{d}[Y]\,|\Delta_p(Y)|^\beta \times \frac{\mathcal{F}_p\left[\Theta\left(Q - \mathbb{1}_{\gamma_2 p}\right) \det{}^{(n-p)/\gamma_1} Q\right](Y)}{\prod_{k=1}^p \det{}^{\beta/2}\left(\mathbb{1}_p + \imath 2\Lambda_k Y/s\beta\right)} . \tag{17}$$

The Fourier integral can be done using the differential operator constructed in appendix B of Ref. [44], but the expression becomes cumbersome and we do not need these details for the following discussion.

Both results (16) and (17) have a p-fold product of determinants in the denominator in common. Due to the exponent $\beta/2$, this eigenvalue integral can be studied, at least for $\beta = 2, 4$, using standard techniques of random matrix theory. For $\beta = 1$ standard methods do not apply as square roots of characteristic polynomials appear. We further evaluate the exact expression (16) elsewhere [40]. Here we focus on the limiting behavior which is more relevant in applications. To this end, we assume that the empirical eigenvalues are random variables according to the distribution $\rho_{\mathrm{emp}}(\Lambda)$ and $t = \mu(\Lambda) + \sigma(\Lambda)\chi$, where the centering and scaling parameters $\mu(\Lambda)$ and $\sigma(\Lambda)$ are assumed to be large. We will show that this is a justified assumption.

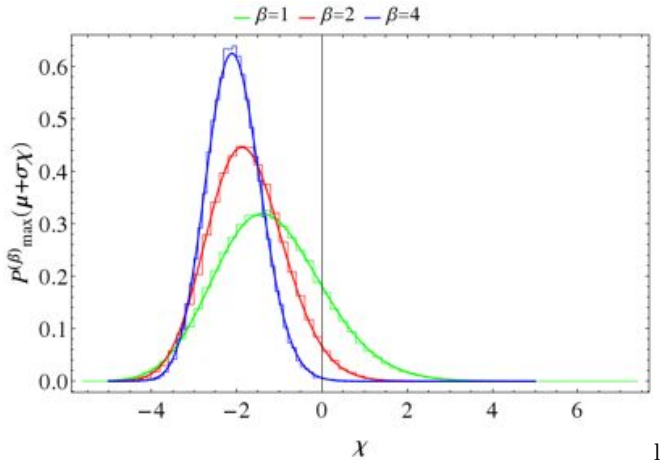

FIG. 1. Comparison of analytic results for the largest eigenvalue distribution (straight lines) with numerical simulations (symbols) for $\beta = 1, 2, 4$. We consider 80000 realizations of 100×300 rectangular matrices, according to the distribution Eq. (4).

For the uncorrelated Wishart ensemble, *i.e.* $\Lambda = \mathbb{1}_p$, previous works [22, 23, 45–58] focus on the exact as well as the limiting distribution of the largest eigenvalue $x_{\max}$ and the smallest eigenvalue $x_{\min}$ of $WW^\dagger$. For n, p tending to infinity, while $p/n = \gamma^2$ is fixed, it was proved that the limiting distribution of $\chi_{\max} = (x_{\max} - \mu_+)\sigma_+^{-1}$ and $\chi_{\min} = (x_{\min} - \mu_-)\sigma_-^{-1}$ is the Tracy-Widom law $f_\beta(\chi)$ [59–61], where

$$\sigma_\pm = \pm\frac{(1 \pm \gamma)^{4/3}}{\gamma^{1/3}} n^{1/3} \quad \text{and} \quad \mu_\pm = (1 \pm \gamma)^2 n \tag{18}$$

$\nu = n - p = (1 - \gamma^2)n$ and γ fixed for $p \to \infty$ [22]. Moreover, if n, p tend to infinity, while $n - p$ is fixed it was shown that the limiting largest eigenvalue distribution is still Tracy-Widom [22, 47, 49].

For the correlated Wishart ensemble, the limiting largest eigenvalue distribution is known for $\beta = 2$ in general [62, 63] and for $\beta = 1, 4$ solely when Λ is a rank one perturbation of the identity matrix [55, 64, 65]. The smallest eigenvalue distribution was already studied in great detail in the microscopic limit, *i.e.* $n, p \to \infty$ while $n - p = \nu$ fixed in Refs. [41, 42], whereas for $n, p \to \infty$ with p/n fixed no results are available yet.

Similar to Refs. [41, 42], we assume that the empiri-

cal eigenvalues Λ_k are of order $\mathcal{O}(1)$ for n, p tending to infinity. It turns out that only the rescaled trace, $\langle\cdot\rangle_{\mathrm{s}} = p^{-1}\mathrm{tr}\,(\cdot)$, of Λ^m, where $m = 1$, does not tend to zero. Moreover, another simple estimate shows $\langle\Lambda^m\rangle_{\mathrm{s}} \sim \mathcal{O}(1)$ such that we cannot determine the exact leading order of the empirical eigenvalue variance $\mathrm{Var}_s(\Lambda)$. Consequently, we impose another requirement on the empirical eigenvalue distribution, namely $\mathrm{Var}_{\mathrm{s}}(\Lambda) \sim \mathcal{O}(1/p^\alpha)$, where $\alpha > 0$ is a free parameter which we fix later. As a consequence of the Tschebyscheff inequality,

$$\mathbb{P}(|\Lambda - \langle\Lambda\rangle_{\mathrm{s}}| \geq x) \leq \frac{\mathrm{Var}_{\mathrm{s}}(\Lambda)}{x^2} \sim \mathcal{O}(1/p^\alpha)\ , \tag{19}$$

we make the following ansatz for the empirical eigenvalues by

$$\Lambda_k = \bar{\Lambda} + p^{-\alpha}\Lambda_k^{(1)}\ , \tag{20}$$

where $\bar{\Lambda} = \langle\Lambda\rangle_{\mathrm{s}}$ and $\Lambda_k^{(1)} \sim \mathcal{O}(1)$. If C is a properly normalized correlation matrix, then $\bar{\Lambda} = \langle\Lambda\rangle_{\mathrm{s}} = 1$.

Substituting Eq. (20) into Eq. (16), and expanding the p-fold product to leading order, under the assumption that t is large, we find for each integration variable y_i

$$\begin{aligned}&\prod_{k=1}^{p}\frac{1}{\left(1+(\imath y_i+1)\,\frac{2}{t\beta}\left(\bar{\Lambda}+p^{-\alpha}\Lambda_k^{(1)}\right)\right)^{\beta/2}}\\ &=\left(1+\frac{t\mathrm{tr}\Lambda^{(1)}}{p^\alpha p\bar{\Lambda}}\frac{\mathrm{d}}{\mathrm{d}t}+\dots\right)\frac{1}{\left(1+(\imath y_i+1)\,\frac{2\bar{\Lambda}}{t\beta}\right)^{p\beta/2}}\ ,\end{aligned} \tag{21}$$

where $i = 1, \dots, n$. The dots correspond to higher powers of $p^{-\alpha}$ times the derivative $t\mathrm{d}/\mathrm{d}t$. If we insert this expansion back into the cumulative distribution function (16) and keep only the leading terms in n, p we find

$$\begin{aligned}E_p^{(\beta)}([0,t];p) = {}& E_p^{(\beta)}([0,t];p)\Big|_{\Lambda=\bar{\Lambda}\mathbb{1}_p}\\ &-\frac{\mathrm{tr}\Lambda^{(1)}}{p^\alpha p\bar{\Lambda}}t\frac{\mathrm{d}}{\mathrm{d}t}\,E_p^{(\beta)}([0,t];p)\Big|_{\Lambda=\bar{\Lambda}\mathbb{1}_p}+\dots\ .\end{aligned} \tag{22}$$

The first term on the right hand side of Eq. (22) is Eq. (11) for an uncorrelated Wishart ensemble with variance $\bar{\Lambda}$. From the discussion above Eq. (18), we conclude that if we center and rescale appropriately, the first term in Eq. (22) converges to the integrated distribution function $F_\beta(\chi) = \int_{-\infty}^{\chi} f_\beta(\chi')\mathrm{d}\chi'$, found by Tracy and Widom. Therefore, we focus our discussion on the second term in Eq. (22).

For the centered and rescaled threshold parameter $t = \mu_+\bar{\Lambda} + \sigma_+\bar{\Lambda}\chi$, we take t times the derivative with respect to t of a function, which in the limit $n, p \to \infty$ and either n/p or $n - p$ fixed converges to $F_\beta(\chi)$. Due to $p^{-1}\mathrm{tr}\Lambda^{(1)} \to$ const. for $p \to \infty$, the prefactor is of order $\mathcal{O}(p^{-\alpha})$. Solely the rescaling of the derivative $t\mathrm{d}/\mathrm{d}t$ can influence this order. A careful analysis shows if α is chosen such that for $n \to \infty$

$$\frac{1}{p^\alpha}\frac{\mu_+}{\sigma_+} = \gamma^{2\alpha-1}(1+\gamma)^{2/3}n^{2/3-\alpha} \to 0\ , \tag{23}$$

the second term in Eq. (22) goes to zero as well. Thus, we require that $\alpha > 2/3$ so that a macroscopic distance between the largest eigenvalue and the empirical eigenvalues is guaranteed only if the fluctuations of the empirical eigenvalues do not overlap with those of the largest eigenvalue.

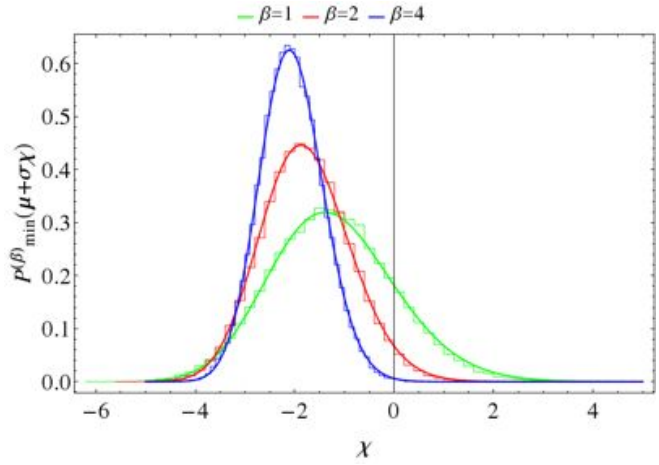

FIG. 2. Comparison of analytic results for the smallest eigenvalue distribution (straight lines) with the same numerical simulations (symbols) as in Fig. 1 for $\beta = 1, 2, 4$.

Like the cumulative distribution function of the largest eigenvalue (16), the dependence of the smallest eigenvalue gap probability (17) on the empirical eigenvalues Λ solely enter in the determinant in the denominator. Hence, we can apply the analysis done for the gap probability corresponding to the largest eigenvalue to that of the smallest one. Eventually, after centering and rescaling the threshold parameter, $s = \tilde{\mu}_-\bar{\Lambda} + \tilde{\sigma}_-\bar{\Lambda}\chi$, where $\tilde{\mu}_-$ and $\tilde{\sigma}_-$ are as in Eq. (18) and assuming the same restrictions on the empirical eigenvalue distribution as above, we obtain that cumulative distribution function is $F_\beta(\chi)$.

To illustrate our findings, we compare our analytical results with Monte Carlo simulations for $\gamma \approx 0.33$. The empirical eigenvalues are random variables with respect to a uniform distribution such that $\mathrm{Var}_s(\Lambda) = p^{-7/4}$, $\langle\Lambda\rangle_s = 1 = \bar{\Lambda}$ and $n^{2/3}\mathrm{Var}_s(\Lambda) \approx 0.013 \ll 1$. The Comparison for the largest and the smallest eigenvalue distribution is shown in Fig. 1 and Fig. 2, respectively. To demonstrate the agreement with the numerical simulations, we properly adjust the centering without changing the limit behavior. For the smallest eigenvalue we even properly adjust the scaling by a constant shift of the order $\mathcal{O}(1/n)$. This is because the smallest eigenvalue always "feels" the presents of a hard wall at zero, whereas the largest eigenvalue does not see any barrier such that the

$1/n$ correction is stronger for the smallest eigenvalue.

In conclusion, we presented a new approach to map observables depending on the eigenvalues of a Wishart matrix only, to an invariant Hermitian matrix model. We demonstrated the concept by applying it to the gap probabilities corresponding to the largest and smallest eigenvalue distributions. Utilizing these invariant matrix model, we showed that for special empirical eigenvalue spectra, the Tracy-Widom distribution persist for the smallest and the largest eigenvalue if n, p tend to infinity while $p/n = \gamma^2$ is fixed. We confirmed our findings by numerical simulations.

A simultaneous but independent study on related issues was very recently put forward in Ref. [66].

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The Smallest Eigenvalue Distribution in the Real Wishart-Laguerre Ensemble with Even Topology

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Abstract. We consider rectangular random matrices of size $p \times n$ belonging to the real Wishart-Laguerre ensemble also known as the chiral Gaussian orthogonal ensemble. This ensemble appears in many applications like QCD, mesoscopic physics, and time series analysis. We are particularly interested in the distribution of the smallest non-zero eigenvalue and the gap probability to find no eigenvalue in an interval $[0, t]$. While for odd topology $\nu = n - p$ explicit closed results are known for finite and infinite matrix size, for even $\nu > 2$ only recursive expressions in p are available. The smallest eigenvalue distribution as well as the gap probability for general even ν is equivalent to expectation values of characteristic polynomials raised to a half-integer power. The computation of such averages is done via a combination of skew-orthogonal polynomials and bosonisation methods. The results are given in terms of Pfaffian determinants both at finite p and in the hard edge scaling limit ($p \to \infty$ and ν fixed) for an arbitrary even topology ν. Numerical simulations for the correlated Wishart ensemble illustrate the universality of our results in this particular limit. These simulations point to a validity of the hard edge scaling limit beyond the invariant case.

keywords: real Wishart matrices, gap probability, smallest eigenvalue distribution, skew-orthogonal polynomials, bosonisation, hard edge scaling limit, characteristic polynomials of half integer power

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1. Introduction

The ensemble of real rectangular $p \times n$ matrices W with independent Gaussian entries is the oldest example of random matrix theory, introduced by Wishart in the context of multivariate statistics [1]. Since then more general ensembles built of complex and quaternionic matrix elements have found a wide area of applications, ranging from physics and mathematics to biology and engineering, e.g. see Refs. [2, 3] for reviews and references. These ensembles got several names and are widely known as Wishart ensembles because of its inventor, Laguerre ensembles because of its relation to the Laguerre polynomials, and chiral Gaussian orthogonal, unitary, or symplectic ensembles hinting to their transformation properties. They are concerned with the singular value statistics of W while in the case of the complex eigenvalue statistics the name Ginibre ensemble is more common.

Despite the fact that the real ensembles are more versatile than their complex and especially their quaternion counterpart, those ensembles are at the same time technically challenging. This is particularly true when correlations among the matrix elements are introduced. The introduction of correlations can be done in three ways, either by a correlation matrix resulting in the correlated Wishart ensemble as it is the case in the analysis of real time series [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], by extending to non-Gaussian probability distributions of the matrix elements [19, 20, 21] or by adding constraints [22, 23, 24].

When analysing the spectral statistics of the positive definite matrix WW^T one has to distinguish between two kinds of correlations. The first kind involves density correlation functions which can be considered on a global or local scale. The second kind comprises correlations only involving individual eigenvalues such as their distribution or spacing. These correlations are by definition local objects since on a large scale individual eigenvalues with a global separation are usually screened and thus uncorrelated. An efficient tool to compute the second kind of correlations are gap probabilities, meaning that a certain interval is void of eigenvalues.

To emphasize the importance of the distribution of individual eigenvalues we summarise a few applications: The condition number of a matrix A is the ratio of the root of the largest over the smallest non-zero eigenvalues of AA^T, which was analysed in a random matrix setting in [25]. The smallest eigenvalues are responsible for chiral symmetry breaking in quantum field theory where real matrices correspond to Quantum Chromodynamics with two colours [26]. Those eigenvalues are very sensitive for fitting lattice QCD data to random matrix results, see [27, 28, 30, 32, 33, 34, 35, 36] and in particular [29, 31] for our symmetry class, for the discussion of the importance of individual eigenvalues in fitting lattice data. In this context, the number of zero-eigenvalues $n - p \equiv \nu \geq 0$ corresponds to the gauge field topology, see [37] for a review on this topic. A further application of these quantities can be found in studying topological insulators, see [38] for a recent review. In multivariate statistics the smallest eigenvalue plays an important role in high dimensional inference [39, 40, 41, 42].

In the uncorrelated Gaussian case all density correlation functions are known most explicitly. For finite p and arbitrary ν the k-point correlation functions of the real Wishart-Laguerre ensemble are given by a Pfaffian determinant of a kernel involving skew-orthogonal polynomials [26, 43]. These are expressed in terms of Laguerre polynomials. In the limit $p \to \infty$ the local kernels are universal and are given by the corresponding Bessel-, sine- or Airy-kernel, for the hard-edge, bulk or

soft-edge scaling limit, see [44] for the corresponding expressions and references. We are particularly interested in the Bessel kernel in the microscopic origin limit (hard edge). The universality of this kernel was shown for non-Gaussian ensembles in [45] and for the some kinds of correlated Wishart ensembles in [14, 18].

Because the individual, e.g. smallest eigenvalue distribution, can be in principle expressed through a Fredholm Pfaffian [46] (see also [27]) of the very same kernel this universality is inherited by the individual eigenvalue distributions. Due to this fact we can restrict ourselves to the Gaussian case. Apart from this relation the universality of the smallest eigenvalue at the soft edge has been proved explicitly [47]. Furthermore, it was shown in [23] that the known expressions for the smallest eigenvalue in the Gaussian ensemble and the one with a fixed trace constraint [22] agree hinting to stronger universal with respect to non-differentiable deformation.

What is know more explicitly about the distribution of the smallest eigenvalue? Closed expressions in the quadratic case $\nu = 0$ were derived in [48] (c.f. [49]). In [25] an exact recursive scheme in the matrix size p was set up, leading to closed results for arbitrary p for $\nu = 0, 1, 2, 3$ only. A distinct structure including both polynomials and Tricomi's confluent hypergeometric functions was observed to hold for ν even and odd, respectively. Exact results for finite p and in the hard-edge scaling limit were derived in [50] for all odd ν fixed and extended to the k-th smallest eigenvalue in [30]. Both results are represented as a Pfaffian determinant, a structure that was unnoticed in the recursion of [25]. Simple approximations following the idea of a Wigner surmise were tested in [51]. An efficient numerical algorithm was used to directly compute the Fredholm Pfaffian expression in [36], and most recently an extension to the correlated Wishart case was presented in [14].

The goal of the present article is the derivation of a Pfaffian representation for the case of even ν at finite and infinite p announced in [52]. Thus we aim at completing the picture of the smallest eigenvalue distribution in the real Wishart-Laguerre ensemble. The idea in [50, 30] that we will use is to represent the smallest eigenvalue and the gap probability as an expectation value of powers of characteristic polynomials. For odd ν half integer powers appear, which is the technical problem we have to solve. The tool we apply are skew-orthogonal polynomials with a non-standard weight function containing a square root. The determination of these polynomials is then based on the method of Grassmann variables and bosonisation [53, 54, 55], a particular case of the supersymmetry method, see [56, 57] and references therein.

The relevance of such expectation values of characteristic polynomials including half integer powers has been advocated independently and solved in a few special cases in [58], motivated mainly from applications to Quantum Chaos. There, the Gaussian orthogonal ensemble is considered. We present results for the chiral Gaussian orthogonal ensemble (chGOE) for characteristic polynomials raised to an arbitrary half-integer power.

The outline of the work is as follows: In section 2, we formulate the problem in terms of expectation values of characteristic polynomials. These can be computed by introducing non-standard polynomials that are skew-orthogonal with respect to a weight function containing a square root, see section 3. When expressed in terms of these polynomials both the gap probability and the smallest eigenvalue distribution exhibit a Pfaffian structure. The building blocks appearing in these expressions, the partition function, the polynomials and their kernel are computed in section 4. Here we also summarise our exact results for arbitrary p and odd ν. In section 5 we take the microscopic origin limit ($p \to \infty$ and ν fixed and even). Furthermore, we illustrate

our findings by numerical simulations, including the correlated Wishart case which follows the same universal predictions, see section 6. Our conclusions and discussion of open problems are presented in section 7.

2. Formulation of the Problem

We consider the singular value statistics close to the origin of the real Wishart-Laguerre ensemble. To this end, we take W to be a rectangular matrix of size $p \times n$ with "rectangularity" $\nu = n-p \geq 0$ and real entries $W_{ij} \in \mathbb{R}$. In QCD the "rectangularity" ν is identified with the index of the Dirac-operator and, thus, with the topological charge of the gauge field configuration, c.f. [37]. The entries of W are drawn from a Gaussian distribution with row-wise correlations [5],

$$P(W|C) \sim \exp\left(-\frac{1}{2}\,\mathrm{tr}\, WW^T C^{-1}\right). \tag{2.1}$$

Sometimes also doubly correlated Wishart random matrices are considered to model spatio-temporal correlations, e.g. see [8, 9, 11, 17]. The singular value statistics of W are completely determined by the eigenvalue statistics of the matrix $WW^\dagger$ which is called the Wishart correlation matrix. The measure on the space of rectangular matrices $\mathrm{d}[W]$ is the flat measure, the product of all independent differentials. In all of our analytic computations we take $C = \mathbf{1}_p$, i.e. we consider the uncorrelated Wishart model. Only in section 6 we argue that generically also the smallest eigenvalue of the correlated model follows the universal distribution derived in section 5.

In order to consider the statistics of the eigenvalues of the Wishart matrix we diagonalize $WW^T = OXO^T$, where $X = \mathrm{diag}(x_1, \ldots, x_p) > 0$ and $O \in \mathrm{O}(p)$ is an orthogonal matrix. This leads to the following normalised joint probability distribution function (jpdf) of the eigenvalues, e.g. see [59, 44]

$$P(X) \equiv \frac{1}{Z_{p,\nu}}\,|\Delta_p(X)| \prod_{i=1}^{p} x_i^{(\nu-1)/2} \exp\left(-x_i/2\right)\ . \tag{2.2}$$

The constant $Z_{p,\nu}$ is the partition function and, hence, the inverse of the normalisation constant. It is a Selberg integral [59] and explicitly reads

$$Z_{p,\nu} \equiv \prod_{i=1}^{p} \int_0^\infty \mathrm{d}x_i x_i^{(\nu-1)/2} e^{-x_i/2}\,|\Delta_p(X)| = 2^{p(p+\nu)/2} \prod_{j=0}^{p-1} \frac{\Gamma\left[(j+3)/2\right]\Gamma\left[(j+\nu+1)/2\right]}{\Gamma\left[3/2\right]} \tag{2.3}$$

for arbitrary $\nu \geq -1$. The term $\Delta_p(X) = \prod_{i>j}^{p}(x_i - x_j)$ denotes the Vandermonde determinant. The variables x_i coincide with the squares of the singular values of W and typically describe the low lying eigenvalues $\pm i\sqrt{x_i}$ of the QCD-Dirac operator [26, 37].

For the smallest non-zero eigenvalues two kinds of large-p limits have to be distinguished. If we take n and p to infinity while $c \equiv p/n$, $c \in (0, 1]$, is kept fixed, the macroscopic density of Marchenko-Pastur detaches from the origin. The level density vanishes with a square root behaviour at both endpoints which are called soft edge. In this situation both, the largest [49, 60] and the smallest eigenvalue, are Tracy-Widom distributed [47].

In contrast to this soft edge scaling we can fix the index $\nu = n - p$ when taking n and p to infinity. Then the macroscopic density behaves as an inverse square root at the

origin, also known as the hard edge. The corresponding scaling is called microscopic origin limit. This limit will be considered in section 5. The microscopic level density [26, 37] and all k-point correlation functions are given by the Bessel-kernel [43] which is universal [45]. In principle all individual eigenvalue distributions including the smallest follow from these density correlations for arbitrary fixed ν. They are given by the Fredholm Pfaffian of the Bessel kernel [46]. In particular the limiting distribution of the smallest eigenvalue, $P_\nu(x)$, follows the microscopic spectral density, $\rho_\nu(x)$, for small x, in particular $P_\nu(x) \approx \rho_\nu(x)$. The level density of the matrix $WW^\dagger$ is [61]

$$\rho_\nu(u) = \frac{1}{4}\left(J_\nu(\sqrt{u})^2 - J_{\nu-1}(\sqrt{u})J_{\nu+1}(\sqrt{u})\right) + \frac{1}{4\sqrt{u}}J_\nu(\sqrt{u})\left(1 - \int_0^{\sqrt{u}} ds J_\nu(s)\right) \tag{2.4}$$

with J_ν the Bessel function of the first kind.

At finite p and ν the distribution of the smallest eigenvalue, $P_{p,\nu}(t)$, can be derived via the gap probability $E_{p,\nu}(t)$. This gap probability is the probability to find no eigenvalue in the interval $[0,t]$. Starting from the jpdf (2.2) we immediately have

$$E_{p,\nu}(t) = \frac{1}{Z_{p,\nu}} \prod_{i=1}^{p} \int_t^\infty \mathrm{d}x_i x_i^{(\nu-1)/2} e^{-x_i/2} \left|\Delta_p(X)\right| \tag{2.5}$$

$$= \frac{1}{Z_{p,\nu}} e^{-pt/2} \prod_{i=1}^{p} \int_0^\infty \mathrm{d}x_i (x_i+t)^{(\nu-1)/2} e^{-x_i/2} \left|\Delta_p(X)\right|. \tag{2.6}$$

In the second line we have shifted $x_i \to x_i + t$ for all $i = 1, \ldots, p$ leaving the Vandermonde determinant invariant.

Let us define the expectation value for any integrable quantity $f(X)$ that only depends on the eigenvalues,

$$\langle f(X)\rangle_{p,\nu} \equiv \frac{1}{Z_{p,\nu}} \prod_{i=1}^{p} \int_0^\infty \mathrm{d}x_i x_i^{(\nu-1)/2} e^{-x_i/2} f(X) \left|\Delta_p(X)\right|. \tag{2.7}$$

Then the gap probability (2.6) can be written as an expectation value of a characteristic polynomial to a certain power with respect to a different partition function with fixed index $\nu = 1$,

$$E_{p,\nu}(t) = e^{-pt/2} \frac{Z_{p,1}}{Z_{p,\nu}} \left\langle \det{}^{(\nu-1)/2}\left(X + t\mathbf{1}_p\right)\right\rangle_{p,1}. \tag{2.8}$$

For $\nu = 2k+1$ odd the determinant is raised to an integer power $\gamma \equiv (\nu-1)/2 = 2k$. Such expectation values have been computed for finite dimension [50] as well as in the microscopic origin limit [30]. For $\nu = 2k$ even the expectation value is taken of a half integer power of a determinant. This average was up to now an open problem and is computed in the present work. The special cases with $\nu = 0$ [49] and $\nu = 2$ [23] were computed in the microscopic limit using a different route, where the latter result is based on the recursive construction of Ref. [25] that only yields closed form expressions for finite p at $\nu = 0$ and $\nu = 2$ even.

Next we turn to the distribution of the smallest eigenvalue. This distribution follows from the gap probability by setting the smallest eigenvalue equal to t, say $x_1 = t$. Hence it is the first derivative in t of the gap probability (2.5),

$$P_{p,\nu}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}E_{p,\nu}(t) \quad \Leftrightarrow \quad E_{p,\nu}(t) = 1 - \int_0^t \mathrm{d}t' P_{p,\nu}(t'), \tag{2.9}$$

where the second relation fixes the normalisation. Rather than first computing $E_{p,\nu}(t)$ and then differentiating with respect to t we can directly start with $P_{p,\nu}(t)$,

$$P_{p,\nu}(t) = \frac{p}{Z_{p,\nu}}\, t^{(\nu-1)/2} e^{-t/2} \prod_{i=2}^{p} \int_t^\infty \mathrm{d}x_i x_i^{(\nu-1)/2} e^{-x_i/2} (x_i - t) |\Delta_{p-1}(x_{j\geq 2})| \tag{2.10}$$

$$= \frac{p}{Z_{p,\nu}}\, t^{(\nu-1)/2} e^{-pt/2} \prod_{i=2}^{p} \int_0^\infty \mathrm{d}x_i x_i (x_i + t)^{(\nu-1)/2} e^{-x_i/2} |\Delta_{p-1}(x_{j\geq 2})|, \tag{2.11}$$

where $\Delta_{p-1}(x_{j\geq 2})$ does not contain x_1. We recall that the smallest eigenvalue is chosen as $x_1 = t$. Therefore the absolute value of these terms can be dropped. In the second line we have again shifted $x_i \to x_i + t$ for all $i = 2, \ldots, p$. Consequently also $P_{p,\nu}(t)$ can be written as an expectation value,

$$P_{p,\nu}(t) = p\, t^{(\nu-1)/2} e^{-pt/2} \frac{Z_{p-1,3}}{Z_{p,\nu}} \left\langle \det{}^{(\nu-1)/2} (X + t\mathbf{1}_{p-1}) \right\rangle_{p-1,3}. \tag{2.12}$$

Here the expectation value is with respect to a partition function of $p-1$ eigenvalues with $\nu = 3$ fixed, accounting for the extra factor x_i in Eq. (2.11).

Once again for $\nu = 2k+1$ the expectation value of determinants to an integer power k is known. Our task reduces to the computation the half integer case $\nu = 2k$. Summarising the problem we consider the following type of expectation values

$$\left\langle \det{}^{-1/2} (X + t\mathbf{1}_p) \prod_{l=1}^{k} \det (X + t_l \mathbf{1}_p) \right\rangle_{p,\nu}. \tag{2.13}$$

Such problems were advocated independently in Ref. [58] for the Gaussian orthogonal ensemble also including more than one square root in the denominator. While special cases have been computed in Ref. [58], having different applications in mind, we determine Eq. (2.13), leading us to the gap probability and the smallest eigenvalue distribution at finite matrix dimension and in the microscopic limit.

As a final remark the gap probability (2.5) has been studied in more detail for even rectangularity $\nu = 2k$ in [14], where a dual supermatrix model was found. Although the supermatrix model is invariant under the action of a particular supergroup, it was not solved in [14], because of non-trivial subtleties related to the necessary diagonalisation. We circumvent these subtleties by combining the method of Grassmann variables with the theory of orthogonal polynomials.

3. Pfaffian Structure and Non-Standard Skew-Orthogonal Polynomials

We tackle the expectation value (2.13) by including the unwanted inverse half-integer power of the characteristic polynomial into the weight function. The remaining integer powers can be expressed in terms of skew orthogonal polynomials (SOP) using standard techniques. The difficulty is thus shifted into finding the SOP with respect to the t-dependent weight

$$w_\gamma(x,t) = \frac{x^\gamma}{\sqrt{x+t}} e^{-\eta x/2}. \tag{3.1}$$

The exponent γ is equal to $\gamma = 0$ for the gap probability and to $\gamma = 1$ for the smallest eigenvalue. Thus we calculate the SOP in a unifying way for both quantities. The

auxiliary parameter η is set to unity unless otherwise stated. It is needed to generate the polynomials of an odd order.

We seek monic, parameter dependent polynomials $R_j^{(\gamma)}(y,t) = y^j + \dots$ that are skew-orthogonal with respect to the anti-symmetric product

$$\begin{aligned}\langle f, g\rangle_t &= \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x \frac{y-x}{|y-x|} w_\gamma(x,t) w_\gamma(y,t) f(x) g(y) \\ &= \int_0^\infty \mathrm{d}y \int_0^y \mathrm{d}x w_\gamma(x,t) w_\gamma(y,t) \left(f(x)g(y) - f(y)g(x)\right) \end{aligned} \tag{3.2}$$

for two arbitrary integrable functions f and g. In particular the polynomials have to fulfill the relations

$$\left\langle R_{2j+1}^{(\gamma)}, R_{2i}^{(\gamma)} \right\rangle_t = r_j^{(\gamma)}(t)\delta_{ij} \ , \quad \left\langle R_{2j}^{(\gamma)}, R_{2i}^{(\gamma)} \right\rangle_t = \left\langle R_{2j+1}^{(\gamma)}, R_{2i+1}^{(\gamma)} \right\rangle_t = 0 \tag{3.3}$$

for $p = 2L$ even. The parameter dependent constants $r_i^{(\gamma)}(t)$ are their normalization constants. In the case $p = 2L+1$ odd the polynomials have to satisfy an additional condition [59]

$$\begin{aligned}&\left\langle \widehat{R}_{2j+1}^{(\gamma)}, \widehat{R}_{2i}^{(\gamma)} \right\rangle_t = r_j^{(\gamma)}(t)\delta_{ij} \ , \quad \left\langle \widehat{R}_{2j}^{(\gamma)}, \widehat{R}_{2i}^{(\gamma)} \right\rangle_t = \left\langle \widehat{R}_{2j+1}^{(\gamma)}, \widehat{R}_{2i+1}^{(\gamma)} \right\rangle_t = 0, \\ &\int_0^\infty \mathrm{d}x \widehat{R}_i^{(\gamma)}(x,t) w_\gamma(t;x) = \delta_{i,2K}, \end{aligned} \tag{3.4}$$

which fixes the normalization of the polynomial of highest order $\widehat{R}_{2K}^{(\gamma)}(x,t)$ with $K \geq L$. Note that the normalization constants $r_j^{(\gamma)}(t)$ are the same as for the case $p = 2L$. The reason is the following relation between the two different kinds of the polynomials

$$\widehat{R}_j^{(\gamma)}(y,t) = R_j^{(\gamma)}(y,t) - \frac{\int_0^\infty \mathrm{d}x w_\gamma(x,t) R_j^{(\gamma)}(x,t)}{\int_0^\infty \mathrm{d}x w_\gamma(x,t) R_{2K}^{(\gamma)}(x,t)} R_{2K}^{(\gamma)}(y,t) \tag{3.5}$$

for $j < 2K$ and $\widehat{R}_{2K}^{(\gamma)}(y,t) = R_{2K}^{(\gamma)}(y,t) / \int_0^\infty \mathrm{d}x w_\gamma(x,t) R_{2K}^{(\gamma)}(x,t)$. Therefore we concentrate on the polynomials $R_j^{(\gamma)}(y,t)$ only. We emphasize that the additional condition $\int_0^\infty \mathrm{d}x \widehat{R}_i^{(\gamma)}(x,t) w_\gamma(t;x) = \delta_{i,2K}$ can be also replaced by other conditions. For example in the framework of [62] the condition would read $\int_0^\infty \mathrm{d}x \widehat{R}_i^{(\gamma)}(x,t) w_\gamma(t;x) = \delta_{i,0}$ for all $i > 0$ which has other advantages in the calculation.

We define a new partition function

$$\begin{aligned} Z_{p,\gamma}(t) &\equiv \prod_{i=1}^p \int_0^\infty \mathrm{d}x_i w_\gamma(x_i;t) \left|\Delta_p(X)\right| \\ &= p! \prod_{j=0}^{\lfloor p/2 \rfloor - 1} r_j^{(\gamma)}(t) \begin{cases} \displaystyle\int_0^\infty \mathrm{d}x w_\gamma(x,t) R_{2L}^{(\gamma)}(x,t), & p = 2L+1, \\ 1, & p = 2L \end{cases} \\ &= Z_{p,2\gamma+1} \left\langle \det{}^{-1/2}(X + t\mathbf{1}_p) \right\rangle_{p,2\gamma+1}, \end{aligned} \tag{3.6}$$

where the second line is a general relation between the normalization constants of the SOP and the partition function [59]. The floor function $\lfloor p/2 \rfloor$ yields the largest integer

smaller than or equal to $p/2$. Employing this partition function we introduce a new parameter dependent expectation value of an integrable observable f,

$$\langle f(X)\rangle_{p,\gamma}^{t} \equiv \frac{1}{Z_{p,\gamma}(t)} \prod_{i=1}^{p} \int_0^\infty \mathrm{d}x_i w_\gamma(x_i;t)\, f(X)\, |\Delta_p(X)|\,. \tag{3.7}$$

The parametric dependence on t is indicated through the superscript.

In this framework the gap probability (2.8) and the distribution of the smallest eigenvalue (2.12) (both for $\nu = 2k$) read

$$E_{p,2k}(t) = e^{-pt/2} \frac{Z_{p,0}(t)}{Z_{p,2k}} \left\langle \det{}^k \left(X + t\mathbf{1}_p\right)\right\rangle_{p,0}^{t}, \tag{3.8}$$

$$P_{p,2k}(t) = p\, t^{(2k-1)/2} e^{-pt/2} \frac{Z_{p-1,1}(t)}{Z_{p,2k}} \left\langle \det{}^k \left(X + t\mathbf{1}_{p-1}\right)\right\rangle_{p-1,1}^{t}. \tag{3.9}$$

It is worth emphasizing that now only integer powers appear in both expressions. Thus, we can apply the results from the literature for general weight functions, c.f. [50, 63, 64]. For $k = 2m$ even, we obtain the following Pfaffian expression with a $2m \times 2m$ dimensional kernel, $\mathcal{K}_{p+k}(\kappa_a, \kappa_b)$,

$$\left\langle \prod_{a=1}^{k} \det(X - \kappa_a \mathbf{1}_p)\right\rangle_{p,\gamma}^{t} = \frac{p! Z_{p+k,\gamma}(t)}{(p+k)! Z_{p,\gamma}(t)} \frac{1}{\Delta_k(\kappa)} \mathrm{pf}_{1\le a,b\le k}\left[\mathcal{K}_{p+k}(\kappa_a,\kappa_b,t)\right], \tag{3.10}$$

while for $k = 2m+1$ we have

$$\begin{aligned}\left\langle \prod_{a=1}^{k} \det(X - \kappa_a \mathbf{1}_p)\right\rangle_{p,\gamma}^{t} &= \frac{p! Z_{p+k+1,\gamma}(t)}{(p+k+1)! Z_{p,\gamma}(t)} \frac{1}{\Delta_k(\kappa)} \\ &\quad \times \mathrm{pf}_{1\le a,b\le k}\begin{bmatrix} \mathcal{K}_{p+k+1}(\kappa_a,\kappa_b) & \mathcal{F}_{p+k+1}(\kappa_a,t) \\ -\mathcal{F}_{p+k+1}(\kappa_a,t) & 0 \end{bmatrix}.\end{aligned} \tag{3.11}$$

The kernels inside the Pfaffians are given by

$$\mathcal{K}_l(\kappa_a,\kappa_b,t) = \begin{cases} \displaystyle\sum_{j=0}^{(l-2)/2} \frac{R_{2j+1}^{(\gamma)}(\kappa_a,t)\, R_{2j}^{(\gamma)}(\kappa_b,t) - R_{2j+1}^{(\gamma)}(\kappa_b,t)\, R_{2j}^{(\gamma)}(\kappa_a,t)}{r_j^{(\gamma)}(t)}, \\ \qquad\qquad\qquad\qquad l \in 2\mathbb{N}, \\ \displaystyle\sum_{j=0}^{(l-3)/2} \frac{\widehat{R}_{2j+1}^{(\gamma)}(\kappa_a,t)\, \widehat{R}_{2j}^{(\gamma)}(\kappa_b,t) - \widehat{R}_{2j+1}^{(\gamma)}(\kappa_b,t)\, \widehat{R}_{2j}^{(\gamma)}(\kappa_a,t)}{r_j^{(\gamma)}(t)}, \\ \qquad\qquad\qquad\qquad l \in 2\mathbb{N}+1. \end{cases} \tag{3.12}$$

The case of $k = 2m+1$ odd is obtained here from the case $k = 2m+2$ even by introducing an additional determinant in the average depending on the dummy variable κ_{2m+2}. This variable is sent to infinity such that the additional row and column in Eq. (3.11) reads

$$\mathcal{F}_l(\kappa_a,t) = -\lim_{\kappa_{2m+2}\to\infty} \frac{\mathcal{K}_l(\kappa_a,\kappa_{2m+2},t)}{\kappa_{2m+2}^{l-1}}. \tag{3.13}$$

This limit is independent of l being even or odd.

Once we have determined the SOP and their kernel we have to take the limit, $\kappa_a \to -t$ for all κ_a. This yields derivatives of the polynomials $R_i^{(\gamma)}(\kappa_b,t)$ because

of l'Hospital's rule. Moreover we could also just take part of the $\kappa_a \to -t$ which is needed to calculate the distributions of the second smallest eigenvalue, the third smallest eigenvalue etc., see Ref. [30]. The distributions of the smallest eigenvalue for QCD with dynamical quarks can be found in this way, too, cf. Refs. [30].

In the next section we explicitly compute the SOP. For this computation it is helpful to understand also these polynomials as expectation values [65, 66],

$$\begin{aligned} R_{2j}^{(\gamma)}(y,t) &= \langle \det\left(y\mathbf{1}_{2j} - X\right)\rangle_{2j,\gamma}^{t}, \qquad (3.14)\\ R_{2j}^{(\gamma)}(y,t) &= \langle \det\left(y\mathbf{1}_{2j+1} - X\right)\left(y + c_j(t) + \operatorname{Tr} X\right)\rangle_{2j+1,\gamma}^{t}\\ &= \left(y + c_j'(t) - 2\frac{\partial}{\partial\eta}\right) R_{2j}^{(\gamma)}(y,t)\Big|_{\eta=1}\\ &= \left(y + \widehat{c}_j(t) - 2y\frac{\partial}{\partial y} - 2t\frac{\partial}{\partial t}\right) R_{2j}^{(\gamma)}(y,t). \qquad (3.15) \end{aligned}$$

These two last relations also hold in a much more general framework where $w_\gamma(x,t)w_\gamma(y,t)(y-x)/|y-x|$ is replaced by an arbitrary anti-symmetric two-point weight $g(x,y) = -g(y,x)$ [67].

The odd polynomials are not unique [65, 59] which is reflected in an ambiguous constant $c_j(t)$. This constant can depend on t and the index j but is independent of y. This dependence is the reason why one could absorb the derivative of the normalization into $c_j'(t)$ and rephrase the derivative in η as a derivative in t and y, yielding a new constant in $\widehat{c}_j(t)$. We will stick to the derivative in η here.

The kernel can be directly expressed as an expectation value, too. Rather than computing the individual polynomials (3.14) and (3.15) and performing the sum (3.12) one can consider the average

$$\mathcal{K}_l(\kappa_a,\kappa_b,t) = \frac{l(l-1)Z_{l-2,\gamma}(t)}{Z_{l,\gamma}(t)}(\kappa_a-\kappa_b)\left\langle \det\left(X-\kappa_a\mathbf{1}_{l-2}\right)\det\left(X-\kappa_b\mathbf{1}_{l-2}\right)\right\rangle_{l-2,\gamma}^{t}, \qquad (3.16)$$

e.g see [68] in the hermitian limit or [64] in the general framework of anti-symmetric two-point weights. This representation is also useful when proving that the large-p limit for even and odd p yields the same answer. The additional row and column in eq. (3.11) is then

$$\mathcal{F}_l(\kappa_a,t) = \frac{l(l-1)Z_{l-2,\gamma}(t)}{Z_{l,\gamma}(t)}\left\langle \det\left(X-\kappa_a\mathbf{1}_{l-2}\right)\right\rangle_{l-2,\gamma}^{t}. \qquad (3.17)$$

We emphasize that for l even this function is equal to the polynomial $R_{l-2}^{(\gamma)}(\kappa_a)$, up to a constant.

4. Calculation of the Finite p Results

We start our calculation by considering the expectation value

$$I_l(\kappa) = \left\langle \prod_{a=1}^{k} \det(X-\kappa_a\mathbf{1}_l)\right\rangle_{l,\gamma}^{t}. \qquad (4.1)$$

This quantity is a polynomial in the variables κ_a. The highest power in these variables determines its normalization, $I_l(\kappa) = (-1)^{lk}\kappa_1^l\cdots\kappa_k^l + \ldots$, such that we can omit the

normalization constants in the intermediate steps of our calculation. The overall constant can be fixed at the end of the calculation.

In the first step we rewrite Eq. (4.1) as an integral over a rectangular real matrix $\widehat{W}$ of dimension $l \times (l + 2\gamma + 1)$, where in this and the next subsection $2\gamma + 1$ can be any integer,

$$\begin{aligned} I_l(\kappa) &\propto \int \mathrm{d}[\widehat{W}] \prod_{a=1}^{k} \det(\kappa_a \mathbf{1}_l - \widehat{W}\widehat{W}^T) \frac{\exp\left(-\operatorname{tr} \widehat{W}\widehat{W}^T/2\right)}{\sqrt{\det\left(\widehat{W}\widehat{W}^T + t\mathbf{1}_l\right)}} \\ &\propto \int \mathrm{d}[\widehat{W}] \prod_{a=1}^{k} \det(\kappa_a \mathbf{1}_l - \widehat{W}\widehat{W}^T) \frac{\exp\left(-\operatorname{tr} \widehat{W}\widehat{W}^T/2\right)}{\sqrt{\det\left(\widehat{W}^T\widehat{W} + t\mathbf{1}_{l+2\gamma+1}\right)}}. \end{aligned} \tag{4.2}$$

We emphasize that the normalization constants in each of the steps may dependent on t. In the second line of Eq. (4.2) we replaced $\widehat{W}\widehat{W}^T \to \widehat{W}^T\widehat{W}$ because of the relation between the determinant and the trace, i.e. $\ln\det A = \operatorname{tr}\ln A$, and the invariance of the trace under circular permutations. The switching of the order of $\widehat{W}$ and $\widehat{W}^T$ allows us to avoid the Efetov-Wegner terms appearing in the superspace dual to this average, see for details Refs. [56, 10, 7, 69].

We introduce a real $(l + 2\gamma + 1)$-dimensional vector v to rewrite the single determinant in the denominator as a Gaussian integral. Additionally we express the product of determinants in the numerator as a Gaussian integral over a rectangular matrix V of dimension $l \times 2k$ whose entries are independent Grassmann variables (anti-commuting variables). For an introduction in supersymmetry we refer to Refs. [70, 56, 57] and for the supersymmetry method with general weight to Refs. [71, 53, 54, 69, 55]. Particularly the bosonisation is described in Refs. [53, 54, 55]. The matrix V satisfies the following symmetry under complex conjugation and under Hermitian conjugation,

$$V^* = V \begin{bmatrix} 0 & \mathbf{1}_k \\ -\mathbf{1}_k & 0 \end{bmatrix} \text{ and } (V^\dagger)^\dagger = -V, \text{ respectively.} \tag{4.3}$$

Then, the average reads

$$\begin{aligned} I_l(\kappa) \propto \int \mathrm{d}[\widehat{W}] \int \mathrm{d}[V] \int \mathrm{d}[v] \exp\left(-\frac{1}{2}\left[\operatorname{tr}\widehat{W}\widehat{W}^T + \operatorname{tr}\widehat{W}\widehat{W}^T VV^\dagger + \operatorname{tr}\widehat{W}vv^T\widehat{W}^T\right]\right) \\ \times \exp\left(-\frac{1}{2}\left[\operatorname{tr} V^\dagger V\kappa + tv^T v\right]\right) \end{aligned} \tag{4.4}$$

with $\kappa = \operatorname{diag}(\kappa_1, \ldots, \kappa_k) \otimes \mathbf{1}_2$. Because of the symmetry (4.3) the dyadic matrix $VV^\dagger$ behaves like a real symmetric matrix such that we can integrate over the matrix $\widehat{W}$ without symmetrizing the other terms. This integration yields

$$\begin{aligned} I_l(\kappa) &\propto \int \mathrm{d}[V] \int \mathrm{d}[v] \det{}^{-1/2}\left(\mathbf{1}_l \otimes \mathbf{1}_{l+2\gamma+1} + VV^\dagger \otimes \mathbf{1}_{l+2\gamma+1} + \mathbf{1}_l \otimes vv^T\right) \\ &\quad \times \exp\left(-\frac{1}{2}\left[\operatorname{tr} V^\dagger V\kappa + tv^T v\right]\right) \\ &\propto \int \mathrm{d}[V] \int \mathrm{d}[v] \det{}^{-1/2}\left([1 + v^T v]\mathbf{1}_l + VV^\dagger\right) \\ &\quad \times \det{}^{-(l+2\gamma)/2}\left(\mathbf{1}_l + VV^\dagger\right) \exp\left(-\frac{1}{2}\left[\operatorname{tr} V^\dagger V\kappa + tv^T v\right]\right) \end{aligned}$$

$$\propto \int \mathrm{d}[V] \int \mathrm{d}[v][1+v^Tv]^{-(l+2k)/2}\det{}^{1/2}\left([1+v^Tv]\mathbf{1}_{2k}+V^\dagger V\right)$$
$$\times \det{}^{(l+2\gamma)/2}\left(\mathbf{1}_{2k}+V^\dagger V\right)\exp\left(-\frac{1}{2}\left[\mathrm{tr}\, V^\dagger V\kappa + tv^Tv\right]\right). \tag{4.5}$$

Again we have used the relation between the determinant and the trace and the invariance of the trace under circular permutations. However we have to remind ourselves that anti-commuting variables are involved such that $\mathrm{tr}(VV^\dagger)^m = -\mathrm{tr}(V^\dagger V)^m$ for any $m \in \mathbb{N}$. This explains the change from negative to positive powers of the determinant in $VV^\dagger$.

In the last step we can choose between two approaches, the generalized Hubbard-Stratonovich transformation [71, 69] and the superbosonization formula [53, 54]. Both approaches are equivalent [55]. We choose the superbosonization formula since it directly leads to a compact expression. Since no supermatrices comprising both, bosonic and fermionic, blocks are involved the superbosonization reduces to bosonisation, only. This means that the norm v^Tv is replaced by a positive variable r (the square of the radial part of an ordinary real vector) and the dyadic matrix $V^\dagger V$ is replaced by a self-dual, unitary matrix,

$$U = \begin{bmatrix} 0 & -\mathbf{1}_k \\ \mathbf{1}_k & 0 \end{bmatrix} U^T \begin{bmatrix} 0 & \mathbf{1}_k \\ -\mathbf{1}_k & 0 \end{bmatrix} \in \mathrm{U}(2k), \tag{4.6}$$

because $V^\dagger V$ is itself self-dual. The set of matrices defined via Eq. (4.6) is the circular symplectic ensemble first studied by Dyson [72]. This set is the coset $\mathrm{CSE}(2k) = \mathrm{U}(2k)/\mathrm{USp}(2k)$ and has a uniquely induced Haar measure $\mathrm{d}\mu(U)$ from the unique, normalized Haar measure of the unitary group $\mathrm{U}(2k)$. In particular up to a normalization constant it is given by $\mathrm{d}\mu(U) \propto \mathrm{d}[U]/\det{}^{k-1/2}U$, with $\mathrm{d}[U]$ the product of differentials of all independent matrix entries of U. The superbosonization formula yields

$$I_l(\kappa) = C_l^{-1} \int\limits_{\mathrm{CSE}(2k)} \mathrm{d}\mu(U)\det{}^{-l/2}U\det{}^{(l+2\gamma)/2}\left(\mathbf{1}_{2k}+U\right)\int_0^\infty \mathrm{d}r \frac{r^{(l+2\gamma-1)/2}}{(1+r)^{(l+2k)/2}}$$
$$\times \det{}^{1/2}\left([1+r]\mathbf{1}_{2k}+U\right)\exp\left(-\frac{1}{2}\left[\mathrm{tr}\, U\kappa + t\, r\right]\right), \tag{4.7}$$

with the normalization constant

$$C_l = \int\limits_{\mathrm{CSE}(2k)} \mathrm{d}\mu(U)\det{}^{-l/2}U e^{\mathrm{tr}\, U/2}\int_0^\infty \mathrm{d}r \frac{r^{(l+2\gamma-1)/2}}{(1+r)^{l/2}} e^{-t\, r/2}. \tag{4.8}$$

The constant follows from the asymptotics for $\kappa \to \infty$. The powers of the additional terms $\det{}^{-l/2}U$ and $r^{(l+2\gamma-1)/2}$ only reflect the nature of the variables from where U and r originate. We underline that the half-integer of the determinants do not cause any problems since the matrices are Kramers degenerate. Therefore the determinants of them are exact squares and the square root is taken such that the result is still a polynomial in the matrix entries.

Starting from expression (4.7) we calculate the partition function $Z_{p,\gamma}(t)$ in subsection 4.1, the polynomials $R_j^{(\gamma)}(y,t)$ and the function $\mathcal{F}_l(\kappa_a)$ in subsection 4.2, and the kernel $\mathcal{K}_l(\kappa_a,\kappa_b)$ in subsection 4.3. In subsection 4.4 we collect everything and give explicit expressions for the gap probability and the distribution of the smallest eigenvalue.

4.1. Normalization Constants

The partition function $Z_{p,\gamma}(t)$ is equal to the case $l \to p$ and $k \to 0$ in the integral (4.7). Therefore we have no integral over a circular unitary ensemble and only the integral over r remains, i.e.

$$Z_{p,\gamma}(t) = \frac{Z_{p,2\gamma+1}}{2^{(p+2\gamma+1)/2}\Gamma[(p+2\gamma+1)/2]} t^{\gamma+1/2} \int_0^\infty \mathrm{d}r \frac{r^{(p+2\gamma-1)/2}}{(1+r)^{p/2}} e^{-t\,r/2}. \tag{4.9}$$

The constant is fixed by the asymptotic behaviour $Z_{p,\gamma}(t) = Z_{p,2\gamma+1} t^{-p/2} + o(t^{-p/2})$ for $t \to \infty$. The remaining integral is a Tricomi confluent hypergeometric function [73],

$$\mathrm{U}(a,b,t) = \frac{1}{\Gamma[a]} \int_0^\infty dz z^{a-1}(1+z)^{b-a-1} e^{-tz}, \text{ with } \mathrm{Re}\, a,\ \mathrm{Re}\, t > 0 \text{ and } a,b,t \in \mathbb{C}. \tag{4.10}$$

This hypergeometric function was already found in the work by Edelman [25] and is a crucial ingredient in his recursive formula. The partition function reads

$$\begin{aligned} Z_{p,\gamma}(t) &= 2^{(p-1)(p+2\gamma+1)/2} \left(\prod_{j=0}^{p-1} \frac{\Gamma[(j+3)/2]\Gamma[(j+2\gamma+2)/2]}{\Gamma[3/2]} \right) \\ &\quad \times t^{\gamma+1/2} \mathrm{U}\left(\frac{p+2\gamma+1}{2}, \frac{2\gamma+3}{2}, \frac{t}{2} \right) \\ &= 2^{p(p+2\gamma)/2} \left(\prod_{j=0}^{p-1} \frac{\Gamma[(j+3)/2]\Gamma[(j+2\gamma+2)/2]}{\Gamma[3/2]} \right) \mathrm{U}\left(\frac{p}{2}, \frac{1-2\gamma}{2}, \frac{t}{2} \right). \end{aligned} \tag{4.11}$$

The second equality follows from the Kummer identity of Tricomi's confluent hypergeometric function [74], $\mathrm{U}(a,b,t) = t^{1-b}\mathrm{U}(a+1-b, 2-b, t)$. Combining (2.3) and (3.6) we have thus obtained the first building block for the Paffian structure,

$$\left\langle \det{}^{-1/2}(X + t\mathbf{1}_p) \right\rangle_{p,\nu} = 2^{-p/2} \mathrm{U}\left(\frac{p}{2}, \frac{2-\nu}{2}, \frac{t}{2} \right), \tag{4.12}$$

which is even valid for any $\nu \in \mathbb{N}_0$.

In the particular case of the gap probability the constant is

$$Z_{p,0}(t) = 2^{(p-1)/2} \left(\prod_{j=1}^{p} j! \right) t^{1/2} \mathrm{U}\left(\frac{p+1}{2}, \frac{3}{2}, \frac{t}{2} \right) = 2^{p/2} \left(\prod_{j=1}^{p} j! \right) \mathrm{U}\left(\frac{p}{2}, -\frac{1}{2}, \frac{t}{2} \right). \tag{4.13}$$

We have used the duplication formula of the Gamma function, $\Gamma[z]\Gamma[z+1/2] = 2^{1-2z}\sqrt{\pi}\Gamma[2z]$, to simplify the expression. Also the partition function needed for the distribution of the smallest eigenvalue the partition function takes a simple form,

$$\begin{aligned} Z_{p-1,1}(t) &= 2^{(p-4)/2} \left(\prod_{j=1}^{p} j! \right) t^{3/2} \mathrm{U}\left(\frac{p+2}{2}, \frac{5}{2}, \frac{t}{2} \right) \\ &= 2^{(p-1)/2} \left(\prod_{j=1}^{p} j! \right) \mathrm{U}\left(\frac{p-1}{2}, -\frac{1}{2}, \frac{t}{2} \right). \end{aligned} \tag{4.14}$$

The second expression of Tricomi's confluent hypergeometric function is employed in Edelman's work [25].

The normalizations $r_j^{(\gamma)}(t)$ of the SOP can be deduced by combining Eqs. (3.6) and (4.11),

$$\begin{aligned} r_j^{(\gamma)}(t) &= \frac{Z_{2j+2,\gamma}(t)}{(2j+2)(2j+1)Z_{2j,\gamma}(t)} \\ &= 2(2j)!\Gamma[2j+2\gamma+2]\frac{\mathrm{U}\left(j+\gamma+3/2,\gamma+3/2,t/2\right)}{\mathrm{U}\left(j+\gamma+1/2,\gamma+3/2,t/2\right)} \\ &= 2(2j)!\Gamma[2j+2\gamma+2]\frac{\mathrm{U}\left(j+1,-\gamma+1/2,t/2\right)}{\mathrm{U}\left(j,-\gamma+1/2,t/2\right)}. \end{aligned} \tag{4.15}$$

This term becomes important in the sum in the kernel.

4.2. Skew-Orthogonal Polynomials

First we concentrate on the function $\mathcal{F}_l(\kappa_a,t)$, see Eq. (3.17). For this case we set $l\to l-2$, $k\to 1$, and $\kappa\to\kappa_a\mathbf{1}_2$. The 2×2 dimensional unitary matrix U only consists of one phase $e^{i\varphi}$ on the diagonal due to its self-duality. Thus we have to calculate the double integral

$$\mathcal{F}_l(\kappa_a,t)\propto\int_0^{2\pi}\mathrm{d}\varphi\frac{\left(1+e^{i\varphi}\right)^{l+2\gamma-2}}{e^{i(l-2)\varphi}}e^{-\kappa_a e^{i\varphi}}\int_0^\infty\mathrm{d}r\frac{r^{(l+2\gamma-3)/2}}{(1+r)^{l/2}}e^{-t\,r/2}\left(1+r+e^{i\varphi}\right). \tag{4.16}$$

The term $(\eta+r+e^{i\varphi})$ is the only coupling between the two integrals and yields a sum of two terms, of which each is a product of two functions. The integrals over r are equal to Tricomi confluent hypergeometric functions (4.10) while the integrals over φ are modified Laguerre polynomials in monic normalization,

$$L_a^{(\mu)}(y)=(-1)^a a!\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}e^{-ai\varphi}(1+e^{i\varphi})^{\mu+a}e^{-ye^{i\varphi}}=y^a+\dots \tag{4.17}$$

Then the function appearing in the additional row and column for odd k, cf. Eq. (3.11), is

$$\begin{aligned} \mathcal{F}_l(\kappa_a,t) = (-1)^l\frac{l(l-1)Z_{l-2,\gamma}(t)}{Z_{l,\gamma}(t)}\Bigg(&L_{l-2}^{(2\gamma)}(\kappa_a) \\ &-(l-2)\frac{\mathrm{U}[\gamma+(l-1)/2,\gamma+1/2,t/2]}{\mathrm{U}[\gamma+(l-1)/2,\gamma+3/2,t/2]}L_{l-3}^{(2\gamma+1)}(\kappa_a)\Bigg). \end{aligned} \tag{4.18}$$

The function was normalized via the known expansion to leading order in κ_a

$$\mathcal{F}_l(\kappa_a,t)=(-1)^l l(l-1)Z_{l-2,\gamma}(t)\kappa_a^{l-2}/Z_{l,\gamma}(t)+\dots \tag{4.19}$$

From the expression (4.18) we can readily read off the polynomials of even order ($l\to 2j+2$),

$$\begin{aligned} R_{2j}^{(\gamma)}(y,t) &= L_{2j}^{(2\gamma)}(y)-2j\frac{\mathrm{U}[j+\gamma+1/2,\gamma+1/2,t/2]}{\mathrm{U}[j+\gamma+1/2,\gamma+3/2,t/2]}L_{2j-1}^{(2\gamma+1)}(y) \\ &= \frac{\mathrm{U}[j+\gamma+1/2,\gamma+1/2,t/2]}{\mathrm{U}[j+\gamma+1/2,\gamma+3/2,t/2]}L_{2j}^{(2\gamma+1)}(y) \\ &\quad+\frac{2j+2\gamma+1}{2}\frac{\mathrm{U}[j+\gamma+3/2,\gamma+3/2,t/2]}{\mathrm{U}[j+\gamma+1/2,\gamma+3/2,t/2]}L_{2j}^{(2\gamma)}(y), \end{aligned} \tag{4.20}$$

for any integer $j \geq 1$ and $R_0^{(\gamma)}(y,t) = 1$. The second expression is the one presented in Ref. [52] and can be found by splitting the term $(1 + r + e^{i\varphi})$ in $(1 + e^{i\varphi})$ and r instead off $e^{i\varphi}$ and $(1 + r)$.

The odd polynomials are determined by their relation (3.15) to the polynomials of even order. For this purpose we recall some recurrence relations of the monic Laguerre polynomials and the Tricomi confluent hypergeometric functions,

$$\left(y - 2y\frac{\partial}{\partial y}\right) L_a^{(\mu)}(y) = L_{a+1}^{(\mu)}(y) + (\mu+1)L_a^{(\mu)}(y) - a(\mu+a)L_{a-1}^{(\mu)}(y), \tag{4.21}$$

$$t\frac{\partial}{\partial t}\mathrm{U}\left(a, b, \frac{t}{2}\right) = (1-b)\mathrm{U}\left(a, b, \frac{t}{2}\right) + (b-a-1)\mathrm{U}\left(a, b-1, \frac{t}{2}\right). \tag{4.22}$$

These relations yield

$$\begin{aligned} R_{2j+1}^{(\gamma)}(y,t) = {} & L_{2j+1}^{(2\gamma)}(y) + (2\gamma + 1 + \widehat{c}_j(t))L_{2j}^{(2\gamma)}(y) - 4j(\gamma+j)L_{2j-1}^{(2\gamma)}(y) \\ & + d_j^{(1)}(t)L_{2j}^{(2\gamma+1)}(y) + d_j^{(2)}(t)L_{2j-1}^{(2\gamma+1)}(y) + d_j^{(3)}(t)L_{2j-2}^{(2\gamma+1)}(y) \end{aligned} \tag{4.23}$$

with the coefficients

$$\begin{aligned} d_j^{(1)}(t) = {} & -2j\frac{\mathrm{U}[j+\gamma+1/2, \gamma+1/2, t/2]}{\mathrm{U}[j+\gamma+1/2, \gamma+3/2, t/2]}, \\ d_j^{(2)}(t) = {} & (2\gamma + \widehat{c}_j(t))d_j^{(1)}(t) + \left(d_j^{(1)}(t)\right)^2 - 4j(j+1)\frac{\mathrm{U}[j+\gamma+1/2, \gamma-1/2, t/2]}{\mathrm{U}[j+\gamma+1/2, \gamma+3/2, t/2]}, \\ d_j^{(3)}(t) = {} & -2(2j-1)(\gamma+j)d_j^{(1)}(t). \end{aligned} \tag{4.24}$$

We have already identified part of the terms in $d_j^{(2)}(t)$ and $d_j^{(3)}(t)$ with the coefficient $d_j^{(1)}(t)$.

We underline that the ambiguous constant $c_j(t)$ for the odd polynomials, cf. Eq. (3.15), is not fixed, yet. Thus we are free to choose the coefficient $c_j(t) = -2\gamma - 1$ such that one of the polynomials drops out in (4.23). Then the polynomials of odd order are a linear combination of only five Laguerre polynomials. This simplifies the result presented in Ref. [52]. Nevertheless we emphasize that both results are correct due to the various relations satisfied by the modified Laguerre polynomials and Tricomi's confluent hypergeometric functions, and the ambiguity in the constant $c_j(t)$ which was chosen differently in Ref. [52] compared to the simpler choice here. In the microscopic origin limit performed in section 5 we choose another constant $c_j(t)$ to simplify the asymptotic result.

The polynomials $\widehat{R}_j^{(\gamma)}(y,t)$, needed for the case $p = 2L + 1$ odd, can be obtained from the polynomials $R_j^{(\gamma)}(y,t)$ with the help of the relations (3.5). Therefore the polynomials of even order, $\widehat{R}_{2j}^{(\gamma)}(y,t)$, are a linear combination of four Laguerre polynomials and the polynomials of odd order, $\widehat{R}_{2j+1}^{(\gamma)}(y,t)$, can be expressed as a sum of six Laguerre polynomials with a suitable choice of the constant $c_j(t)$.

4.3. Kernel

The kernel $\mathcal{K}_l(\kappa_a, \kappa_b, t)$ can be first of all understood as a sum over the SOP, see Eq. (3.12). Plugging the results of subsection 4.2 into this sum we are done. However we can also start from the representation (3.16) and take the general result (4.7) for

$l \to l-2$, $k \to 2$, and $\kappa \to \mathrm{diag}(\kappa_a, \kappa_b) \otimes \mathbf{1}_2$. Then we find

$$\frac{\mathcal{K}_l(\kappa_a, \kappa_b, t)}{(\kappa_a - \kappa_b)} \propto \int\limits_{\mathrm{CSE}(4)} \mathrm{d}\mu(U) \mathrm{det}^{-(l-2)/2} U \mathrm{det}^{(l+2\gamma-2)/2} \left(\mathbf{1}_4 + U\right) \tag{4.25}$$
$$\times \int_0^\infty \mathrm{d}r \frac{r^{(l+2\gamma-3)/2}}{(1+r)^{(l+2)/2}} \mathrm{det}^{1/2} \left([1+r]\mathbf{1}_4 + U\right) \exp\left(-\frac{1}{2}\left[\mathrm{tr}\, U\kappa + t\, r\right]\right).$$

To evaluate the integrals we first expand the determinant coupling U and r, i.e.

$$\mathrm{det}^{1/2}\left([1+r]\mathbf{1}_4 + U\right) = (1+r)^2 + (1+r)\,\mathrm{tr}\, U/2 + \mathrm{det}^{1/2} U\ , \tag{4.26}$$

and arrive at

$$\frac{\mathcal{K}_l(\kappa_a, \kappa_b, t)}{(\kappa_a - \kappa_b)} \tag{4.27}$$
$$\propto \left[\int_0^\infty \mathrm{d}r \frac{r^{(l+2\gamma-3)/2} e^{-t\, r/2}}{(1+r)^{(l-2)/2}} - \int_0^\infty \mathrm{d}r \frac{r^{(l+2\gamma-3)/2} e^{-t\, r/2}}{(1+r)^{l/2}} \left(\frac{\partial}{\partial \kappa_a} + \frac{\partial}{\partial \kappa_b}\right)\right]$$
$$\times \int\limits_{\mathrm{CSE}(4)} \mathrm{d}\mu(U) \mathrm{det}^{-(l-2)/2} U \mathrm{det}^{(l+2\gamma-2)/2} \left(\mathbf{1}_4 + U\right) e^{-\,\mathrm{tr}\, U\kappa/2}$$
$$+ \int_0^\infty \mathrm{d}r \frac{r^{(l+2\gamma-3)/2} e^{-t\, r/2}}{(1+r)^{(l+2)/2}} \int\limits_{\mathrm{CSE}(4)} \mathrm{d}\mu(U) \mathrm{det}^{-(l-3)/2} U \mathrm{det}^{(l+2\gamma-2)/2} \left(\mathbf{1}_4 + U\right) e^{-\,\mathrm{tr}\, U\kappa/2}.$$

The derivatives in κ_a and κ_b generate the trace of U.

Next we diagonalize the matrix $U = V \mathrm{diag}(e^{i\varphi_1}, e^{i\varphi_2}) \otimes \mathbf{1}_2 V^\dagger = V \Phi V^\dagger$ with $V \in \mathrm{USp}(4)/\mathrm{USp}^2(2)$. The normalized measure becomes

$$\mathrm{d}\mu(U) = |e^{i\varphi_1} - e^{i\varphi_2}|^4 \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \mathrm{d}\mu_{\mathrm{Haar}}(V)/(6(2\pi)^2) \tag{4.28}$$

with $\mathrm{d}\mu_{\mathrm{Haar}}(V)$ the normalized Haar measure on $\mathrm{USp}(4)/\mathrm{USp}^2(2)$. The integral over V is an Itzykson-Zuber integral which is well-known [75],

$$\int_{\mathrm{USp}(4)/\mathrm{USp}^2(2)} \mathrm{d}\mu_{\mathrm{Haar}}(V) \exp\left[-\frac{1}{2}\,\mathrm{tr}\, V\Phi V^\dagger \kappa\right] \tag{4.29}$$
$$= 6\left[\frac{(\kappa_a - \kappa_b)(e^{i\varphi_1} - e^{i\varphi_2}) + 2}{(\kappa_a - \kappa_b)^3 (e^{i\varphi_1} - e^{i\varphi_2})^3} e^{-\kappa_a e^{i\varphi_1} - \kappa_b e^{i\varphi_2}} + \{\varphi_1 \leftrightarrow \varphi_2\}\right].$$

We plug this result into the integral over U and have for two arbitrary $p, q \in \mathbb{N}$

$$\int\limits_{\mathrm{CSE}(4)} \mathrm{d}\mu(U) \mathrm{det}^{-p/2} U \mathrm{det}^{(p+q)/2} \left(\mathbf{1}_4 + U\right) e^{-\,\mathrm{tr}\, U\kappa/2}$$
$$= \frac{2}{(\kappa_a - \kappa_b)^3} \int_0^{2\pi} \frac{d\varphi_1}{2\pi} \int_0^{2\pi} \frac{d\varphi_2}{2\pi} e^{-i(p+2)(\varphi_1+\varphi_2)} (1 + e^{i\varphi_1})^{p+q} (1 + e^{i\varphi_2})^{p+q}$$
$$\times \left[(\kappa_a - \kappa_b)(e^{i\varphi_1} - e^{i\varphi_2})^2 + 2(e^{i\varphi_1} - e^{i\varphi_2})\right] e^{-\kappa_a e^{i\varphi_1} - \kappa_b e^{i\varphi_2}}$$
$$= \frac{2}{(p+1)!(p+2)!} \frac{1}{(\kappa_a - \kappa_b)^3}$$
$$\times \Big[(\kappa_a - \kappa_b)\Big((p+1) L_p^{(q)}(\kappa_a) L_{p+2}^{(q-2)}(\kappa_b) - 2(p+2) L_{p+1}^{(q-1)}(\kappa_a) L_{p+1}^{(q-1)}(\kappa_b)$$
$$+ (p+1) L_{p+2}^{(q-2)}(\kappa_a) L_p^{(q)}(\kappa_b)\Big) - 2 L_{p+1}^{(q-1)}(\kappa_a) L_{p+2}^{(q-2)}(\kappa_b) + 2 L_{p+2}^{(q-2)}(\kappa_a) L_{p+1}^{(q-1)}(\kappa_b)\Big]$$
$$= \frac{2}{([p+2]!)^2} \left[\frac{1}{\kappa_a - \kappa_b}\left(\frac{\partial}{\partial \kappa_a} - \frac{\partial}{\partial \kappa_b}\right)\right]^2 L_{p+2}^{(q-2)}(\kappa_a) L_{p+2}^{(q-2)}(\kappa_b). \tag{4.30}$$

The second equality results from an expansion of the polynomials in the bracket in the phases $e^{i\varphi_1}$ and $e^{i\varphi_2}$. The third equality is a compact representation by rewriting the polynomial in the brackets as derivatives in κ_a and κ_b. We remark that Eq. (4.30) is normalized to the leading order term $2(\kappa_a\kappa_b)^p/[p!(p+2)!]$.

We combine the intermediate result (4.30), the definition of Tricomi's confluent hypergeometric function (4.10), and the derivative of monic Laguerre polynomials $\partial_y L_a^{(\mu)}(y) = aL_{a-1}^{(\mu+1)}(y)$. Then we find for the kernel

$$\begin{aligned}\mathcal{K}_l(\kappa_a,\kappa_b,t) = \frac{Z_{l-2,\gamma}(t)}{Z_{l,\gamma}(t)}(\kappa_a-\kappa_b)\left[\frac{1}{\kappa_a-\kappa_b}\left(\frac{\partial}{\partial\kappa_a}-\frac{\partial}{\partial\kappa_b}\right)\right]^2 \\ \times\left[1-\frac{\mathrm{U}\left(\gamma+(l-1)/2,\gamma+1/2,t/2\right)}{\mathrm{U}\left(\gamma+(l-1)/2,\gamma+3/2,t/2\right)}\left(\frac{\partial}{\partial\kappa_a}+\frac{\partial}{\partial\kappa_b}\right)\right. \\ \left.+\frac{\mathrm{U}\left(\gamma+(l-1)/2,\gamma-1/2,t/2\right)}{\mathrm{U}\left(\gamma+(l-1)/2,\gamma+3/2,t/2\right)}\frac{\partial^2}{\partial\kappa_a\partial\kappa_b}\right] L_l^{(2\gamma-2)}(\kappa_a)L_l^{(2\gamma-2)}(\kappa_b).\end{aligned} \tag{4.31}$$

Although we have not identified the recurrence relation of the skew-orthogonal polynomials we have been able to derive the Christoffel-Darboux formula corresponding to the sum (3.12). It expresses the whole sum as a finite small number of terms. Each term is a product of two Laguerre polynomials. The total number of these terms is twelve after differentiating and ordering the Laguerre polynomials with respect to there index. This is a fixed number of terms which shows that this expression of the kernel is ideal to study the large l behaviour including the asymptotic behaviour in the bulk and the soft edge scaling limit. Nonetheless we employ the expression as a sum over the skew-orthogonal polynomials, see Eq. (3.12), in the derivation of the gap probability and the distribution of the smallest eigenvalue in the microscopic limit. The reasons are the additional derivatives we have to perform resulting from degeneracy of the variables $\kappa_j \to -t$, see subsection 4.4. Then the result (4.31) becomes quite nasty due to a $1/(\kappa_a-\kappa_b)$ term in the differential operator in front of the product of the two Laguerre polynomials.

4.4. Gap Probability and Distribution of the Smallest Eigenvalue

To obtain the gap probability or the distribution of the smallest eigenvalue itself we have to set all variables equal, $\kappa_1 = \ldots = \kappa_k = -t$. Hence we have to apply l'Hôpital's rule in Eqs. (3.10) and (3.11) yielding

$$\begin{aligned}\left\langle \det^k(X+t\mathbf{1}_p)\right\rangle_{p,\gamma}^t &= \frac{(-1)^{k(k-1)/2}p!Z_{p+k,\gamma}(t)}{(p+k)!Z_{p,\gamma}(t)\prod_{j=0}^{k-1}j!} \\ &\quad\times \mathrm{pf}_{1\le a,b\le k}\left[\partial_{\kappa_1}^{a-1}\partial_{\kappa_2}^{b-1}\mathcal{K}_{p+k}(\kappa_1,\kappa_2,t)|_{\kappa_1=\kappa_2=-t}\right]\end{aligned} \tag{4.32}$$

for $k=2m$ even and

$$\begin{aligned}\left\langle \det^k(X+t\mathbf{1}_p)\right\rangle_{p,\gamma}^t &= \frac{(-1)^{k(k-1)/2}p!Z_{p+k+1,\gamma}(t)}{(p+k+1)!Z_{p,\gamma}(t)\prod_{j=0}^{k-1}j!} \\ \times \mathrm{pf}_{1\le a,b\le k}&\begin{bmatrix}\partial_{\kappa_1}^{a-1}\partial_{\kappa_2}^{b-1}\mathcal{K}_{p+k+1}(\kappa_1,\kappa_2,t)|_{\kappa_1=\kappa_2=-t} & \partial_\kappa^{a-1}\mathcal{F}_{p+k+1}(\kappa,t)|_{\kappa=-t} \\ -\partial_\kappa^{b-1}\mathcal{F}_{p+k+1}(\kappa,t)|_{\kappa=-t} & 0\end{bmatrix}\end{aligned} \tag{4.33}$$

for $k=2m+1$ odd. The additional sign and the product of inverse factorials in front of the Pfaffians result from differentiating the Vandermonde determinant $\Delta_k(\kappa)$, cf. Eqs. (3.10) and (3.11).

The prefactor in front of the averages for the kernels $K_{p+k}(\kappa_1,\kappa_2,t)$ and polynomials $F_{p+k+1}(\kappa,t)$, have to be considered together with the prefactors in Eqs. (3.16) and (3.17). Hence we can normalize the two kernels such that the microscopic origin limit of these kernels is finite in preparation of section 5. We find the result for arbitrary γ,

$$\left\langle \det{}^k(X+t\mathbf{1}_p)\right\rangle^t_{p,\gamma} = C^{(\gamma)}_{pk}(t)\mathrm{pf}_{0\le a,b\le k-1}\left[\Xi^{(\gamma,p+k)}_{ab}(t)\right] \tag{4.34}$$

for $k=2m$ even and

$$\left\langle \det{}^k(X+t\mathbf{1}_p)\right\rangle^t_{p,\gamma} = C^{(\gamma)}_{pk}(t)\mathrm{pf}_{0\le a,b\le k-1}\begin{bmatrix} \Xi^{(\gamma,p+k+1)}_{ab}(t) & \xi^{(\gamma,p+k+1)}_{a}(t) \\ -\xi^{(\gamma,p+k+1)}_{b}(t) & 0 \end{bmatrix} \tag{4.35}$$

for $k=2m+1$ odd. The functions serving as the kernels are

$$\begin{aligned}\Xi^{(\gamma,l)}_{ab}(t) &= \frac{(-1)^{a+b}t^{2\gamma+a+b+1}l(l-1)Z_{l-2,\gamma}(t)}{Z_{l,\gamma}(t)}\partial^a_{\kappa_1}\partial^b_{\kappa_2}(\kappa_1-\kappa_2) \\ &\quad\times\left\langle \det\left(X-\kappa_1\mathbf{1}_{l-2}\right)\det\left(X-\kappa_2\mathbf{1}_{l-2}\right)\right\rangle^t_{l-2,\gamma}|_{\kappa_1=\kappa_2=-t} \\ &= (-1)^{a+b}t^{2\gamma+a+b+1} \\ &\times\begin{cases}\displaystyle\sum_{j=0}^{(l-2)/2}\frac{\partial^a_{\kappa_1}R^{(\gamma)}_{2j+1}(\kappa_1,t)\,\partial^b_{\kappa_2}R^{(\gamma)}_{2j}(\kappa_2,t)-\partial^b_{\kappa_2}R^{(\gamma)}_{2j+1}(\kappa_2,t)\,\partial^a_{\kappa_1}R^{(\gamma)}_{2j}(\kappa_1,t)}{r^{(\gamma)}_j(t)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad l\in 2\mathbb{N}, \\ \displaystyle\sum_{j=0}^{(l-3)/2}\frac{\partial^a_{\kappa_1}\widehat{R}^{(\gamma)}_{2j+1}(\kappa_1,t)\,\partial^b_{\kappa_2}\widehat{R}^{(\gamma)}_{2j}(\kappa_2,t)-\partial^b_{\kappa_2}\widehat{R}^{(\gamma)}_{2j+1}(\kappa_2,t)\,\partial^a_{\kappa_1}\widehat{R}^{(\gamma)}_{2j}(\kappa_1,t)}{r^{(\gamma)}_j(t)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad l\in 2\mathbb{N}+1,\end{cases}\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}\xi^{(\gamma,l)}_a(t) &= \frac{(-1)^a t^{2\gamma+a}}{(l-2)!}\partial^a_\kappa\left\langle\det\left(X-\kappa\mathbf{1}_{l-2}\right)\right\rangle^t_{l-2,\gamma}|_{\kappa=-t} \\ &= (-1)^{a+l}t^{2\gamma+a}\left(\frac{L^{(2\gamma+a)}_{l-a-2}(-t)}{(l-a-2)!}-\frac{\mathrm{U}[\gamma+(l-1)/2,\gamma+1/2,t/2]}{\mathrm{U}[\gamma+(l-1)/2,\gamma+3/2,t/2]}\frac{L^{(2\gamma+a+1)}_{l-a-3}(-t)}{(l-a-3)!}\right).\end{aligned} \tag{4.37}$$

Due to this particular normalization we have the overall constant

$$C^{(\gamma)}_{pk}(t) = \begin{cases}\dfrac{p!}{(p+k)!\prod_{j=0}^{k-1}j!}\dfrac{Z_{p+k,\gamma}(t)}{t^{k(k+2\gamma)/2}Z_{p,\gamma}(t)}, & k\in 2\mathbb{N}_0, \\ \dfrac{p!}{\prod_{j=0}^{k-1}j!}\dfrac{Z_{p+k-1,\gamma}(t)}{t^{k(k+2\gamma)/2+(2\gamma-1)/2}Z_{p,\gamma}(t)}, & k\in 2\mathbb{N}_0+1.\end{cases} \tag{4.38}$$

Note that we have not included the prefactors shown in Eqs. (3.8) and (3.9) for the full expressions of the gap probability and the distribution of the smallest eigenvalue. For the definition of the limiting quantities see Eqs. (5.1) and (5.2) where all factors in p are accounted for.

First we show the explicit expression of the gap probability at finite N. We multiply the results (4.34) and (4.35) for $\gamma=0$ with the factor $e^{-pt/2}Z_{p,0}(t)/Z_{p,2k}$. Then we find the first of our main results,

$$\begin{aligned}E_{p,2k}(t) &= \mathcal{C}^{(0)}_{p,k}(4pt)^{-k^2/2+1/2}e^{-pt/2}\frac{\Gamma[(p+k+1)/2]\mathrm{U}\left((p+k+1)/2,3/2;t/2\right)}{2\sqrt{2p}} \\ &\quad\times\mathrm{pf}_{0\le a,b\le k-1}\left[\Xi^{(0,p+k)}_{ab}(t)\right]\end{aligned} \tag{4.39}$$

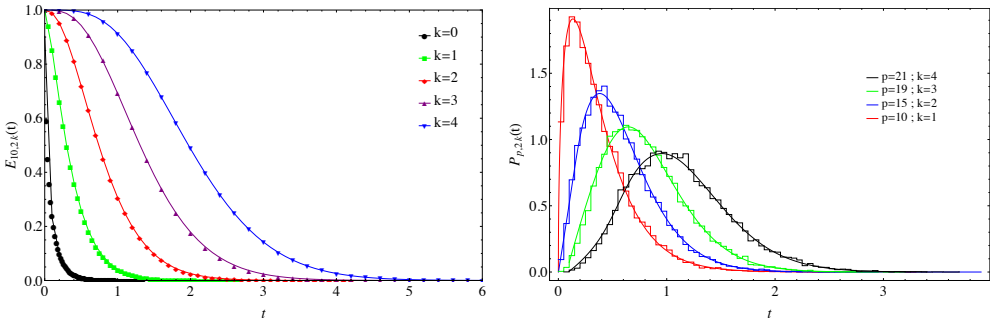

Figure 1. Visualization of our analytical results (4.39), (4.40), (4.42), and (4.43) for finite $p = 10$ (solid curves) compared to Monte Carlo simulations (symbols, histogram) for the gap probability (left plot) and the distribution of the smallest eigenvalue (right plot). For the gap probability we generated 10000 real Wishart matrices of size $10 \times (10 + \nu)$ with an index $\nu = 2k = 0, 2, 4, 6, 8$. For the distribution of the smallest eigenvalue we simulated 20000 random matrices with various dimensions p shown in the inset corresponding to $\nu = 2k = 2, 4, 6, 8$.

for even k and

$$E_{p,2k}(t) = \mathcal{C}_{p,k}^{(0)} (4pt)^{-k^2/2+1} e^{-pt/2} \frac{\Gamma[(p+k)/2]\mathrm{U}\left((p+k)/2, 3/2; t/2\right)}{2\sqrt{2p}} \times \mathrm{pf}_{0\le a,b\le k-1} \begin{bmatrix} \Xi_{ab}^{(0,p+k+1)}(t) & \xi_a^{(0,p+k+1)}(t) \\ -\xi_b^{(0,p+k+1)}(t) & 0 \end{bmatrix} \tag{4.40}$$

for odd k, where we introduce the global normalization constant

$$\mathcal{C}_{p,k}^{(0)} = \prod_{l=0}^{k-1} \frac{4^{l+1}(2l)!\Gamma[p+l+2]p^{l-1}}{l!\Gamma[p+2l+1]} \times \begin{cases} \dfrac{1}{\sqrt{\pi}} \dfrac{2^{-k/2} p! p^{3k/2} \Gamma[(p+1)/2]}{(p+k)!\Gamma[(p+k+1)/2]} & , k \in 2\mathbb{N}, \\ \dfrac{1}{\sqrt{\pi}} \dfrac{2^{-(k+3)/2} p! p^{(3k-1)/2} \Gamma[(p+1)/2]}{(p+k)!\Gamma[(p+k)/2]} & , k \in 2\mathbb{N}+1 \end{cases} \tag{4.41}$$

which is also ideal to take the limit $p \to \infty$.

The explicit expression of the kernels $\Xi_{ab}^{(0,p+k)}(t)$ and $\xi_b^{(0,p+k+1)}(t)$ are not much more enlightening at finite p than the expressions (4.36) and (4.37) for general γ. Therefore we skip their expressions here and show them explicitly for the particular case $\gamma = 0$ and $\gamma = 1$ in the microscopic limit in section 5. For finite p we visualize the gap probability in Fig. 1.

We skip the explicit kernels for the distribution of the smallest eigenvalue, too, and only explicitly show the Pfaffian structure with its normalization constant. For this reason we multiply the results (4.34) and (4.35) for $\gamma = 1$ and $p \to p-1$ with the factor $pt^{(2k-1)/2} e^{-pt/2} Z_{p-1,1}(t)/Z_{p,2k}$ from Eq. (3.9)

$$P_{p,2k}(t)\mathrm{d}t = \mathcal{C}_{p,k}^{(1)} (4pt)^{-k^2/2+1} e^{-pt/2} \frac{\Gamma[(p+k+2)/2]\mathrm{U}\left((p+k+2)/2, 5/2; t/2\right)}{2(2p)^{3/2}} \times \mathrm{pf}_{0\le a,b\le k-1} \left[\Xi_{ab}^{(1,p+k-1)}(t)\right] \mathrm{d}(4pt) \tag{4.42}$$

for even k and

$$P_{p,2k}(t)\mathrm{d}t = \mathcal{C}^{(1)}_{p,k}(4pt)^{-k^2/2+1/2}e^{-pt/2}\frac{\Gamma[(p+k+1)/2]\mathrm{U}\left((p+k+1)/2,5/2;t/2\right)}{2(2p)^{3/2}}$$
$$\times \mathrm{pf}_{0\leq a,b\leq k-1}\begin{bmatrix} \Xi^{(1,p+k)}_{ab}(t) & \xi^{(1,p+k)}_{a}(t) \\ -\xi^{(1,p+k)}_{b}(t) & 0 \end{bmatrix}\mathrm{d}(4pt) \tag{4.43}$$

for odd k. We multiplied the differential $\mathrm{d}t$ to emphasize that this quantity is a density and transforms as a pseudo scalar under changes of coordinates. The constant

$$\mathcal{C}^{(1)}_{p,k} = \prod_{l=0}^{k-1}\frac{4^{l+1}(2l)!\Gamma[p+l+2]p^{l-1}}{l!\Gamma[p+2l+1]}$$
$$\times\begin{cases} \frac{1}{\sqrt{\pi}}\frac{2^{-(k+5)/2}p!p^{(3k-1)/2}\Gamma[(p+1)/2]}{(p+k)!\Gamma[(p+k)/2]} & ,k\in 2\mathbb{N} \\ \frac{1}{\sqrt{\pi}}\frac{2^{-(k+6)/2}p!p^{3k/2}\Gamma[(p+1)/2]}{(p+k)!\Gamma[(p+k+1)/2]} & ,k\in 2\mathbb{N}+1 \end{cases} \tag{4.44}$$

it is t independent and converges to a finite number in the limit $p\to\infty$.

Equations (4.42) and (4.43) are our second main result. The distributions at finite p are visualized in Fig. 1. The two simplest cases for $k=0$,

$$P_{p,0}(t) = \frac{p!}{2^{p-1/2}\Gamma[p/2]}\frac{1}{\sqrt{t}}e^{-pt/2}\mathrm{U}\left(\frac{p-1}{2},-\frac{1}{2},\frac{t}{2}\right), \tag{4.45}$$

and $k=1$

$$P_{p,2}(t) = \frac{\Gamma[(p+1)/2]}{\sqrt{2\pi}}\sqrt{t}e^{-pt/2} \tag{4.46}$$
$$\times\left[\mathrm{U}\left(\frac{p-1}{2},-\frac{1}{2},\frac{t}{2}\right)\frac{(-1)^{p-1}L^{(2)}_{p-1}(-t)}{(p-1)!}+\mathrm{U}\left(\frac{p+1}{2},\frac{1}{2},\frac{t}{2}\right)\frac{(-1)^{p-2}L^{(3)}_{p-2}(-t)}{(p-2)!}\right],$$

obviously agree with the results by Edelman [25]. We underline that he has employed the standard normalization of the Laguerre polynomials while we have chosen the monic normalization. Moreover we have used Kummer's identity of Tricomi's confluent hypergeometric function and the duplication formula of the Gamma function to obtain Eqs. (4.45) and (4.46) from Eqs. (4.42) and (4.43), respectively.

5. Microscopic Origin Limit

In the microscopic origin limit of the gap probability (2.5) we have to perform the limit $p\to\infty$ while keeping $2k=\nu=n-p$ and $t=u/4p$ fixed. Hence we zoom into a region of scale $1/p$ around the origin. In this region chiral random matrix theory is identical with physical theories like QCD below the critical temperature [26, 37] and condensed matter theory of disordered system for particular topological insulators and superconductors [38]. In this regime the limiting gap probability and distribution of the first eigenvalue are defined as follows:

$$\mathcal{E}_\nu(u) := \lim_{p\to\infty}E_{p,\nu}\left(\frac{u}{4p}\right) \tag{5.1}$$

and

$$\mathcal{P}_\nu(u)\mathrm{d}u := \lim_{p\to\infty}P_{p,\nu}\left(\frac{u}{4p}\right)\mathrm{d}\left(\frac{u}{4p}\right). \tag{5.2}$$

We underline that the scaling factor resulting from the differential in the second definition is crucial to obtain a finite limit.

The first question we have to address is the dependence of the limit on the parity of p (if p is even or odd) because the skew-orthogonal polynomials crucially depend on it, cf. Eqs. (3.3) and (3.4). We can circumvent this problem by the fact that all important quantities can be written in terms of averages of characteristic polynomials to some powers, in particular the normalization (3.6) and the kernels (3.16) and (3.17). Therefore it is sufficient to show that the limit of the average (4.1) is independent of the parity of p. For this reason we consider the intermediate result (4.7). The microscopic limit can be readily performed by rescaling the integral $U \to 2pU$ and $r \to 2pr$ yielding

$$\lim_{p\to\infty} I_p\left(\frac{\kappa}{4p}\right) \propto \int\limits_{\mathrm{CSE}(2k)} \mathrm{d}\mu(U)\det{}^{\gamma}U \int_0^{\infty} \mathrm{d}r r^{(2\gamma-2k-1)/2}\det{}^{1/2}\left(r\mathbf{1}_{2k}+U\right) \times \exp\left(-\frac{1}{4}\left[\mathrm{tr}\,U\kappa - \mathrm{tr}\,U^{-1} + t\,r + r^{-1}\right]\right), \tag{5.3}$$

up to a p independent normalization constant. In particular this result is independent if the limit has been approached by an even p or an odd one. Therefore we do the asymptotic analysis for even p, only.

The behaviour of the Laguerre polynomials and Tricomi's confluent hypergeometric function in the microscopic limit determines the whole asymptotics. The asymptotics of both kinds of functions are given by

$$\lim_{p\to\infty} \frac{(-1)^{ap+c}p^{-b}}{\Gamma\left(ap+c+1\right)} L_{ap+c}^{(b)}\left(-\frac{u}{4p}\right) = \left(\frac{4a}{u}\right)^{b/2} I_b\left(\sqrt{au}\right), \tag{5.4}$$

and

$$\lim_{p\to\infty} \frac{\Gamma\left(ap+c\right)}{p^{b-1}} \mathrm{U}\left(ap+c,b;\frac{u}{8p}\right) = 2\left(\frac{8a}{u}\right)^{(b-1)/2} K_{b-1}\left(\sqrt{\frac{au}{2}}\right), \tag{5.5}$$

respectively. Here we employ the modified Bessel functions of the first and second kind, I_a and K_a, respectively. The auxiliary parameters a, b, c take certain values in the particular limits below.

The limits (5.4) and (5.5) imply the following asymptotics of the derivatives of the polynomials for even order

$$\begin{aligned}\xi_a^{(\gamma,\infty)}\left(xu\right) &:= \lim_{p\to\infty} \xi_a^{(\gamma,xp)}\left(\frac{u}{4p}\right) = \lim_{p\to\infty} \frac{(-1)^{xp-a}}{\Gamma[xp+1]}\left(\frac{u}{4p}\right)^{2\gamma+a} \partial_\kappa^a R_{xp}^{(\gamma)}\left(\kappa,\frac{u}{4p}\right)\Big|_{\kappa=-u/(4p)} \\ &= \left(\frac{xu}{4}\right)^{(2\gamma+a)/2}\left[I_{2\gamma+a}(\sqrt{xu}) + \frac{K_{\gamma-1/2}(\sqrt{xu/4})}{K_{\gamma+1/2}(\sqrt{xu/4})} I_{2\gamma+a+1}(\sqrt{xu})\right]\end{aligned} \tag{5.6}$$

and for odd order

$$\begin{aligned}&\lim_{p\to\infty} \frac{(-1)^{xp-a}}{\Gamma[xp+1]}\left(\frac{u}{4p}\right)^{2\gamma+a} \partial_\kappa^a R_{xp+1}^{(\gamma)}\left(\kappa,\frac{u}{4p}\right)\Big|_{\kappa=-u/(4p)} \\ &= \lim_{p\to\infty} \frac{(-1)^{xp-a}}{\Gamma[xp+1]}\left(\frac{u}{4p}\right)^{2\gamma+a} \\ &\quad\times \partial_\kappa^a\left(\kappa + \widehat{c}_{px/2}\left(\frac{u}{4p}\right) - 2\kappa\partial_\kappa - 2u\partial_u\right) R_{xp}\left(\kappa,\frac{u}{4p}\right)\Big|_{\kappa=-u/(4p)}\end{aligned}$$

$$= [C(xu) - 2a - 2u\partial_u] \lim_{p\to\infty} \xi_a^{(\gamma,xp)}\left(\frac{u}{4p}\right)$$
$$= \left(\frac{xu}{4}\right)^{(2\gamma+a)/2}\left[I_{2\gamma+a}(\sqrt{xu})\left(C(xu) - 2a - \sqrt{xu}\frac{K_{\gamma-1/2}(\sqrt{xu/4})}{K_{\gamma+1/2}(\sqrt{xu/4})}\right) + I_{2\gamma+a+1}(\sqrt{xu})\right.$$
$$\left.\times\left(\left[C(xu) + 2\gamma + 1 - \sqrt{\frac{xu}{4}}\frac{K_{\gamma-1/2}(\sqrt{xu/4})}{K_{\gamma+1/2}(\sqrt{xu/4})}\right]\frac{K_{\gamma-1/2}(\sqrt{xu/4})}{K_{\gamma+1/2}(\sqrt{xu/4})} - \sqrt{\frac{xu}{4}}\right)\right]. \quad (5.7)$$

Here we have used the rescaled function (4.37) which is one of the polynomials for $k = 2m + 1$ odd. The shift in the order of the polynomial from $l - 2$ to xp, cf. Eqs. (4.37) and (5.6), has no effect on the limit since it can be absorbed in the limit $p \to \infty$. Furthermore we used the recurrence relation between the polynomials of even order and the polynomials of odd order, see Eq. (3.15), to derive the limit for the latter. The ambiguous function $\widehat{c}_{px/2}(u/(4p))$ is chosen such that its limit exists and converges to $C(xu)$. Indeed this function only depends on the combination xu because a scaling in x yields an inverse scaling in u by absorbing the scaling factor in the p limit. We underline that also the constant $C(xu)$ is ambiguous and we may choose it in such a way that the prefactor in front of the Bessel function $I_{2\gamma+a+1}(\sqrt{xu})$ vanishes, i.e.

$$\lim_{p\to\infty}\frac{(-1)^{xp-a}}{\Gamma[xp+1]}\left(\frac{u}{4p}\right)^{2\gamma+a}\partial_\kappa^a R_{xp+1}^{(\gamma)}\left(\kappa,\frac{u}{4p}\right)\Big|_{\kappa=-u/(4p)} = \left(\frac{xu}{4}\right)^{(2\gamma+a)/2} I_{2\gamma+a}(\sqrt{xu})$$
$$\times\left[\sqrt{\frac{xu}{4}}\left(\frac{K_{\gamma+1/2}(\sqrt{xu/4})}{K_{\gamma-1/2}(\sqrt{xu/4})} - \frac{K_{\gamma-1/2}(\sqrt{xu/4})}{K_{\gamma+1/2}(\sqrt{xu/4})}\right) - 2\gamma - 1 - 2a\right]. \quad (5.8)$$

This expression is more compact than the one in Eq. (5.7) such that we stick with this intermediate result (which is again simpler than in [52]).

The normalization constants of the polynomials, see Eq. (4.15), have the limit

$$\lim_{p\to\infty}\frac{1}{\Gamma^2[xp+1](xp)^{2\gamma}} r_{xp/2}^{(\gamma)}\left(\frac{u}{4p}\right) = 4. \quad (5.9)$$

Moreover we need to express the limit of the sum of p terms in order to deal with the two-point kernel (3.12). Let us consider a function f_j where the limit $\lim_{p\to\infty} f_{px} = f(x)$ exists for all $x \in [0,1]$. Then the sum becomes

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=0}^{\lfloor(p+k-1)/2\rfloor} f_j = \frac{1}{2}\int_0^1 \mathrm{d}x f(x). \quad (5.10)$$

Combining this limit with Eqs. (5.6), (5.8), and (5.9) we find the asymptotics of the kernel

$$\Xi_{ab}^{(\gamma,\infty)}(u) = \lim_{p\to\infty}\Xi_{ab}^{(\gamma,p)}\left(\kappa,\frac{u}{4p}\right) \quad (5.11)$$
$$= \lim_{p\to\infty}(-1)^{a+b}\left(\frac{u}{4p}\right)^{2\gamma+a+b+1}\partial_{\kappa_1}^a\partial_{\kappa_2}^b\mathcal{K}_p\left(\kappa_1,\kappa_2,\frac{u}{4p}\right)\Big|_{\kappa_1=\kappa_2=-u/(4p)}$$
$$= \frac{1}{4}\int_0^{\sqrt{u}/2}\mathrm{d}x x^{a+b+1}\Big[2(b-a)I_{2\gamma+a}(2x)I_{2\gamma+b}(2x)$$

$$+\frac{K_{\gamma-1/2}(x)}{K_{\gamma+1/2}(x)}\left[x\left(\frac{K_{\gamma+1/2}(x)}{K_{\gamma-1/2}(x)}-\frac{K_{\gamma-1/2}(x)}{K_{\gamma+1/2}(x)}\right)-2\gamma-1\right]$$
$$\times\left(I_{2\gamma+a}(2x)I_{2\gamma+b+1}(2x)-I_{2\gamma+a+1}(2x)I_{2\gamma+b}(2x)\right)$$
$$+\frac{K_{\gamma-1/2}(x)}{K_{\gamma+1/2}(x)}\left(2bI_{2\gamma+a+1}(2x)I_{2\gamma+b}(2x)-2aI_{2\gamma+a}(2x)I_{2\gamma+b+1}(2x)\right)\Bigg].$$

Recall that the asymptotics for even and odd p yields the same answer, and so we have obtained this limit by choosing $p=2L$ even.

When considering the particular cases of the gap probability ($\gamma=0$) and the distribution of the smallest eigenvalue ($\gamma=1$) it is more enlightening to express the Bessel functions involved in terms of more explicit functions,

$$K_{1/2}(z)=K_{-1/2}(z)=\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{z}}e^{-z}, \tag{5.12}$$

$$K_{3/2}(z)=\left(1+\frac{1}{z}\right)K_{1/2}(z)=\sqrt{\frac{\pi}{2}}\left(\frac{1}{z^{1/2}}+\frac{1}{z^{3/2}}\right)e^{-z}. \tag{5.13}$$

Then the gap probability reads

$$\mathcal{E}_{2k}(u)=\left(\prod_{l=0}^{k-1}\frac{4^{l+1}(2l)!}{l!}\right)u^{-k^2/2}e^{-u/8-\sqrt{u/4}}$$
$$\times\begin{cases}\mathrm{pf}_{0\le a,b\le k-1}\left[\Xi_{ab}^{(0,\infty)}(u)\right], & k\in 2\mathbb{N}_0,\\ \frac{\sqrt{u}}{4}\mathrm{pf}_{0\le a,b\le k-1}\begin{bmatrix}\Xi_{ab}^{(0,\infty)}(u) & \xi_a^{(0,\infty)}(u)\\ -\xi_b^{(0,\infty)}(u) & 0\end{bmatrix}, & k\in 2\mathbb{N}_0+1\end{cases} \tag{5.14}$$

with

$$\xi_a^{(0,\infty)}(u)=\left(\frac{u}{4}\right)^{a/2}\left[I_a(\sqrt{u})+I_{a+1}(\sqrt{u})\right], \tag{5.15}$$

$$\Xi_{ab}^{(0,\infty)}(u)=\frac{1}{4}\int_0^{\sqrt{u}/2}\mathrm{d}x x^{a+b+1}\Big[2(b-a)I_a(2x)I_b(2x) \tag{5.16}$$
$$+(2b+1)I_{a+1}(2x)I_b(2x)-(2a+1)I_a(2x)I_{b+1}(2x)\Big].$$

The results in Ref. [52], have three typos. First, the factor x^{a+b+1} ($z^{(a+b)/2}$ in the notation therein) is missing in the integral for the two-point kernel see Eqs. (25) therein. Second, the indices of the Pfaffian in the case of odd k should go from 0 to $k-1$, see Eqs. (27) and (29) therein.

The gap probability seems to diverge at $u\to 0$ due to the terms $u^{-k^2/2}$ and $u^{-(k^2-1)/2}$. However the kernels vanish as $\Xi_{ab}^{(0,\infty)}(u)\propto u^{a+b+1}$ and $\xi_a^{(0,\infty)}(u)\propto u^a$ for $u\ll 1$ such that both terms cancel and the gap probability behaves as a constant, especially it is normalized to $\mathcal{E}_{2k}(0)=1$.

The distribution of the first eigenvalue is

$$\mathcal{P}_{2k}(u)=\frac{1}{8}\left(\prod_{l=0}^{k-1}\frac{4^{l+1}(2l)!}{l!}\right)(\sqrt{u}+2)u^{-(k^2+1)/2}e^{-u/8-\sqrt{u/4}}$$
$$\times\begin{cases}\mathrm{pf}_{0\le a,b\le k-1}\left[\Xi_{ab}^{(1,\infty)}(u)\right], & k\in 2\mathbb{N}_0,\\ \frac{1}{\sqrt{u}}\mathrm{pf}_{0\le a,b\le k-1}\begin{bmatrix}\Xi_{ab}^{(1,\infty)}(u) & \xi_a^{(1,\infty)}(u)\\ -\xi_b^{(1,\infty)}(u) & 0\end{bmatrix}, & k\in 2\mathbb{N}_0+1.\end{cases} \tag{5.17}$$

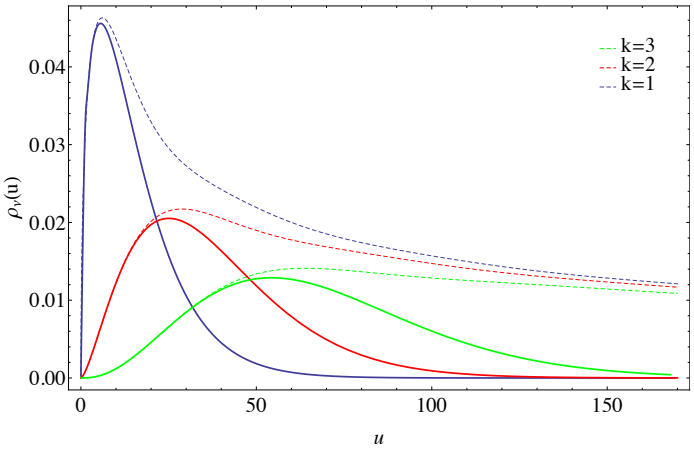

Figure 2. The distribution of the smallest eigenvalue (5.17) (solid curves) smoothly fits with the microscopic level density (2.4) (dotted curves) around the origin. Only for larger values of u deviations due to the contribution of the second and larger eigenvalues become visible. A similar plot can be made for $\nu = 0$ which we have omitted for a better readability of the figure.

The behaviour of the distribution at the origin can be read off from the kernels which are

$$\xi_a^{(1,\infty)}(u) = \left(\frac{u}{4}\right)^{a/2+1}\left[I_{a+2}(\sqrt{u}) + \frac{\sqrt{u}}{\sqrt{u}+2}I_{a+3}(\sqrt{u})\right], \tag{5.18}$$

$$\Xi_{ab}^{(1,\infty)}(u) = \frac{1}{4}\int_0^{\sqrt{u}/2} \mathrm{d}x x^{a+b+1}\Big[2(b-a)I_{a+2}(2x)I_{b+2}(2x) \tag{5.19}$$

$$+ \frac{x}{x+1}\left(\left(2b+\frac{x+2}{x+1}\right)I_{a+3}(2x)I_{b+2}(2x) - \left(2a+\frac{x+2}{x+1}\right)I_{a+2}(2x)I_{b+3}(2x)\right)\Big].$$

Since $\Xi_{ab}^{(1,\infty)}(u) \propto u^{a+b+3}$ and $\xi_a^{(1,\infty)}(u) \propto u^{a+2}$ for $|u| \ll 1$ we have $\mathcal{P}_{2k}(u) \propto u^{k-1/2}$ agreeing with the behaviour of the microscopic level density (2.4). We remark that the term $J_\nu(\sqrt{u})/\sqrt{16u} \sim u^{(\nu-1)/2}$ in the expression (2.4) is the dominant term in the limit $u \to 0$. The same behaviour of both distributions around the origin is inherent because the level density is governed by the smallest eigenvalue in this regime. Only for larger argument u the other eigenvalues start to contribute to the level density, cf. Fig. 2.

The results (5.14) and (5.17) are our third main result. We emphasize that these results do not only describe the smallest eigenvalue of an artificial system, namely real chiral Gaussian random matrices with an even index $\nu = 2k$, but also of physical systems. Due to universality not only the level density (2.4) has to agree with those from physical systems like QCD or mesoscopic systems. Also the distributions of the smallest eigenvalues of those physical systems have to follow the same distributions of random matrix theory in the limit of its applicability, e.g. see [37, 38, 3] and references therein. This agreement should already happen at moderate system sizes. In Fig. 3 we compare the analytic results (4.42) and (4.43) of the distribution of the smallest eigenvalue at finite p with the microscopic limit (5.17). This comparison underlines how fast the convergence to the universal result happens. Although this comparison

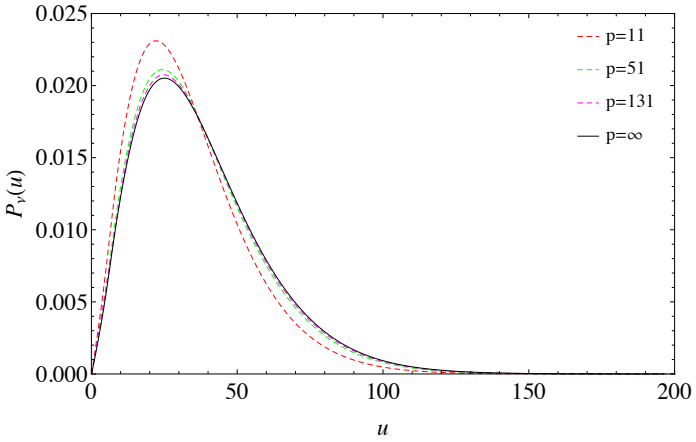

Figure 3. Illustration of the rate of convergence by comparing the finite p result (4.42) for $p = 11, 51, 131$ with the universal limit $p \to \infty$ (5.17) for the distribution of the smallest eigenvalue. The index of the Wishart matrix is chosen $\nu = 2k = 4$ as an example.

is done for random matrices which have particular advantages in comparison to real systems we expect that also physical systems should display a rapid convergence to the universal result.

6. The Correlated Wishart-Laguerre Ensemble

We are now interested in the effects of a fixed, non-trivial correlation matrix C not proportional to the identity matrix on the distribution of the smallest eigenvalue, see Eq. (2.1). Such a correlation matrix can naturally encode system specific information. In time series analysis such a correlation may encode correlations between companies in finance [11], seasonal effects in climate research [8] or organized crime in criminal defence [9]. However usually such correlation matrices have no microscopic limit in time series. But also in QCD and mesoscopic systems which exhibit such a microscopic limit correlations may appear. These correlations encode the structure of space-time and the choice of the gauge theory for example. They are system specific informations and may have an influence on the smallest eigenvalues.

We expect that the correlation matrix has no influence on the smallest eigenvalues as long as its eigenvalues have a finite distance to the origin. Then the screening of the infinitely many eigenvalues between the smallest eigenvalue of W and the eigenvalues of C is strong enough. This was also shown in [14]. For this purpose we choose a non-trivial empirical correlation matrix $C \neq \mathbf{1}_p$. In Fig. 3 we compare Monte-Carlo simulations with such an empirical correlation matrix C and the universal result (5.17). The matrix size is chosen such that $(n-p)/p = \nu/p \ll 1$ where $p = 200$ and $\nu = 2k = 0, 2, 4$. The perfect agreement underlines that correlations in the Wishart matrix have a very weak effect on the spectral statistics of the smallest eigenvalue.

Again we emphasize that we have not looked at the situation where C develops a spectrum where some eigenvalues lie on the scale $1/p$. Nor have we looked at the situation of doubly correlated Wishart-Laguerre ensembles. However for the latter

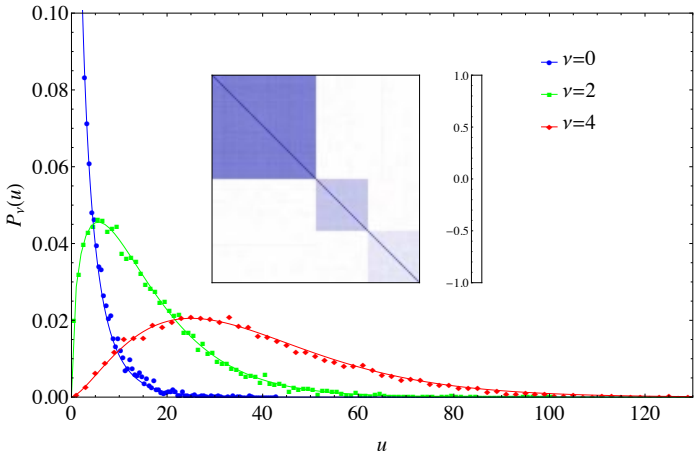

Figure 4. Comparison of Monte Carlo simulations (symbols) with a non-trivial empirical correlation matrix C at finite $p = 200$, and the microscopic limit $p \to \infty$ of the distribution of the smallest eigenvalue (5.17) (solid lines). The empirical correlation matrix is shown in the inset. We have generated 10000 correlated Wishart random matrices of sizes $200 \times (200 + \nu)$ with $\nu = 2k = 0, 2, 4$.

kind of ensembles we expect a similar if not exactly the same behaviour like one-sided correlated Wishart-Laguerre ensembles as considered here.

7. Conclusion

We have addressed and solved an open problem in the real Wishart-Laguerre ensemble also known as the chiral Gaussian orthogonal ensemble of rectangular $p \times n$ random matrices. We computed the distribution of the smallest eigenvalue and its integral, the gap probability that the vicinity of the origin is empty of eigenvalues. To this aim we have established that an integrable Pfaffian structure holds also when $p - n = \nu$ measuring the rectangularity (or topology in the field theory application) is even. Such a Pfaffian structure was previously only known when $\nu = 2k + 1$ is odd. So far for an even rectangularity a recursive construction in p led to closed form expressions for $\nu = 0, 2$ only. In view of the various applications it would be unnatural to restrict oneself to odd ν, and not to expect for such an integrable structure to exist. However, the recursive construction (and closed results for $\nu = 0, 2$) already revealed the appearance of special functions for finite p which are absent for ν odd, namely Tricomi's confluent hypergeometric functions. From our construction we now better understand why they appear through the expectation value of the square root of characteristic polynomials which are among the building blocks for the quantities in question.

On a technical level our computation was possible due to the combination of the method of skew-orthogonal polynomials, though with a non standard weight, and the map of our building blocks onto invariant co-set integrals derived by bosonisation. In an initial step the computation of the the gap probability and the distribution of the smallest eigenvalue requires the evaluation of expectation values of characteristic polynomials raised to integer/half-integer powers for ν odd/even, respectively. Hence the problem exhibits an increased level of difficulty in computing these objects for

even ν. By including the square root for even ν into the weight function we were back to expectation values of integer powers, which are known to be expressible through Pfaffian determinants of kernels and skew-orthogonal polynomials. The price we had to pay was to compute the latter for a non-standard weight including the square root. This was done by expressing the polynomials and kernel themselves through expectation values, mapping these back to matrix integrals and computing them via bosonisation.

Indeed one can also consider the distribution and the cumulative distribution of the second to smallest eigenvalue, third to smallest eigenvalue etc. These quantities can be simply deduced from our results, too, because we calculated a quite explicit expression of the kernel at finite p. In particular we found a Christoffel-Darboux-like formula which expresses the sum over $(p+k)$ terms in a sum over twelve terms, only.

The Pfaffian structure enabled us to take the microscopic large-p limit at the origin, while keeping $\nu = 2k$ fixed. In this limit we could show that the distinction between even and odd p for finite p becomes immaterial. We found results in terms of a Pfaffian comprising the limiting kernel for even k, plus an additional column and row for k odd, both for the gap probability and the smallest eigenvalue.

Our results are universal for non-Gaussian potentials, as inherited from the universality of the known density correlation functions. We have checked that our findings follow the microscopic density for small argument, and that our finite-p results, which we have confirmed through numerical simulations, converge towards the universal limit. Furthermore, we have also studied numerically the distribution of the smallest eigenvalue in an example of a correlated Wishart-Laguerre ensemble. We found that for moderate p it already follows the universal limiting distribution for several values of ν.

The computation of products of ratios of characteristic polynomials that also include square roots is in general an open question in random matrix theory. The structure of the results we obtained on a subset of such correlators should be relatively easy to translate to the Gaussian orthogonal ensemble, where such correlation functions enjoy further applications, e.g. in Quantum Chaos. It is very plausible that our universal result will also apply when introducing a fixed trace constraint, as it is known for odd ν. What is less clear is whether a corresponding representation in terms of hypergeometric functions of matrix arguments exist, having the advantage that they can be continued to real $\beta > 0$. It would also be very interesting to see, if and when the universality at the origin breaks down for the correlated Wishart-Laguerre ensemble. A further open question is the computation of the gap probability and smallest eigenvalue distribution in the chiral Gaussian symplectic ensemble with $\beta = 4$. Apart from $\nu = 0$ only Taylor expansions exist so far. However, following similar ideas as in the present work, Pfaffian structures for these quantities exist and should be universal. Work in this direction is currently under way.

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C Articles summarized in Subsection 3.3

Singular value correlation functions for products of Wishart random matrices

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Abstract

We consider the product of M quadratic random matrices with complex elements and no further symmetry, where all matrix elements of each factor have a Gaussian distribution. This generalises the classical Wishart-Laguerre Gaussian Unitary Ensemble with $M = 1$. In this paper we first compute the joint probability distribution for the singular values of the product matrix when the matrix size N and the number M are fixed but arbitrary. This leads to a determinantal point process which can be realised in two different ways. First, it can be written as a one-matrix singular value model with a non-standard Jacobian, or second, for $M \geq 2$, as a two-matrix singular value model with a set of auxiliary singular values and a weight proportional to the Meijer G-function. For both formulations we determine all singular value correlation functions in terms of the kernels of biorthogonal polynomials which we explicitly construct. They are given in terms of hypergeometric and Meijer G-functions, generalising the Laguerre polynomials for $M = 1$. Our investigation was motivated from applications in telecommunication of multi-layered scattering MIMO channels. We present the ergodic mutual information for finite-N for such a channel model with $M - 1$ layers of scatterers as an example.

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1 Introduction

Random matrix theory (RMT) remains a very active field of research, after many decades of work. While originally being conceived in the area of mathematical statistics and nuclear physics, today's applications of RMT extend beyond the mathematical and physical sciences even in a broad sense, and we refer to [1] for a recent overview.

One of the topics in RMT that has caught recent attention is that of products of random matrices. Having one of its original motivations in statistical physics in the description of chaotic and disordered systems [2], among more recent applications are combinatorics [3] and telecommunications [4], which has also been one of our motivations. In particular, we consider MIMO (multiple-input and multiple-output) communication networks, where multi-antenna transceivers are utilised to improve the system capacity in a rich scattering environment.

Products of matrices loose much of the symmetry of the individual matrices and are generically complex. For simplicity we will consider the individual matrices to be complex, too, with independent Gaussian distributions. The spectral properties of matrix ensembles carry important information. It is encoded in the eigenvalue decomposition as well as in the singular value decomposition. We will focus on the latter here.

A striking property of RMT is its universality, that is the independence of the underlying distribution of the individual matrix elements. It is usually manifest in the limit of large matrix size. However, if we study the local, microscopic behaviour of the spectrum on the scale of the mean level spacing between singular values, it is often vital to have a detailed knowledge of the joint distribution of singular values (or eigenvalues) at hand for finite matrix size. In particular to derive a determinantal or Pfaffian structure of the correlation functions of the random matrix ensemble has proven very useful for universality studies. Some of the most powerful proofs of universality start from the knowledge of orthogonal polynomials of these determinantal or Pfaffian point processes, in order to perform the asymptotic analysis. We refer to [5] for a standard reference.

The aim of this work is to provide such a starting point, by deriving the joint probability density function (jpdf) of the singular values of the product matrix at finite matrix size N, for a finite product of M matrices. We consider the simplest case of M quadratic $N \times N$ matrices of Wishart type, independently and identically distributed by Gaussians with unit variance. In telecommunications this is also the setting often encountered, with both N and M finite. The singular value distribution of products of complex Wishart matrices is then the setup for the calculation of several information-theoretic quantities.

In previous works the spectral density of singular values as well as the moments of such product matrices were derived in the macroscopic large-N limit. They use probabilistic methods such as free random variables [6, 7, 8], field theoretic methods such as planar diagrams [9], and inverse Mellin transforms [3]. The limit of infinitely many matrices in a product were studied in other works, either for finite-size [10, 11, 12, 13] or for infinitely large matrices [14, 15], where the problem was mapped to a differential equation.

Very recently the jpdf and its correlation functions of the complex eigenvalues of the matrix ensemble we are considering has been derived for finite N and M [16, 17, 18]. Also here the macroscopic large-N density in the complex plane was known previously, see [19, 20, 9] for a collection of works. However the corresponding question about the singular value correlation functions for finite N and M was still open and is addressed in our work.

This article is organised as follows. In section 2 we determine the jpdf of singular values, using two different ways. Section 3 is devoted to the computation of the correlation functions, by first determining the biorthogonal polynomials associated to this problem in subsection 3.1, and then the kernel(s) leading to all k-point correlators in subsection 3.2. The spectral density itself is discussed

in more detail in subsection 3.3, where we compute all its moments and identify the correct rescaling for the macroscopic large-N limit of our results. In particular we compare our results to the known large-N asymptotic spectral density e.g. from [9]. Section 4 illustrates how our results can help in applications in telecommunications, by computing the ergodic mutual information and comparing it to numerical simulations. After presenting our concluding remarks and open questions in section 5 we collect some technical tools in two appendices.

2 Joint probability distribution of singular values

We are interested in the singular values of the product P_M of M independent matrices with complex matrix elements of size $N \times N$, $X_j \in \mathrm{Gl}(N, \mathbb{C})$ for all $j = 1, \ldots, M$:

$$P_M \equiv X_M X_{M-1} \ldots X_1 \ . \tag{2.1}$$

Furthermore the matrices X_j are distributed by identical, independent Gaussians,

$$\mathcal{P}(X_j) = \exp\left[-\mathrm{Tr}\, X_j^\dagger X_j\right] \ . \tag{2.2}$$

The partition function $\mathcal{Z}_N^{(M)}$ is then defined as

$$\mathcal{Z}_N^{(M)} = C \int \prod_{j=1}^{M} d[X_j] P(X_j) \ , \tag{2.3}$$

where $d[X_j] = \prod_{\alpha,\beta=1}^{N} d(X_j)_{\alpha\beta} d(X_j^*)_{\alpha\beta}$ denotes the flat measure over all independent matrix elements. In this section we compute the jpdf of the singular values of P_M. For $M = 1$ this is the well known Wishart-Laguerre (also called chiral Gaussian) Unitary Ensemble, and for most of the following we will thus restrict ourselves to $M > 1$. We will not keep track of the normalisation constant C here, and will only specify it later, once we have changed to singular values.

In the first step we perform the following successive change of variables from X_j to Y_j:

$$Y_1 \equiv X_1 \ , \quad \text{and} \quad Y_j \equiv X_j Y_{j-1} \quad \text{for} \quad j = 2, \ldots, M \ , \tag{2.4}$$

e.g. $Y_2 = X_2 X_1$, $Y_3 = X_3 X_2 X_1$ etc. In the new variables $Y_M = P_M$ is the product matrix we are aiming at. While the first one, $Y_1 = X_1$, is a trivial relabelling, each subsequent change of variables carries a non-trivial Jacobian given by $1/\det[Y_{j-1}^\dagger Y_{j-1}]^N$ for $j = 2, \ldots, M$. This can be seen as follows. Due to $d(Y_j)_{\alpha\beta} = \sum_{\gamma=1}^{N} d(X_j)_{\alpha\gamma} (Y_{j-1})_{\gamma\beta}$ every column vector of the matrix Y_j acquires a factor $1/\det[Y_{j-1}]$ from the change of variables, and likewise its complex conjugate. Taking into account all N column vectors and their complex conjugates we find the given Jacobian for each $j = 2, \ldots, M$, and we arrive at

$$\mathcal{Z}_N^{(M)} = C \int \prod_{i=1}^{M} d[Y_i] \, \exp\left[-\mathrm{Tr}\, Y_1^\dagger Y_1\right] \prod_{j=2}^{M} \frac{1}{\det[Y_{j-1}^\dagger Y_{j-1}]^N} \exp\left[-\mathrm{Tr}\, Y_j^\dagger Y_j (Y_{j-1}^\dagger Y_{j-1})^{-1}\right] \ . \tag{2.5}$$

In writing this we assume that the matrices X_j and thus their products Y_j are invertible[1]. In the second step we decompose each matrix $Y_j = V_j \Lambda_j U_j$, $j = 1, \ldots, M$ in its angles and singular values, where $\Lambda_j = \mathrm{diag}(\lambda_1^{(j)}, \ldots, \lambda_N^{(j)})$ contains the positive singular values $\lambda_a^{(j)} \in \mathbb{R}_+$, $a = 1, \ldots, N$, and $U_j \in \mathrm{U}(N)$

[1] Our restriction from general complex $N \times N$ matrices to $\mathrm{Gl}(N, \mathbb{C})$ removes only a set of measure zero.

and $V_j \in \mathrm{U}(N)/\mathrm{U}(1)^N$ are unitary. The Jacobian resulting from the singular value decomposition of each matrix Y_j is well known and is given in terms of the Vandermonde determinant,

$$\Delta_N(\Lambda_j) \equiv \prod_{N\geq a>b\geq 1} (\lambda_a^{(j)} - \lambda_b^{(j)}) = \det_{1\leq a,b\leq N} \left[(\lambda_a^{(j)})^{b-1} \right]. \tag{2.6}$$

Since we encounter the matrices Y_j in the combination $Y_j^\dagger Y_j$, only, the unitary matrices V_j completely drop out which leads to

$$\begin{aligned}
\mathcal{Z}_N^{(M)} &= C' \int \prod_{i=1}^{M} \left\{ d[V_i] d[U_i] \prod_{a=1}^{N} d\lambda_a^{(i)} \lambda_a^{(i)} \right\} \exp\left[-\mathrm{Tr}\, \Lambda_1^2 \right] \Delta_N(\Lambda_1^2)^2 \\
&\quad \times \prod_{j=2}^{M} \frac{1}{\det[\Lambda_{j-1}^2]^N} \exp\left[-\mathrm{Tr}\, U_{j-1} U_j^\dagger \Lambda_j^2 U_j U_{j-1}^\dagger \Lambda_{j-1}^{-2} \right] \Delta_N(\Lambda_j^2)^2 \\
&= C' \int \prod_{i=1}^{M} \left\{ d[V_i] d[U_i] \prod_{a=1}^{N} d\lambda_a^{(i)} \lambda_a^{(i)} \right\} \exp\left[-\mathrm{Tr}\, \Lambda_1^2 \right] \Delta_N(\Lambda_1^2)^2 \\
&\quad \times \prod_{j=2}^{M} \frac{1}{\det[\Lambda_{j-1}^2]^N} \exp\left[-\mathrm{Tr}\, U_j^\dagger \Lambda_j^2 U_j \Lambda_{j-1}^{-2} \right] \Delta_N(\Lambda_j^2)^2 \,.
\end{aligned} \tag{2.7}$$

In the second step we employed the invariance of the Haar measure $d[U_j]$ under $U_j \to U_j U_{j-1}$, which leads to the decoupling of the integrations over $d[U_1]$ and all the $d[V_j]$ from the rest of the integrals. The remaining unitary integrations in the last line of eq. (2.7) can be performed using the so-called Harish-Chandra–Itzykson-Zuber (HCIZ) integral [21, 22][2]

$$\int d[U_j] \exp\left[-\mathrm{Tr}\, \left(U_j^\dagger \Lambda_j^2 U_j \Lambda_{j-1}^{-2} \right) \right] = \frac{1}{\Delta_N(\Lambda_j^2)\Delta_N(\Lambda_{j-1}^{-2})} \det_{1\leq a,b\leq N} \left[\exp\left(-\frac{(\lambda_a^{(j)})^2}{(\lambda_b^{(j-1)})^2} \right) \right] . \tag{2.8}$$

The Vandermonde determinant of inverse powers is proportional to the ordinary one with positive powers due to the following identity:

$$\Delta_N(\Lambda_j^{-2}) = \det_{1\leq a,b\leq N} \left[\frac{1}{(\lambda_a^{(j)})^{2b-2}} \right] = (-1)^{N(N-1)/2} \frac{\Delta_N(\Lambda_j^2)}{\det[\Lambda_j^2]^{N-1}} \,. \tag{2.9}$$

This leads to many cancellations in eq. (2.7), in particular of almost all Vandermonde determinants:

$$\begin{aligned}
\mathcal{Z}_N^{(M)} &= C'' \int_0^\infty \prod_{a=1}^{N} \left\{ d\lambda_a^{(M)} \lambda_a^{(M)} \prod_{j=1}^{M-1} \frac{d\lambda_a^{(j)}}{\lambda_a^{(j)}} \right\} \exp\left[-\sum_{b=1}^{N} (\lambda_b^{(1)})^2 \right] \Delta_N(\Lambda_1^2) \Delta_N(\Lambda_M^2) \\
&\quad \times \prod_{i=2}^{M} \det_{1\leq c,d\leq N} \left[\exp\left(-\frac{(\lambda_c^{(i)})^2}{(\lambda_d^{(i-1)})^2} \right) \right] .
\end{aligned} \tag{2.10}$$

We expand the determinant comprising $\lambda_d^{(1)}$ and $\lambda_c^{(2)}$ in $N!$ terms. Each of these terms yields the same contribution since the permutation involved in the definition of the determinant can be absorbed in the determinant comprising $\lambda_d^{(2)}$ and $\lambda_c^{(3)}$ due to the antisymmetry of determinants, and a relabelling of the integration variables. Next we expand the determinant comprising $\lambda_d^{(2)}$ and $\lambda_c^{(3)}$ whose permutations

[2] The Haar measure is normalised such that there is no further proportionality constant on the right hand side.

can be absorbed in the determinant comprising $\lambda_d^{(3)}$ and $\lambda_c^{(4)}$, and so on. This interplay of expansion and absorption of the permutations of the determinants can be continued untill all determinants stemming from the HCIZ integral are replaced by their diagonal part. Note that we do not require any symmetrisation in the variables $\lambda_a^{(M)}$ here. Hence our argument applies to correlation functions of the $\lambda_a^{(M)}$ in the next section 3, too, where we integrate the joint probability distribution function (jpdf) only over a subset of these singular values.

Almost all remaining multiple integrals can be simplified as follows:

$$\begin{aligned}
&\prod_{a=1}^{N}\left\{\int_0^\infty \frac{d\lambda_a^{(1)}}{\lambda_a^{(1)}}\,\exp\left[-(\lambda_a^{(1)})^2\right]\left(\prod_{j=2}^{M-1}\int_0^\infty\frac{d\lambda_a^{(j)}}{\lambda_a^{(j)}}\,\exp\left[-\frac{(\lambda_a^{(j)})^2}{(\lambda_a^{(j-1)})^2}\right]\right)\,\exp\left[-\frac{(\lambda_a^{(M)})^2}{(\lambda_a^{(M-1)})^2}\right]\right\}\\
&\times\det_{1\le c,d\le N}\left[(\lambda_c^{(1)})^{2d-2}\right]\\
=\;&\det_{1\le c,d\le N}\left[\int_0^\infty\frac{d\lambda_c^{(1)}}{(\lambda_c^{(1)})^{3-2d}}\exp\left[-(\lambda_c^{(1)})^2\right]\left(\prod_{j=2}^{M-1}\int_0^\infty\frac{d\lambda_c^{(j)}}{\lambda_c^{(j)}}\,\exp\left[-\frac{(\lambda_c^{(j)})^2}{(\lambda_c^{(j-1)})^2}\right]\right)\exp\left[-\frac{(\lambda_c^{(M)})^2}{(\lambda_c^{(M-1)})^2}\right]\right]\\
=\;&\det_{1\le c,d\le N}\left[\frac{1}{2^{M-1}}G^{M,0}_{0,M}\left(\begin{matrix}-\\0,\ldots,0,d-1\end{matrix}\,\middle|\,(\lambda_c^{(M)})^2\right)\right]\;. \qquad (2.11)
\end{aligned}$$

Notice that we have left out the integration over the variables $\lambda_a^{(M)}$. In the second line of eq. (2.11) we have pulled all the integrations into the corresponding rows of the determinants, and in the third line we have used the integral identity (A.10) in the squared singular values, that is derived in appendix A. The special function appearing here is the so-called Meijer G-function, see eq. (A.1) for its definition [23]. The number of zeros in the bottom line of the Meijer G-function is $M-1$. Our first main result is thus the following singular value representation of the partition function, after changing to squared singular values $s_a \equiv (\lambda_a^{(M)})^2$ with $ds_a = 2\lambda_a^{(M)}d\lambda_a^{(M)}$, $a=1,\ldots,N$, in eq. (2.10):

$$\mathcal{Z}_N^{(M)} = C_N^{(M)}\int_0^\infty\prod_{a=1}^N ds_a\;\Delta_N(s)\det_{1\le c,d\le N}\left[G^{M,0}_{0,M}\left(\begin{matrix}-\\0,\ldots,0,d-1\end{matrix}\,\middle|\,s_c\right)\right] = \int_0^\infty\prod_{a=1}^N ds_a\mathcal{P}_{\rm jpdf}(s)\;, \qquad (2.12)$$

$$\mathcal{P}_{\rm jpdf}(s) \equiv C_N^{(M)}\Delta_N(s)\det_{1\le c,d\le N}\left[G^{M,0}_{0,M}\left(\begin{matrix}-\\0,\ldots,0,d-1\end{matrix}\,\middle|\,s_c\right)\right]\;, \qquad (2.13)$$

where $\mathcal{P}_{\rm jpdf}$ is the jpdf. We will show later that it corresponds to a determinantal point process. The constant in front of eq. (2.13),

$$(C_N^{(M)})^{-1} \equiv N!\prod_{a=1}^N\Gamma(a)^{M+1}\;, \qquad (2.14)$$

has been chosen such that the partition function is normalised to unity. This can be seen as follows. Applying the Andréief integral identity,

$$\int\prod_{a=1}^N ds_a\;\det_{1\le c,d\le N}[\phi_c(s_d)]\det_{1\le c,d\le N}[\psi_c(s_d)] = N!\det_{1\le c,d\le N}\left[\int ds\,\phi_c(s)\psi_d(s)\right]\;, \qquad (2.15)$$

which applies to any two sets of functions ϕ_c and ψ_c such that all integrals exist, we obtain for the

partition function

$$\begin{aligned}\mathcal{Z}_N^{(M)} &= C_N^{(M)} N! \det_{1\le c,d\le N}\left[\int_0^\infty ds s^{c-1} G^{M,0}_{0,M}\left(\begin{matrix}-\\ 0,\ldots,0,d-1\end{matrix}\Big| s\right)\right] \\ &= C_N^{(M)} N! \det_{1\le c,d\le N}\left[\Gamma(c)^{M-1}\Gamma(c+d-1)\right] \\ &= C_N^{(M)} N! \prod_{c=1}^N \Gamma(c)^{M-1} \prod_{b=1}^N \Gamma(b)^2 = 1 \ . \end{aligned} \tag{2.16}$$

Here we have used another integral identity from the appendix, eq. (A.7), and pulled out factors of Gamma-functions from the rows of the determinant. The remaining determinant is nothing but the normalisation of the Wishart-Laguerre ensemble (at $M = 1$) which is well-known [24].

We refer to the result (2.12) as a "one-matrix" singular value model because it is of the form that would result from the singular value decomposition of a single random matrix, however with a non-standard Jacobian $\neq \Delta_N(s)^2$. As an easy check we can see that due to

$$G^{1,0}_{0,1}\left(\begin{matrix}-\\ d-1\end{matrix}\Big| s_c\right) = s_c^{d-1}\exp[-s_c] \ , \tag{2.17}$$

the expression in eq. (2.12) reduces to the standard Wishart-Laguerre ensemble when setting $M = 1$, and taking the exponentials out of the second determinant.

In principle we could now try to compute all singular value k-point correlation functions defined as

$$R_k^{(M)}(s_1,\ldots,s_k) \equiv \frac{N!}{(N-k)!}\int_0^\infty \prod_{a=k+1}^N ds_a \ \mathcal{P}_{\rm jpdf}(s) \ . \tag{2.18}$$

For example for $k = 1$ this gives the spectral density which is normalised to N in our convention following [24],

$$N = \int ds_1 R_{k=1}^{(M)}(s_1) \ , \tag{2.19}$$

whereas for $k = N$ we have the jpdf itself, $R_N^{(M)}(s_1,\ldots,s_N) = N!\mathcal{P}_{\rm jpdf}(s)$. However, due to the matrix inside the second determinant in eq. (2.12) being labelled by indices of the Meijer G-function, the computation of the $R_k^{(M)}$ is a highly nontrivial task. We postpone this computation to section 3 using a second "two-matrix" formulation, that is introduced in the next subsection.

2.1 An alternative jpdf with auxiliary variables

We introduce a formalism that is more convenient to handle when computing correlation functions. Let us step back by considering eq. (2.10), taking for simplicity $M = 2$:

$$\begin{aligned}\mathcal{Z}_N^{(M=2)} &= C'\int_0^\infty \prod_{a=1}^N d\lambda_a^{(2)}\lambda_a^{(2)} \prod_{b=1}^N \frac{d\lambda_b^{(1)}}{\lambda_b^{(1)}} \exp\left[-(\lambda_b^{(1)})^2\right] \ \Delta_N(\Lambda_1^2)\Delta_N(\Lambda_2^2) \\ &\quad \times \det_{1\le c,d\le N}\left[\exp\left(-\frac{(\lambda_c^{(2)})^2}{(\lambda_d^{(1)})^2}\right)\right] \ . \end{aligned} \tag{2.20}$$

This is precisely of the form of a "two-matrix" singular value model (2mm) that results from the singular value decomposition of a two-matrix model, see e.g. in [25]. The advantage is that now we have the *standard* form of the Jacobian given by one Vandermonde per set of variables and an

additional determinant of a function that couples the two sets of variables. Such a setting can be tackled using the known biorthogonal polynomial technique, as reviewed in [26]. In order to apply this technique we have to bring eq. (2.10) into such a form, but for arbitrary values of $M \geq 2$. This can be readily achieved by taking the same steps as from eq. (2.10) to eq. (2.11), but this time excluding both sets of integrations over the variables $\lambda_a^{(M)}$ and $\lambda_a^{(1)}$.

The symmetry argument goes along the same lines replacing determinants by their diagonal parts, see the discussion after eq. (2.10), but this time we keep the determinant containing the exponential with $\lambda_d^{(1)}$ and $\lambda_d^{(M)}$. The integrations are:

$$\begin{aligned}
&\prod_{a=1}^{N}\prod_{j=2}^{M-1}\left\{\int_0^\infty \frac{d\lambda_a^{(j)}}{\lambda_a^{(j)}}\,\exp\left[-\frac{(\lambda_a^{(j+1)})^2}{(\lambda_a^{(j)})^2}\right]\right\}\det_{1\leq c,d\leq N}\left[\exp\left(-\frac{(\lambda_c^{(2)})^2}{(\lambda_d^{(1)})^2}\right)\right] \\
&= \det_{1\leq c,d\leq N}\left[\prod_{j=2}^{M-1}\left\{\int_0^\infty \frac{d\lambda_c^{(j)}}{\lambda_c^{(j)}}\,\exp\left[-\frac{(\lambda_c^{(j+1)})^2}{(\lambda_c^{(j)})^2}\right]\right\}\exp\left(-\frac{(\lambda_c^{(2)})^2}{(\lambda_d^{(1)})^2}\right)\right] \\
&= \det_{1\leq c,d\leq N}\left[\frac{1}{2^{M-2}}G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\,\middle|\,\frac{(\lambda_c^{(M)})^2}{(\lambda_d^{(1)})^2}\right)\right]\,, \qquad (2.21)
\end{aligned}$$

where we have used again the identity (A.10) from appendix A, see also [16] for a related recurrence relation. We therefore arrive at our second main result, the following 2mm representation for the jpdf, after changing again to squared singular values $s_a \equiv (\lambda_a^{(M)})^2$, $t_a \equiv (\lambda_a^{(1)})^2$, $a = 1, \ldots, N$:

$$\begin{aligned}
\mathcal{Z}_N^{(M)} &= \frac{C_N^{(M)}}{N!}\int_0^\infty \prod_{a=1}^{N} ds_a \frac{dt_a}{t_a}\,\mathrm{e}^{-t_a}\;\Delta_N(s)\Delta_N(t)\det_{1\leq c,d\leq N}\left[G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\,\middle|\,\frac{s_c}{t_d}\right)\right] \\
&= \int_0^\infty \prod_{a=1}^{N} ds_a\, dt_a \mathcal{P}_{\rm jpdf}^{\rm 2mm}(s,t)\,, \\
\mathcal{P}_{\rm jpdf}^{\rm 2mm}(s,t) &\equiv \frac{C_N^{(M)}}{N!}\prod_{a=1}^{N} t_a^{-1}\mathrm{e}^{-t_a}\;\Delta_N(s)\Delta_N(t)\det_{1\leq c,d\leq N}\left[G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\,\middle|\,\frac{s_c}{t_d}\right)\right]. \qquad (2.22)
\end{aligned}$$

It is normalised to unity as we will check below.

The crucial advantage in comparison to $\mathcal{P}_{\rm jpdf}$ is the matrix inside the determinant which has indices that label the integration variables, and not the indices of the Meijer G-function as in eq. (2.12). This 2mm describes the correlations among the singular values $\lambda_a^{(1)}$ of a single matrix X_1 and the singular values $\lambda_a^{(M)}$ of the entire product matrix P_M, to be computed in the next section 3.

As a check for $M = 2$ we get back to eq. (2.20), using

$$G^{1,\,0}_{0,\,1}\left(\begin{matrix}-\\ 0\end{matrix}\,\middle|\,\frac{(\lambda_c^{(2)})^2}{(\lambda_d^{(1)})^2}\right) = \exp\left[-\frac{(\lambda_c^{(2)})^2}{(\lambda_d^{(1)})^2}\right]\,. \qquad (2.23)$$

Confirming the normalisation in eq. (2.22) is at the same time a check that this representation can be mapped back to eq. (2.12) in a different way. Applying once again the Andréief formula (2.15) to

eq. (2.22), but this time only to the t-integration, we obtain the following:

$$\begin{aligned}
&\int_0^\infty \prod_{a=1}^N \frac{dt_a}{t_a} \mathrm{e}^{-t_a}\ \Delta_N(t) \det_{1\le c,d\le N}\left[G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s_c}{t_d}\right)\right] \\
&= N!\det_{1\le c,d\le N}\left[\int_0^\infty dt\, t^{d-2}\mathrm{e}^{-t} G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s_c}{t}\right)\right] \\
&= N!\det_{1\le c,d\le N}\left[G^{M,\,0}_{0,\,M}\left(\begin{matrix}-\\ 0,\ldots,0,d-1\end{matrix}\middle|\ s_c\right)\right]\ ,
\end{aligned} \tag{2.24}$$

upon using the identity (A.9) with $m = M-1$ from appendix A. This brings us back to the "one-matrix" model representation eq. (2.12) in terms of a single set of singular values, with the proper normalisation.

3 Singular value correlation functions

We are now prepared to compute arbitrary k-point correlation functions of the singular values. Rather than using the definition (2.18) we will consider the more general correlation functions of the jpdf in the 2mm representation (2.22):

$$R^{(M)}_{k,l}(s_1,\ldots,s_k;t_1,\ldots,t_l) \equiv \frac{N!^2}{(N-k)!(N-l)!}\int_0^\infty \prod_{a=k+1}^N ds_a \prod_{b=l+1}^N dt_b\ \mathcal{P}^{\mathrm{2mm}}_{\mathrm{jpdf}}(s,t)\ . \tag{3.1}$$

The k-point functions of the singular values of the product matrix P_M in eq. (2.18) can be obtained by integrating out all auxiliary variables, or by setting $l=0$: $R^{(M)}_{k,0}(s_1,\ldots,s_k;-) = R^{(M)}_k(s_1,\ldots,s_k)$.

Let us introduce the following set of biorthogonal polynomials (bOP) in monic normalisation, $p_i(x) = x^i + \ldots$ and $q_j(x) = x^j + \ldots$,

$$\int_0^\infty ds\, dt\, w^{(M)}(s,t)\, p^{(M)}_i(s) q^{(M)}_j(t) = \delta_{ij} h^{(M)}_j\ , \tag{3.2}$$

with squared norms $h^{(M)}_j$ and the weight function defined as

$$w^{(M)}(s,t) \equiv t^{-1}\mathrm{e}^{-t} G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s}{t}\right)\ , \tag{3.3}$$

for $M>1$. These polynomials are guaranteed to exist following the general theory of bOP that was very recently further developed [27], see also [26] for a recent review. We will explicitly construct the bOP for general $M>1$. The general (k,l)-point correlation functions eq. (3.1) are given in terms of four kernels that are constructed from the kernel of bOP [28, 29, 30]:

$$R^{(M)}_{k,l}(s_1,\ldots,s_k;t_1,\ldots,t_l) = \det\begin{bmatrix} H_{01}(s_a,s_b) & H_{00}(s_a,t_j) \\ {\scriptstyle 1\le a,b\le k} & {\scriptstyle 1\le a\le k,\ 1\le j\le l} \\ H_{11}(t_i,s_b) & H_{10}(t_i,t_j) \\ {\scriptstyle 1\le b\le k,\ 1\le i\le l} & {\scriptstyle 1\le i,j\le l}\end{bmatrix}\ . \tag{3.4}$$

The kernel of bOP is defined as

$$K_N(s,t) \equiv \sum_{j=0}^{N-1} \frac{p^{(M)}_j(s) q^{(M)}_j(t)}{h^{(M)}_j}\ . \tag{3.5}$$

All four kernels H_{ab} are based on this relation,

$$H_{01}(s_a,s_b) \equiv \int_0^\infty dt\ K_N(s_a,t)t^{-1}\mathrm{e}^{-t}G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s_b}{t}\right)\ , \tag{3.6}$$

$$H_{00}(s_a,t_j) \equiv t_j^{-1}\mathrm{e}^{-t_j}K_N(s_a,t_j)\ , \tag{3.7}$$

$$\begin{aligned} H_{11}(t_i,s_b) \equiv & \int_0^\infty ds\int_0^\infty dtK_N(s,t)t^{-1}\mathrm{e}^{-t}G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s}{t_i}\right)G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s_b}{t}\right) \\ & - G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s_b}{t_i}\right), \end{aligned} \tag{3.8}$$

$$H_{10}(t_i,t_j) \equiv t_j^{-1}\mathrm{e}^{-t_j}\int_0^\infty ds\,K_N(s,t_j)G^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s}{t_i}\right)\ . \tag{3.9}$$

Note that eq. (3.4) implies that both the one- and two-matrix model jpdf represent determinantal point processes, i.e. for the former eq. (2.13) becomes

$$\mathcal{P}_{\mathrm{jpdf}}(s) = \frac{1}{N!}\det_{1\le a,b\le N}\left[H_{01}(s_a,s_b)\right]\ . \tag{3.10}$$

3.1 The biorthogonal polynomials

In order to compute the bOP let us first determine the bimoment matrix

$$\begin{aligned} I_{ij} &\equiv \int_0^\infty ds\int_0^\infty dt\,w^{(M)}(s,t)s^it^j \\ &= \int_0^\infty dtt^j\mathrm{e}^{-t}\int_0^\infty\frac{ds}{t}s^iG^{M-1,\,0}_{0,\,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\ \frac{s}{t}\right) \\ &= \int_0^\infty dt\,t^{j+i}\mathrm{e}^{-t}(i!)^{M-1} \\ &= (i+j)!(i!)^{M-1}\ , \end{aligned} \tag{3.11}$$

which follows again from an identity for Meijer G-functions, see eq. (A.7) appendix A. The bOP as well as their norms are determined by this bimoment matrix (see e.g. [27]):

$$p_n^{(M)}(s) = \frac{1}{D_n^{(M)}}\det\begin{bmatrix} I_{00} & I_{10} & \ldots & I_{n0} \\ I_{01} & I_{11} & \ldots & I_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ I_{0n-1} & I_{1n-1} & \ldots & I_{nn-1} \\ 1 & s & \ldots & s^n \end{bmatrix}, \tag{3.12}$$

$$q_n^{(M)}(t) = \frac{1}{D_n^{(M)}}\det\begin{bmatrix} I_{00} & I_{10} & \ldots & I_{n-10} & 1 \\ I_{01} & I_{11} & \ldots & I_{n-11} & t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{0n} & I_{1n} & \ldots & I_{n-1n} & t^n \end{bmatrix}, \tag{3.13}$$

where

$$D_n^{(M)} \equiv \det_{0\le i,j\le n-1}[I_{ij}] = \prod_{i=0}^{n-1}(i!)^{M-1}\det_{0\le i,j\le n-1}[(i+j)!]\ , \tag{3.14}$$

$$h_n^{(M)} = D_{n+1}^{(M)}/D_n^{(M)}\ . \tag{3.15}$$

In order to have more explicit expressions it is instructive to compare these equations with the standard Laguerre polynomials $L_n(x)$. We need them in monic normalisation denoted by $\tilde{L}_n(x)$:

$$\tilde{L}_n(x) \equiv (-1)^n n!\, L_n(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \left(\frac{n!}{k!}\right)^2 x^k \ , \tag{3.16}$$

with squared norms

$$\int_0^\infty dx\, \mathrm{e}^{-x} \tilde{L}_n(x)\tilde{L}_m(x) = \delta_{nm}(n!)^2 \equiv \delta_{nm}\, h_n^{(M=1)} \ , \tag{3.17}$$

and a symmetric bimoment matrix

$$I_{ij}\big|_{M=1} \equiv \int_0^\infty dt\, t^{i+j}\, \mathrm{e}^{-t} = (i+j)! \ , \quad \tilde{L}_n(x) = \frac{1}{D_n^{(M=1)}} \det \begin{bmatrix} I_{00}\big|_{M=1} & \dots & I_{0n}\big|_{M=1} \\ \vdots & \vdots & \vdots \\ I_{0n-1}\big|_{M=1} & \dots & I_{nn-1}\big|_{M=1} \\ 1 & \dots & x^n \end{bmatrix} . \tag{3.18}$$

The former equals eqs. (3.11) with $M = 1$. As can be seen the monic Laguerre polynomials also have a determinant representation from the Gram-Schmidt procedure, which is exactly the one in eq. (3.12) at $M = 1$ (or eq. (3.13) as they become equal then).

From the comparison of eqs. (3.12) and (3.13) to eq. (3.18) we can read off the following. For the $q_n(t)$ we take out the common factors $(i!)^{M-1}$ from the first n columns of the determinant, $i = 0, 1, \dots, n-1$, with the remaining determinant being identical to that of the monic Laguerre polynomials. The determinant of the bimoment matrix (3.14) is already written to be proportional to the corresponding one of the Laguerre ensemble. We thus have

$$q_n^{(M)}(t) = \frac{\prod_{i=0}^{n-1}(i!)^{M-1}}{\prod_{i=0}^{n-1}(i!)^{M-1}} \tilde{L}_n(t) = \tilde{L}_n(t) \ , \tag{3.19}$$

for all values of M. Likewise we can read off the squared norms by comparing them to the Laguerre case eq. (3.17):

$$h_n^{(M)} = \frac{\prod_{i=0}^{n}(i!)^{M-1}}{\prod_{i=0}^{n-1}(i!)^{M-1}} h_n^{(M=1)} = (n!)^{M+1} \ . \tag{3.20}$$

This equation is formally redundant for $M = 1$.

For the polynomials $p_n(s)$ the case is slightly more complicated. Also for these polynomials we can take out common factors from all $n+1$ columns, however this will modify the arguments in the last row of the determinant in the numerator: $s^i \to s^i/(i!)^{M-1}$. Expanding with respect to the last row we get a polynomial with the same coefficients as the monic Laguerre polynomials, but now with $s^i/(i!)^{M-1}$ instead of the monomial s^i alone, resulting into

$$\begin{aligned} p_n^{(M)}(s) &= \frac{\prod_{i=0}^{n}(i!)^{M-1}}{\prod_{i=0}^{n-1}(i!)^{M-1}} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \left(\frac{n!}{k!}\right)^2 \frac{s^k}{(k!)^{M-1}} \\ &= \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \left(\frac{n!}{k!}\right)^{M+1} s^k \\ &= (-1)^n (n!)^M \ {}_1F_M(-n; 1, \dots, 1; s). \end{aligned} \tag{3.21}$$

This function can be interpreted as a generalisation of the monic Laguerre polynomials reobtained when setting $M = 1$, see eq. (3.16). Moreover we could express it in terms of the generalised hypergeometric function ${}_1F_M$ which has M arguments equal to 1 in the second set of its indices.

The kernel (3.5) is now completely determined. We proceed by computing the various kernels (3.6), (3.8) and (3.9) by integrating the kernel of bOP.

3.2 The kernels and all correlation functions

We start with the kernel H_{01} that is relevant for the correlation functions of all singular values $s_a = (\lambda_a^{(M)})^2$. We have

$$\begin{aligned} H_{01}(s_a, s_b) &= \sum_{j=0}^{N-1} \frac{1}{h_j^{(M)}} p_j^{(M)}(s_a) \int_0^\infty dt' \tilde{L}_j(t') t'^{-1} \mathrm{e}^{-t'} G^{M-1,0}_{0,\,M-1}\left(\begin{matrix} - \\ 0,\ldots,0 \end{matrix} \middle| \frac{s_b}{t'} \right) \\ &\equiv \sum_{j=0}^{N-1} \frac{1}{h_j^{(M)}} p_j^{(M)}(s_a) \chi_j^{(M)}(s_b) \ , \end{aligned} \tag{3.22}$$

where we introduce the following integral transform

$$\begin{aligned} \chi_j^{(M)}(s_b) &\equiv \int_0^\infty dt' \tilde{L}_j(t') t'^{-1} \mathrm{e}^{-t'} G^{M-1,0}_{0,\,M-1}\left(\begin{matrix} - \\ 0,\ldots,0 \end{matrix} \middle| \frac{s_b}{t'} \right) \\ &= \sum_{i=0}^{j} \frac{(-1)^{j-i}}{(j-i)!} \left(\frac{j!}{i!} \right)^2 \int_0^\infty dt' (t')^{i-1} \, \mathrm{e}^{-t'} G^{M-1,0}_{0,\,M-1}\left(\begin{matrix} - \\ 0,\ldots,0 \end{matrix} \middle| \frac{s_b}{t'} \right) \\ &= \sum_{i=0}^{j} \frac{(-1)^{j-i}}{(j-i)!} \left(\frac{j!}{i!} \right)^2 G^{M,0}_{0,\,M}\left(\begin{matrix} - \\ 0,\ldots,0,i \end{matrix} \middle| s_b \right) \ , \end{aligned} \tag{3.23}$$

upon using the identity eq. (A.9) for $d-1=i$ and $m=M-1$. A more compact expression of χ_j can be found by combining the Rodrigues formula

$$\tilde{L}_j(t') = e^{t'} \left(-\frac{d}{dt'} \right)^j \left(t'^j e^{-t'} \right) \ , \tag{3.24}$$

and the identity (A.13). We substitute $t' \to s_b/t'$ in eq. (3.23) and express the derivative in eq. (3.24) as a derivative in s_b. The integration over t' yields

$$\chi_j^{(M)}(s_b) = \left(-\frac{d}{ds_b} \right)^j \left(s_b^j G^{M,0}_{0,\,M}\left(\begin{matrix} - \\ 0,\ldots,0 \end{matrix} \middle| s_b \right) \right) = (-1)^j G^{M,1}_{1,\,M+1}\left(\begin{matrix} -j \\ 0,\ldots,0 \end{matrix} \middle| s_b \right) . \tag{3.25}$$

The second equality can be obtained by applying the definition (A.1) of Meijer's G-function. We can thus write down the full answer for all k-point correlation functions of the singular values:

$$\begin{aligned} R_k^{(M)}(s_1,\ldots,s_k) &= R_{k,0}^{(M)}(s_1,\ldots,s_k;-) = \det_{1\le a,b\le k} \left[H_{01}(s_a,s_b) \right] \\ &= \det_{1\le a,b\le k} \left[\sum_{j=0}^{N-1} \frac{1}{j!} \, {}_1F_M(-j;1,\ldots,1;s_a) G^{M,1}_{1,\,M+1}\left(\begin{matrix} -j \\ 0,\ldots,0 \end{matrix} \middle| s_b \right) \right] . \end{aligned} \tag{3.26}$$

The simplest example is the density or 1-point correlation function of singular values which is given by

$$R_1^{(M)}(s) = H_{01}(s,s) = \sum_{j=0}^{N-1} \frac{1}{j!} \, {}_1F_M(-j;1,\ldots,1;s) G^{M,1}_{1,\,M+1}\left(\begin{matrix} -j \\ 0,\ldots,0 \end{matrix} \middle| s \right) \ . \tag{3.27}$$

This example will be further discussed in the next subsection 3.3.

The next kernel H_{00} is readily given from its definition eq. (3.7) together with eqs. (3.21), (3.19) and (3.20). We therefore turn to H_{10} from eq. (3.9)

$$\begin{aligned} H_{10}(t_i,t_j) &= {t_j}^{-1}\mathrm{e}^{-t_j}\sum_{l=0}^{N-1}\frac{1}{h_l^{(M)}}\left(\int_0^\infty ds\; p_l^{(M)}(s)G^{M-1,0}_{0,\,M-1}\left(\begin{matrix}-\\0,\ldots,0\end{matrix}\middle|\,\frac{s}{t_i}\right)\right)\tilde{L}_l(t_j) \\ &\equiv {t_j}^{-1}\mathrm{e}^{-t_j}\sum_{l=0}^{N-1}\frac{1}{h_l^{(M)}}\psi_l^{(M)}(t_i)\tilde{L}_l(t_j)\;, \end{aligned} \tag{3.28}$$

where we define the following integral transform

$$\begin{aligned} \psi_l^{(M)}(t) &\equiv \int_0^\infty ds\; p_l^{(M)}(s)G^{M-1,0}_{0,\,M-1}\left(\begin{matrix}-\\0,\ldots,0\end{matrix}\middle|\,\frac{s}{t_i}\right) \\ &= \sum_{i=0}^{l}\frac{(-1)^{l-i}}{(l-i)!}\left(\frac{l!}{i!}\right)^{M+1}\int_0^\infty ds\; s^i\, G^{M-1,0}_{0,\,M-1}\left(\begin{matrix}-\\0,\ldots,0\end{matrix}\middle|\,\frac{s}{t}\right) \\ &= \sum_{i=0}^{l}\frac{(-1)^{l-i}}{(l-i)!}\left(\frac{l!}{i!}\right)^{M+1}t^{i+1}(i!)^{M-1} \\ &= (l!)^{M-1}t\tilde{L}_l(t)\;. \end{aligned} \tag{3.29}$$

In the second step we have used the identity (A.7), which leads us back to the standard Laguerre polynomials. Taking into account the normalisation (3.20) we arrive at the following final result,

$$H_{10}(t_i,t_j) = \frac{t_i}{t_j}\mathrm{e}^{-t_j}\sum_{l=0}^{N-1}\frac{1}{(l!)^2}\tilde{L}_l(t_i)\tilde{L}_l(t_j)\;, \tag{3.30}$$

which is proportional to the kernel of ordinary Laguerre polynomials (3.17). The remaining kernel H_{11} can be expressed in terms of the two integral transforms which we have already computed,

$$H_{11}(t,s) = \sum_{l=0}^{N-1}\frac{1}{h_l^{(M)}}\psi_l^{(M)}(t)\chi_l^{(M)}(s) - \; G^{M-1,0}_{0,\,M-1}\left(\begin{matrix}-\\0,\ldots,0\end{matrix}\middle|\,\frac{s}{t}\right)\;. \tag{3.31}$$

This completes the computation of all (k,l)-point correlation functions in the 2mm, together with eq. (3.4).

Although we will postpone the detailed analysis of the large-N limit to future work let us mention the following nontrivial identity with respect to the kernel H_{11}:

$$\sum_{l=0}^{\infty}\frac{1}{(l!)^2}t\tilde{L}_l(t)\chi_l^{(M)}(s) = G^{M-1,0}_{0,\,M-1}\left(\begin{matrix}-\\0,\ldots,0\end{matrix}\middle|\,\frac{s}{t}\right)\;, \tag{3.32}$$

implying that $\lim_{N\to\infty}H_{11}(t,s)=0$. Assuming that the sum converges and can be integrated piecewise this can be shown as follows. In appendix B we verify that $p_j^{(M)}(s)$ and $\chi_l^{(M)}(s)$ form a set of orthogonal functions with respect to the flat measure, see eq. (3.34). After multiplying both sides of eq. (3.32) with $p_j^{(M)}(s)$ and integrating s over $\mathbb{R}_+$ we obtain

$$(j!)^{M-1}t\tilde{L}_j(t) = \int_0^\infty ds\; p_j^{(M)}(s)G^{M-1,0}_{0,\,M-1}\left(\begin{matrix}-\\0,\ldots,0\end{matrix}\middle|\,\frac{s}{t}\right) = \psi_j^{(M)}(t)\;, \tag{3.33}$$

which is consistent with eq. (3.29).

The identity (3.32) most likely implies that in the naive large-N limit, meaning s, t and M fixed, the correlation function $R_{k,l}^{(M)}$ factorises into s- and t-dependent parts regardless if H_{00} vanishes or not since the determinant (3.4) factorises into a $k \times k$ determinant incorporating H_{01} and a $l \times l$ determinant comprising the block H_{10}. Therefore the correlation functions of the singular values t_j of the matrix X_1 decouple and become the ones of the standard Wishart-Laguerre type. There may be other ways to obtain a nontrivial coupling between singular values s_a and t_j in a more sophisticated large-N limit (like in the so-called weak limit in [25]).

We finally make contact again to the one-matrix model formulation (2.12). The following orthogonality relation which follows from eq. (3.2) is explicitly verified in appendix B

$$\int_0^\infty ds\, p_i^{(M)}(s)\, \chi_j^{(M)}(s) = (i!)^{M+1} \delta_{ij} \ , \tag{3.34}$$

in other words $p_n^{(M)}(s)$ and $\chi_l^{(M)}(s)$ constitute a set of biorthogonal functions with respect to a single variable with flat measure. In particular this relation results into the following property of the kernel H_{01}, see eq. (3.22), that contains the two:

$$\int_0^\infty ds' H_{01}(s,s') H_{01}(s',s'') = H_{01}(s,s'') \ . \tag{3.35}$$

We have thus closed the circle back to the jpdf (2.13) where we could directly replace

$$\begin{aligned} \Delta_N(s) \det_{1\le c,d\le N} \left[G^{M,0}_{0,M} \left(\begin{matrix} - \\ 0,\ldots,0,d-1 \end{matrix} \middle| s_c \right) \right] &= \det_{1\le a,b\le N} [p_{a-1}^{(M)}(s_b)] \det_{1\le c,d\le N} [\chi_{c-1}^{(M)}(s_d)] \\ &= \prod_{j=0}^{N-1} h_j^{(M)} \det_{1\le a,b\le N} [H_{01}(s_a,s_b)] \ . \end{aligned} \tag{3.36}$$

This uses the invariance property of determinants under addition of columns, and then proceeds with standard techniques to deduce the correlation functions. This directly leads from eq. (2.13) to eq. (3.10) for the jpdf. With the property (3.35) of the kernel we deduce eq. (3.26) from Dyson's theorem [24].

3.3 Spectral density, its moments and large-N scaling

In this subsection we discuss in more detail the implications of our results for the spectral density of singular values. Starting from the expression eq. (3.27) which we repeat here in two equivalent forms,

$$\begin{aligned} R_1^{(M)}(s) &= \sum_{l=0}^{N-1} \sum_{i,j=0}^{l} \frac{(-1)^{j+i}(l!)^2}{(l-j)!(l-i)!(i!)^2(j!)^{M+1}} \, G^{M,0}_{0,M} \left(\begin{matrix} - \\ j,\ldots,j,i+j \end{matrix} \middle| s \right) \\ &= \sum_{j=0}^{N-1} \frac{1}{j!} \, {}_1F_M(-j;1,\ldots,1;s) G^{M,1}_{1,M+1} \left(\begin{matrix} -j \\ 0,\ldots,0 \end{matrix} \middle| s \right) \ , \end{aligned}$$

we can explicitly compute expectation values for the moments for finite-N. Starting from the first expression we obtain

$$\mathbb{E}[s^k] \equiv \frac{1}{N} \int_0^\infty ds\ s^k R_1^{(M)}(s) \tag{3.37}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} \sum_{i,j=0}^{l} \frac{(-1)^{j+i}(l!)^2 (i+j+k)! ((j+k)!)^{M-1}}{(l-j)!(l-i)!(i!)^2 (j!)^{M+1}} \ . \tag{3.38}$$

Here we have normalised by eq. (2.19) and we have used again the identity eq. (A.7) for moments of the Meijer G-function. On the other hand using the compact expression for $\chi_j^{(M)}(s)$ in the second formulation of the density we can obtain a more concise result in the following way:

$$\begin{aligned}
\mathbb{E}[s^k] &= \frac{1}{N}\sum_{j=0}^{N-1}\frac{1}{h_j^{(M)}}\sum_{l=0}^{j}\frac{(-1)^{j-l}}{(j-l)!}\left(\frac{j!}{l!}\right)^{M+1}\int_0^\infty ds\, s^{l+k}(-1)^j G^{M,1}_{1,\,M+1}\left(\begin{matrix}-j\\0,\ldots,0\end{matrix}\middle|\, s\right)\\
&= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{l=0}^{j}\frac{(-1)^{j-l}}{(j-l)!(k+l-j)!}\left(\frac{(k+l)!}{l!}\right)^{M+1}\\
&= \frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{(k+l)!}{l!}\right)^{M+1}\sum_{j=0}^{N-1-l}\frac{(-1)^j}{j!(k-j)!}\\
&= \frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{(k+l)!}{l!}\right)^{M+1}\int_0^{2\pi}\frac{d\varphi}{2\pi}\frac{1}{k!}(1-\mathrm{e}^{\imath\varphi})^k\frac{1-\mathrm{e}^{-\imath(N-l)\varphi}}{1-\mathrm{e}^{-\imath\varphi}}\\
&= \frac{1}{N}\sum_{l=0}^{N-1}\frac{(-1)^{N-l-1}}{k!}\left(\frac{(k+l)!}{l!}\right)^{M+1}\binom{k-1}{N-l-1}.
\end{aligned} \tag{3.39}$$

We use the convention that inverse powers of factorials of negative integers give zero, rather than using the Gamma-function everywhere. In the first step we have employed eq. (A.8). Interestingly, the remaining sum can be further expressed in terms of a hypergeometric function if $k \geq N$,

$$\mathbb{E}[s^k] = (-1)^{N-1}\frac{(k!)^{M-1}(k-1)!}{N!\Gamma(k-N+1)}\;{}_{M+2}F_{M+1}(k+1,\ldots,k+1,1-N;1,\ldots,1,k-N+1;1)\;, \tag{3.40}$$

by extending the sum to infinity and comparing their Taylor series. Indeed this relation can be generalised to $k < N$. Notice that in this case the singular contributions in the hypergeometric function cancel with those in the Gamma-function in the denominator.

In the ensuing discussion we will need in particular the first moment $F_N^{(M)}$ for $k = 1$ when rescaling the density, which can be readily read off

$$\mathbb{E}[s] \equiv F_N^{(M)} = N^M\;. \tag{3.41}$$

It agrees with the known case for $M = 1$. An alternative short derivation for the first moment is sketched in appendix B. Higher moments easily follow from eq. (3.39), e.g. for the second moment we have

$$\mathbb{E}[s^2] = \frac{1}{2}N^M\left((N+1)^{M+1} - (N-1)^{M+1}\right)\;. \tag{3.42}$$

We illustrate our results for the density (3.27) by plotting it for various values of N and M. As a first example in fig. 1 the density is shown for $M = 1, 2, 3$ at fixed $N = 4$. Clearly it is mandatory to know the right scale dependence of the correlation functions on N and M. This means that after properly rescaling the bulk of the singular values is of order one, in order to be able to compare the density for finite N at different values of N. In particular it is important to check the finite N-results against the limiting large-N behaviour for different M, which has been derived for products of quadratic [6, 7] and rectangular matrices [8, 9].

Let us explain our procedure. First we normalise our density to unity, using eq. (2.19). Then we rescale the density by its first moment,

$$\hat{R}_1^{(M)}(x) \equiv \frac{1}{N}F_N^{(M)}R_1^{(M)}\left(F_N^{(M)}x\right)\;, \tag{3.43}$$

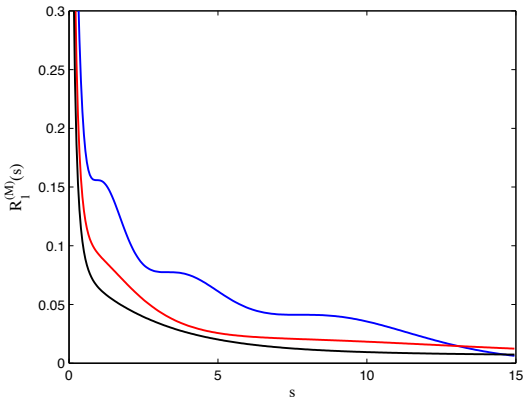

Figure 1: Comparison of the density eq. (3.27) $R_1^{(M)}(s)$ for fixed $N = 4$ without rescaling. The values $M = 1, 2, 3$ correspond to the top (blue), middle (red) and bottom (black) curve, respectively.

so that the new density $\hat{R}_1^{(M)}(x)$ has norm and first moment equal to unity[3]. Notice that this rescaling is an alternative to the unfolding procedure onto the scale of the local mean level spacing. Instead of fixing the mean distance between two successive singular values, we fix here the singular values themselves, such that they are always of order one. This is exactly the macroscopic limit.

Inserting eq. (3.41) we thus obtain a limiting density, which we denote by,

$$\hat{\rho}^{(M)}(x) \equiv \lim_{N\to\infty} \hat{R}_1^{(M)}(x) = \lim_{N\to\infty} N^{M-1} R_1^{(M)}\left(N^M x\right) \ , \tag{3.44}$$

that also has norm and first moment unity. Note that the known limiting density from the literature may still have to be rescaled accordingly. We can then compare $\hat{\rho}^{(M)}(x)$ and $\hat{R}_1^{(M)}(x)$ for different values of N at fixed M.

This procedure can be illustrated with the simplest case $M = 1$, the well known Wishart Laguerre ensemble. At large-N we have

$$\lim_{N\gg 1} R_1^{(M=1)}(s) \approx \frac{1}{2\pi}\sqrt{\frac{4N-s}{s}}\Theta(4N-s), \tag{3.45}$$

which is the N-dependent Marchenko-Pastur density, and $\Theta(x)$ denotes the Heaviside function. With the first moment being given by $F_N^{(M=1)} = N$ we thus have

$$\hat{\rho}^{(M=1)}(x) = \lim_{N\to\infty} N^0 R_1^{(M=1)}\left(Nx\right) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\ \Theta(4-x) \ , \tag{3.46}$$

for the rescaled and normalised density. This is the Marchenko-Pastur density with compact support on $(0, 4]$. A comparison between $\hat{\rho}^{(M=1)}(x)$ and $\hat{R}_1^{(M=1)}(x)$ for various values of $N = 3, 4, 5$ and 10 is given in fig. 2 (left figure).

For $M > 1$ the limiting expression for the density is not as explicit as in eq. (3.46). Here we will follow the notation of [9] where a polynomial equation for the resolvent $G(z)$ was derived, which we display for the case of quadratic matrices only:

$$\left(zG^{(M)}(z)\right)^{M+1} = z\left(zG^{(M)}(z) - 1\right) \ . \tag{3.47}$$

[3]All densities satisfying this property will be denotes with a hat "$\widehat{\ }$".

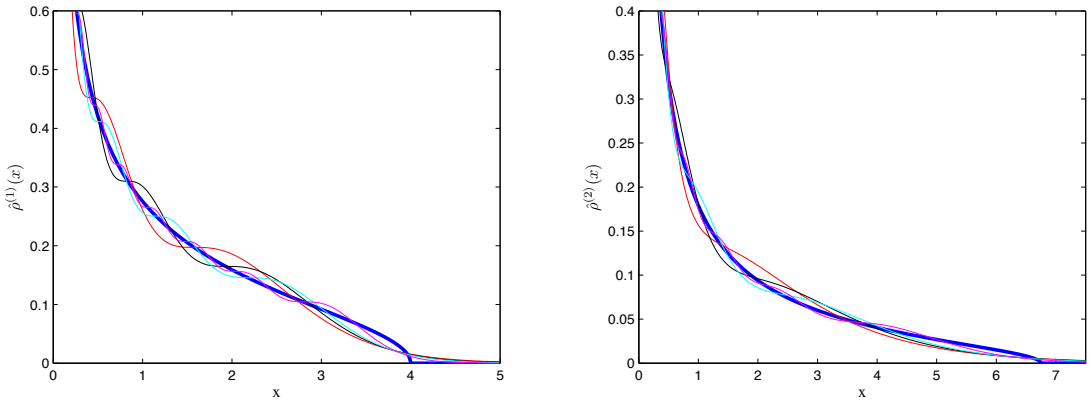

Figure 2: Left plot: the rescaled density $R_1^{(M=1)}(Nx)$ eq. (3.27) for $M = 1$ and $N = 3, 4, 5, 10$ (corresponding to red, black, cyan, and magenta, respectively) versus the limiting Marchenko-Pastur density eq. (3.46) (thick line). Right plot: the density eq. (3.27) for $M = 2$ rescaled as $NR_1^{(M=2)}(N^2 s)$, for $N = 3, 4, 5, 10$ compared to the limiting density $\hat{\rho}^{(M=2)}(x)$ (thick line).

The resolvent is related to the limiting spectral density by

$$G^{(M)}(z) \equiv \int_0^\infty d\lambda \frac{\rho^{(M)}(\lambda)}{z - \lambda} \ , \tag{3.48}$$

where z is outside the support of the spectral density. This relation can be inverted as follows:

$$\rho^{(M)}(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \Im m \left(G^{(M)}(\lambda + i\epsilon) \right) \ . \tag{3.49}$$

For $M = 1$ eq. (3.47) reduces to a quadratic equation, which after taking the discontinuity along the support eq. (3.49) leads to eq. (3.46), without further rescaling.

Increasing to $M = 2$ the equation becomes cubic and we can still write out its solution, which is chosen subject to boundary conditions and to yield a real density on the support $(0, 3^3/2^2]$:[4]

$$\begin{aligned} G^{(M=2)}(z) &= \frac{1}{\sqrt{3z}} \left(A_-^{-1/3}(z) + A_-^{1/3}(z) \right) = \frac{1}{\sqrt{3z}} \left((-A_+(z))^{1/3} + A_-^{1/3}(z) \right) \ , \\ A_\pm(z) &\equiv \sqrt{\frac{27}{4z} - 1} \pm \sqrt{\frac{27}{4z}} \ . \end{aligned} \tag{3.50}$$

The density $\hat{\rho}^{(M=2)}(s)$ that is obtained from eq. (3.50) by taking the discontinuity according to eq. (3.49) (which happens to have the first moment equal to unity without further rescaling) is shown in fig. 2 (right figure) in comparison to our rescaled finite-N result $\hat{R}_1^{(M=2)}(s)$ for various values of N. As in the known case $M = 1$ we obtain a nice agreement for $M = 2$. An alternative derivation of $\hat{\rho}^{(M=2)}(s)$ via multiple orthogonal polynomials was recently presented in Refs. [3, 31], with which our density agrees.

As a final remark for larger M it might be useful to resolve the singularity of the density at the origin [9], $\lim_{s \to 0} \hat{\rho}^{(M)}(s) \sim s^{-M/(M+1)}$, by changing variables, just as it is well known that for $M = 1$ a change to squared variables maps the Marchenko-Pastur density eq. (3.46) to the semi-circle. A more detailed comparison to existing large-N results for the density, its moments and support would require a careful asymptotic analysis of the special functions constituting our finite-N density, and is postponed to future work.

[4] We thank Z. Burda for indicating how to determine the limiting support.

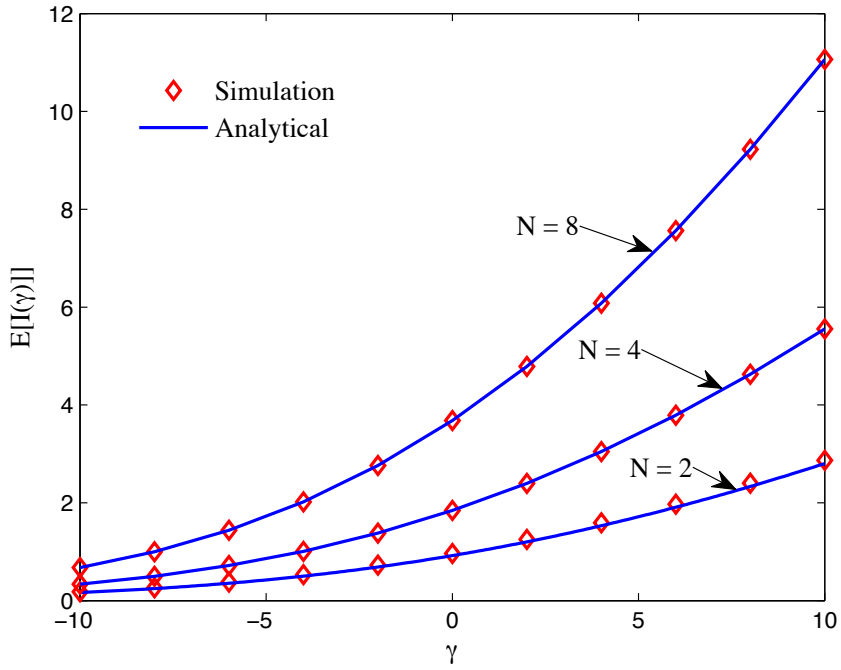

Figure 3: The ergodic mutual information of multi-layered scattering MIMO channels with a fixed number of 2 clusters ($M = 3$) with a different number of scatters $N = 2, 4, 8$.

4 Application to telecommunication

Consider a MIMO network with a single source and destination, both equipped with N antennas. Information transmitted by the source is conveyed to the destination via $M - 1$ successive clusters of scatterers, where each cluster (layer) is assumed to have N scattering objects. Such a channel model proposed in [8] is typical in modelling the indoor propagation of information between different floors [32].

We assume that the vector-valued transmitted signal propagates from the transmitter array to the first cluster, from the first to the second cluster, and so on, until it is received from the $(M-1)$-st cluster by the receiver antenna array. Each communication channel is described by a random complex Gaussian matrix, and as a result the effective channel of this multi-layered model equals the product matrix P_M, see eq. (2.1).

For the described communication channels, the mutual information measured in units of the natural logarithm (nats) per second per Hertz is defined as

$$\mathcal{I}(\gamma) \equiv \ln\det\left(I_N + \frac{\gamma}{N^M} P_M P_M^\dagger\right) = \sum_{a=1}^{N} \ln\left(1 + \frac{\gamma}{N^M} s_a\right), \tag{4.1}$$

where γ defines the average received signal-to-noise ratio per antenna which is a constant. We employ the distribution (3.27) of squared singular values to compute its average. The quantity of interest is called the ergodic mutual information of such channels. It is given by the expectation value of the

random variable $\mathcal{I}(\gamma)$. Using the analogue of the expression (3.37) from the previous section we have

$$\begin{aligned}\mathbb{E}\left[\mathcal{I}(\gamma)\right] &= N\mathbb{E}\left[\ln\left(1+\frac{\gamma}{N^M}s\right)\right] \\ &= \sum_{l=0}^{N-1}\sum_{i,j=0}^{l}\frac{(-1)^{i+j}(l!)^2}{(l-j)!(l-i)!(i!)^2(j!)^{M+1}}\int_0^\infty ds\; G_{0,\,M}^{M,\,0}\left(\begin{array}{c}-\\ j,\ldots,j,i+j\end{array}\,\middle|\, s\right)\ln\left(1+\frac{\gamma}{N^M}s\right) \\ &= \sum_{l=0}^{N-1}\sum_{i,j=0}^{l}\frac{(-1)^{i+j}(l!)^2}{(l-j)!(l-i)!(i!)^2(j!)^{M+1}}G_{2,\,M+2}^{M+2,\,1}\left(\begin{array}{c}0,1\\ 0,0,j+1,\ldots,j+1,i+j+1\end{array}\,\middle|\,\frac{N^M}{\gamma}\right). \quad (4.2)\end{aligned}$$

The last step is obtained by first replacing the logarithm by a Meijer G-function, eq. (A.5), and then applying the integral identity (A.14) from the appendix. Note that the corresponding ergodic mutual information for the traditional MIMO channel model, i.e. $M = 1$, was derived in [33].

In order to get an independent confirmation of our analytical result (4.2) we compare it to numerical simulations as follows. We plot the ergodic mutual information (4.2) in fig. 3 against Monte-Carlo simulations as a function of γ in decibel (dB) for a given number of clusters, $M-1$, as an example, with different numbers of scatters per cluster, N. Each simulated curve is obtained by averaging over 10^6 independent realisations of P_M. The statistical error bar is smaller than the symbol for the simulation in our plots. The comparison in fig. 3 shows a 2-layered scattering channel, i.e. $M = 3$, with the number of scatters per layer varying from $N = 2,\ 4,$ to 8. We have also compared our results to simulations for other values of N and M.

5 Conclusions and open questions

In this paper we have derived the joint probability distribution function (jpdf) of singular values for any finite product of M quadratic random matrices of finite size $N \times N$, with complex elements distributed according to a Gaussian distribution. This generalises the Wishart-Laguerre (also called chiral Gaussian) Unitary Ensemble which we recover for $M = 1$. Starting from the jpdf we have computed all k-point density correlation functions of the singular values, by taking a detour over a two-matrix model like representation of the same model. In that way we showed that the jpdf being proportional to a Vandermonde times the determinant of Meijer G-functions represents a determinantal point process. Its kernel of orthogonal functions generates all k-point functions in the standard way using Dyson's theorem. We also solved the auxiliary two-matrix model that couples a single matrix to the product of M matrices, by constructing the biorthogonal polynomials explicitly, as well as the corresponding four kernels with their integral transforms. On the way we found some nontrivial identities and integral representations of the Meijer G-function.

The density of the singular values are discussed in more detail at finite N and M, including all its moments. We identified the macroscopic scaling to match the density with the known large-N results for the macroscopic density of singular values. As a further application we have computed the averaged mutual information for multi-layered scattering of MIMO channels and have compared them to Monte-Carlo simulations for small M and N.

Previous results for the macroscopic large-N density of the singular values of quadratic or rectangular matrices and its expectation values of traces have mainly been obtained from probabilistic methods, in particular using free random variables. The explicit results that we have obtained for the jpdf and all correlation functions thereof open up the possibility of another direction. One can now investigate the microscopic scaling limits zooming into various parts of the spectrum, by performing the asymptotic analysis of the orthogonal polynomials and their integral transforms that we computed. Since the ensemble represents a determinantal point process one can also investigate the

limiting distribution of individual eigenvalues, e.g. by considering their Fredholm determinant representation. Moreover, one can now study the distributions of linear statistics such as the trace of the derived ensemble as well.

Based on known results for the universality of the spectrum of random matrices we expect the following outcome of such an analysis. The bulk and the soft edge behaviour of the spectrum should be governed by the universal Sine- and Airy-kernel, respectively, after unfolding and scaling appropriately. This includes that at the soft edge we expect to find the Tracy-Widom distribution, and it will be very interesting to identify the right scaling for that. In contrast at the hard edge we expect to find new universality classes labelled by M. Already the way the macroscopic density diverges at the origin depends on it. This would fit into what was found recently for the complex eigenvalue spectrum of products of independent Ginibre matrices.

Other open problems include a generalisation of our construction to products of rectangular matrices, which seems quite feasible. Also the inclusion of determinants (or characteristic polynomials) into the weight, which should be related to rectangular matrices, seems within reach. On the other hand going to non-Gaussian weight functions or investigating other symmetry classes with real or quaternion real matrix elements are very challenging problems. The reason is that our method depends crucially on the Harish-Chandra–Itzykson-Zuber group integral to eliminate the angular variables. For the other symmetry classes no such explicit tool is presently at hand.

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A Some integral identities for Meijer G-functions

In this appendix we collect a few integral representations and identities for the so-called Meijer G-function. It is defined as [23]

$$G^{m,n}_{p,q}\left(\begin{array}{cccc} a_1, & a_2, & \ldots, & a_p \\ b_1, & b_2, & \ldots, & b_q \end{array} \middle| z \right) = \frac{1}{2\pi i} \int\limits_{\mathcal{C}} du \, \frac{\prod_{j=1}^{m} \Gamma(b_j - u) \prod_{j=1}^{n} \Gamma(1 - a_j + u)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + u) \prod_{j=n+1}^{p} \Gamma(a_j - u)} \, z^u \; . \tag{A.1}$$

The contour of integration $\mathcal{C}$ goes from $-i\infty$ to $+i\infty$ such that all poles of the Gamma functions related to the b_j lie to the right of the path, and all poles related to the a_j to the left of the path. We are in particular interested in the case

$$G^{m,\,0}_{0,\,m}\left(\begin{matrix} - \\ b_1, \ldots, b_m \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int\limits_{\mathcal{C}} du \, z^u \prod_{j=1}^{m} \Gamma(b_j - u) \; . \tag{A.2}$$

Note that the function is symmetric in all its indices $b_1, \ldots, b_m$. Special cases for small m are given by [23]

$$G^{1,\,0}_{0,\,1}\left(\begin{matrix} - \\ b_1 \end{matrix} \middle| z \right) = z^{b_1} \mathrm{e}^{-z} \; , \tag{A.3}$$

$$G^{2,\,0}_{0,\,2}\left(\begin{matrix} - \\ b_1, b_2 \end{matrix} \middle| z \right) = 2 z^{(b_1+b_2)/2} K_{b_1 - b_2}(2\sqrt{z}) \; , \tag{A.4}$$

and [34]

$$G^{1,\,2}_{2,\,2}\left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\,\middle|\, z\right) \;=\; \ln(1+z)\;, \tag{A.5}$$

where K is the modified Bessel function of the second kind equivalently known as the Macdonald function. It is possible to absorb powers of the argument of the Meijer G-function, due to the following shift [23]

$$z^k G^{m,n}_{p,q}\left(\begin{array}{ccc} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{array}\,\middle|\, z\right) = G^{m,n}_{p,q}\left(\begin{array}{ccc} a_1+k, & \ldots, & a_p+k \\ b_1+k, & \ldots, & b_q+k \end{array}\,\middle|\, z\right). \tag{A.6}$$

The integral identities we collect here are for the moments of the Meijer G-function:

$$\int_0^\infty dt\, t^{n-1} G^{m,\,0}_{0,\,m}\left(\begin{smallmatrix} - \\ b_1,\ldots,b_m \end{smallmatrix}\,\middle|\, t\right) \;=\; \prod_{j=1}^m \Gamma(b_j+n)\;, \tag{A.7}$$

$$\int_0^\infty ds\, s^n G^{M,\,1}_{1,\,M+1}\left(\begin{smallmatrix} -j \\ 0,\ldots,0 \end{smallmatrix}\,\middle|\, s\right) \;=\; \lim_{\varepsilon\to 0}\frac{(n!)^M\Gamma(j-n+\varepsilon)}{\Gamma(-n+\varepsilon)} = (-1)^j\frac{(n!)^{M+1}}{\Gamma(n-j+1)}\;. \tag{A.8}$$

We only state two particular cases, for the most general setting see e.g. [34]. In the last step of the second identity we have employed the recursion of the Gamma-function, $\Gamma(j-n+\varepsilon) = \Gamma(-n+\varepsilon)\prod_{l=0}^{j-1}(l-n+\varepsilon)$.

The second integral identity needed in section 2 follows easily from the representation eq. (A.2):

$$\begin{aligned} &\int_0^\infty dt\, t^{d-2}\mathrm{e}^{-t} G^{m,\,0}_{0,\,m}\left(\begin{smallmatrix} - \\ b_1,\ldots,b_m \end{smallmatrix}\,\middle|\, \frac{s}{t}\right) = \int_0^\infty \frac{dv}{v}\left(\frac{s}{v}\right)^{d-1}\mathrm{e}^{-s/v} G^{m,\,0}_{0,\,m}\left(\begin{smallmatrix} - \\ b_1,\ldots,b_m \end{smallmatrix}\,\middle|\, v\right) \\ =\;& \frac{1}{2\pi i}\int_{\mathcal{C}} du \prod_{j=1}^m \Gamma(b_j-u)\int_0^\infty dt\, t^{d-2}\mathrm{e}^{-t}\left(\frac{s}{t}\right)^u = \frac{1}{2\pi i}\int_{\mathcal{C}} du \prod_{j=1}^m \Gamma(b_j-u)\, s^u\Gamma(d-1-u) \\ =\;& G^{m+1,\,0}_{0,\,m+1}\left(\begin{smallmatrix} - \\ b_1,\ldots,b_m,d-1 \end{smallmatrix}\,\middle|\, s\right)\;. \end{aligned} \tag{A.9}$$

In the first line we simply substituted $t \to v = s/t$ to obtain a second version of the identity. The rest follows from the definition of the Meijer G- and the Gamma-function. Notice that for $d = 1$ and $b_1 = \ldots = b_m = 0$ this recursion for the Meijer G-function was already derived in [16].

We are now prepared to show the multiple integral representation of the Meijer G-function that we need in the derivation of the jpdf in section 2. It slightly generalises the representation found in [16]. The statement is that

$$G^{m,\,0}_{0,\,m}\left(\begin{smallmatrix} - \\ 0,\ldots,0,b \end{smallmatrix}\,\middle|\, \frac{x_m}{x_0}\right) = \int_0^\infty \frac{dx_1}{x_1}\left(\frac{x_1}{x_0}\right)^b \int_0^\infty \frac{dx_2}{x_2}\cdots\int_0^\infty \frac{dx_{m-1}}{x_{m-1}}\prod_{j=1}^m \mathrm{e}^{-x_j/x_{j-1}}\;, \tag{A.10}$$

for $m > 1$. Comparing this equation to eq. (2.11) with $b = d-1$, we work with $m = M$ squared singular values, $x_j = (\lambda_c^{(j)})^2$, where we introduce a dummy variable x_0. The variable x_0 will be set to unity for our original purposes. However, it will be useful when applying the identity to eq. (2.21). Our proof goes by induction in m. For $m = 2$ [23] we have

$$\int_0^\infty \frac{dx_1}{x_1}\left(\frac{x_1}{x_0}\right)^b \mathrm{e}^{-\frac{x_1}{x_0}-\frac{x_2}{x_1}} = 2\left(\frac{x_2}{x_0}\right)^{b/2} K_{-b}\left(2\sqrt{x_2/x_0}\right) = G^{2,\,0}_{0,\,2}\left(\begin{smallmatrix} - \\ 0,b \end{smallmatrix}\,\middle|\, \frac{x_2}{x_0}\right), \tag{A.11}$$

where the last step is due to eq. (A.4). This leads to

$$\int_0^\infty \frac{dx_2}{x_2} G^{2,0}_{0,2}\left(\begin{matrix}-\\ 0,b\end{matrix}\middle|\,\frac{x_2}{x_0}\right) \mathrm{e}^{-x_3/x_2} = G^{3,0}_{0,3}\left(\begin{matrix}-\\ 0,b,0\end{matrix}\middle|\,\frac{x_3}{x_0}\right) \ , \tag{A.12}$$

for $m = 3$, where we have applied the identity eq. (A.9) in its second form shown in the first line, with $d = 1$ and $v = x_2/x_0$. For the induction step $m - 1 \to m$ we simply have to repeat the same procedure, which follows easily from the very same identity

$$\int_0^\infty \frac{dx_m}{x_m} G^{m,0}_{0,m}\left(\begin{matrix}-\\ 0,\ldots,0,b\end{matrix}\middle|\,\frac{x_m}{x_0}\right) \mathrm{e}^{-x_{m+1}/x_m} = G^{m+1,0}_{0,m+1}\left(\begin{matrix}-\\ 0,\ldots,0,b\end{matrix}\middle|\,\frac{x_{m+1}}{x_0}\right) \ , \tag{A.13}$$

which completes the proof.

Note that the same identity (A.10) can be used to provide the second step in eq. (2.21), when setting $b = 0$, $m = M - 1$ and shifting the indices of the variables $x_{j-1} = (\lambda_c^{(j)})^2$ for $j = 1, \ldots, M$. This is the reason why x_0 is useful.

Finally we state an integral identity needed in section 4, concerning the integral of two Meijer G-functions,

$$\int_0^\infty ds\; G^{1,2}_{2,2}\left(\begin{matrix}1 & 1\\ 1 & 0\end{matrix}\middle|\, s\frac{\gamma}{N}\right) G^{M,0}_{0,M}\left(\begin{matrix}-\\ j,\ldots,j,i+j\end{matrix}\middle|\, s\right) = \frac{N}{\gamma} G^{M+2,1}_{2,M+2}\left(\begin{matrix}-1,0\\ -1,-1,j,\ldots,j,i+j\end{matrix}\middle|\,\frac{N}{\gamma}\right). \tag{A.14}$$

Notice that it is a particular choice of a general formula [34]. In order to arrive at eq. (4.2) we apply the shift (A.6).

B Orthogonality check and first moment

In this appendix we explicitly confirm both the orthogonality (3.2) of the bOP $p_n^{(M)}(s)$ and $q_l^{(M)}(t)$ with respect to two variables, as well as the fact that $p_n^{(M)}(s)$ and $\chi_l^{(M)}(s)$ constitute a set of biorthogonal functions with respect to one variable with flat measure, eq. (3.34). Although being true by construction we will see the orthogonality ultimately boils down to the standard orthogonality of Laguerre polynomials.

The biorthogonal polynomials that were constructed in subsection 3.1 using the bimoment matrix must automatically satisfy the orthogonality relation (3.2). We will check this here independently, which implies at the same time that one of the polynomials and the integral transform (3.23) of the other are orthogonal functions (as they should be, in order to constitute proper kernels):

$$\begin{aligned}
&\int_0^\infty ds\, p_i^{(M)}(s)\, \chi_j^{(M)}(s) = \int_0^\infty ds \int_0^\infty dt\, w^{(M)}(s,t)\, p_i^{(M)}(s) q_j^{(M)}(t)\\
=\ &\int_0^\infty dt\, \mathrm{e}^{-t} \tilde{L}_j(t) \sum_{k=0}^{i} \frac{(-1)^{i-k}}{(i-k)!} \left(\frac{i!}{k!}\right)^{M+1} \int_0^\infty \frac{ds}{t} G^{M-1,0}_{0,M-1}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\middle|\,\frac{s}{t}\right) s^k\\
=\ &\int_0^\infty dt\, \mathrm{e}^{-t} \tilde{L}_j(t) \sum_{k=0}^{i} \frac{(-1)^{i-k}}{(i-k)!} \left(\frac{i!}{k!}\right)^{M+1} t^k (k!)^{M-1} = (i!)^{M-1} \int_0^\infty dt\, \mathrm{e}^{-t} \tilde{L}_j(t) \tilde{L}_i(t)\\
=\ &(i!)^{M+1} \delta_{ij} \ .
\end{aligned} \tag{B.1}$$

Here we have used the identity (A.7) for moments of the Meijer G-function, which cancels the extra factorials in the generalised Laguerre polynomials $p_i^{(M)}(s)$, see eq. (3.21) after the first integration over $s' = s/t$. The last step follows from the known orthogonality of Laguerre type.

In the second part of this appendix we provide a much simpler, probabilistic argument that leads to the first moment eq. (3.41). Noting that

$$\mathbb{E}\left[X_j X_j^\dagger\right] = N I_N \ , \quad \text{for} \quad j = 1, \ldots, L \tag{B.2}$$

we have that

$$\begin{aligned} \mathbb{E}[s] &= \frac{1}{N}\mathbb{E}\left[\sum_{a=1}^{N} s_a\right] = \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\,(P_M P_M^\dagger)\right] \\ &= \frac{1}{N}\mathrm{Tr}\left(\prod_{j=1}^{M}\mathbb{E}\left[X_j X_j^\dagger\right]\right) = \frac{1}{N}N^M \mathrm{Tr}\,(I_N) = N^M \ , \end{aligned} \tag{B.3}$$

where we have reordered successively the X_j under the trace in the first line.

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Products of Rectangular Random Matrices: Singular Values and Progressive Scattering

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We discuss the product of M rectangular random matrices with independent Gaussian entries, which have several applications including wireless telecommunication and econophysics. For complex matrices an explicit expression for the joint probability density function is obtained using the Harish-Chandra–Itzykson–Zuber integration formula. Explicit expressions for all correlation functions and moments for finite matrix sizes are obtained using a two-matrix model and the method of bi-orthogonal polynomials. This generalises the classical result for the so-called Wishart–Laguerre Gaussian unitary ensemble (or chiral unitary ensemble) at $M = 1$, and previous results for the product of square matrices. The correlation functions are given by a determinantal point process, where the kernel can be expressed in terms of Meijer G-functions. We compare the results with numerical simulations and known results for the macroscopic level density in the limit of large matrices. The location of the endpoints of support for the latter are analysed in detail for general M. Finally, we consider the so-called ergodic mutual information, which gives an upper bound for the spectral efficiency of a MIMO communication channel with multi-fold scattering.

I. INTRODUCTION

Random Matrix Theory has existed for more than half a century, and its success is undeniable. A vast number of applications is known within the mathematical and physical sciences, and beyond; we refer to [1] for a recent overview. A direction within Random Matrix Theory, which has recently caught renewed attention is the study of products of random matrices. Among others, products of matrices have been applied to disordered and chaotic systems [2], matrix-valued diffusions [3, 4], quantum chromodynamics at finite chemical potential [5, 6], Yang–Mills theory [7–9], finance [10] and wireless telecommunication [11]. In this paper, our attention will be directed towards the latter.

When considering products of matrices we are faced with the fact that the product often possesses less symmetries than the individual matrices. For example a product of symmetric matrices will not be symmetric in general. For simplicity, we will look at matrices with a minimum of symmetry. Our discussion will concern products of matrices drawn from the Wishart ensemble. Thus the matrices have independently, identically distributed Gaussian entries. Also other proposals exist, e.g. by multiplying matrices that are chosen from a set of *fixed* matrices with a given probability. This problem has applications in percolation as was pointed out in [12]. However it considerably differs from our approach, notably due to the lack of invariance.

The statistical properties of the complex eigenvalues and real singular values of a product of matrices from the Wishart ensemble have been discussed in several papers (in the former case they are usually called Ginibre matrices). Macroscopic properties for eigenvalues of complex ($\beta = 2$) matrices have been discussed in the limit of large matrices using diagrammatic methods [4, 13, 14], while proofs are given in [15, 16]. The macroscopic behaviour of the singular values and their moments have also been discussed in the literature using probabilistic methods [17–19] as well as diagrammatic methods [14].

Recently, the discussion of products of matrices from Wishart ensembles has been extended to matrices of finite size [20–23], but this discussion has so far been limited to the case of square matrices. We want to extend this discussion to include products of rectangular matrices. In particular, we consider the product matrix

$$\mathbb{Y}_M = \mathbb{X}_M \mathbb{X}_{M-1} \cdots \mathbb{X}_1, \tag{1}$$

where $\mathbb{X}_m$ are $N_m \times N_{m-1}$ real ($\beta = 1$), complex ($\beta = 2$) or quaternion ($\beta = 4$) matrices from the Wishart ensemble. This paper is concerned with the singular values of such matrices, and the spectral correlation functions of $\mathbb{Y}_M \mathbb{Y}_M^\dagger$. A discussion of the complex eigenvalues is postponed to a future publication [24].

Matrix products like $\mathbb{Y}_M$ have direct applications in finance [10] , wireless telecommunication [17] and quantum entanglement [25, 26]. The importance of the generalisation from square to rectangular matrices is evident from its applications to e.g. wireless telecommunication. Let us consider a MIMO (Multiple–Input Multiple–Output) communication channel from a single source to a single destination via $M - 1$ clusters of scatterers. The source and destination are assumed to be equipped with N_0 transmitting and N_M receiving antennas, respectively. Each cluster of scatterers is assumed to have N_m ($1 \leq m \leq M-1$) scattering objects. Such a communication link is canonically described by a channel matrix identical to the complex version of the product matrix (1). Here the Gaussian nature of the matrix entries models a Rayleigh fading environment. This model was proposed in [17], while the single channel model ($M = 1$)

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goes back to [27–29]. There is no reason to assume that the number of scattering object at each cluster in such a communication channel should be identical, which illustrates the importance of the generalisation to rectangular matrices.

This paper will be organised as follows: In section II we will find the joint probability density function for the singular values of the product matrix (1) in the complex case. Starting with general $\beta = 1, 2, 4$ it turns out that the restriction to complex ($\beta = 2$) matrices is necessary, since our method relies on the Harish-Chandra–Itzykson–Zuber integration formula for the unitary group [30, 31]. An explicit expression for all k-point correlation functions for the singular values will be derived in section III using a two-matrix model and the method of bi-orthogonal polynomials. The spectral density and its moments will be discussed further in section IV, while we return to the above mentioned communication channel in section V. Section VI is devoted to conclusions and outlook. Some properties and identities for the special functions we encounter are collected in appendix A.

II. JOINT PROBABILITY DISTRIBUTION OF SINGULAR VALUES

As mentioned in the introduction we are interested in the statistical properties of the singular values of the product matrix (1), which is governed by the following partition function,

$$Z_\beta^M = \prod_{m=1}^{M} \int |D\mathbb{X}_m| \exp[-\operatorname{Tr} \mathbb{X}_m \mathbb{X}_m^\dagger]. \tag{2}$$

Here $D\mathbb{X}_m$ denotes the Euclidean volume, i.e. the exterior product of all independent one-forms, while $|D\mathbb{X}_m|$ is the corresponding unoriented volume element.

Let us assume that the smallest dimension is $N_0 = N_{\min}$. We stress that the properties of the non-zero singular values of $\mathbb{Y}_M$ are completely independent of this choice, see [24]. Thus, the product matrix, $\mathbb{Y}_M = \mathbb{X}_M \cdots \mathbb{X}_1$, has maximally rank N_0. It follows that the product matrix can be parameterised as [24]

$$\mathbb{Y}_M = \mathcal{U}_M \begin{pmatrix} Y_M \\ 0 \end{pmatrix}, \tag{3}$$

where Y_M is a square $N_0 \times N_0$ matrix with real, complex or quaternion entries, while $\mathcal{U}_M$ is an orthogonal, a unitary or a unitary symplectic matrix for $\beta = 1, 2, 4$, respectively. From equation (3) it is immediate that the non-zero singular values of the rectangular matrix $\mathbb{Y}_M$ are identical to the singular values of the square matrix Y_M. The ultimate goal is to derive the joint probability density function for these singular values. In [24] the invariance of the matrix measure for Y_M under permutations of the matrix dimensions, N_m, was shown This invariance carries over to the joint probability density function of the singular values as we will see.

The parametrisation (3) follows directly from a parametrisation of each individual matrix,

$$\mathbb{X}_m = \mathcal{U}_m \begin{pmatrix} X_m & A_m \\ 0 & B_m \end{pmatrix} \mathcal{U}_{m-1}^{-1}, \tag{4}$$

where $\mathcal{U}_0 = \mathbb{1}_{N_0}$. The matrices X_m, A_m and B_m have the dimensions $N_0 \times N_0$, $N_0 \times (N_{m-1} - N_0)$ and $(N_m - N_0) \times (N_{m-1} - N_0)$, respectively. The entries of these matrices are real for $\beta = 1$, complex for $\beta = 2$ and quaternion for $\beta = 4$. Accordingly, we have

$$\mathcal{U}_m \in \begin{cases} \mathrm{O}(N_m)/[\mathrm{O}(N_0) \times \mathrm{O}(N_m - N_0)], \\ \mathrm{U}(N_m)/[\mathrm{U}(N_0) \times \mathrm{U}(N_m - N_0)], \\ \mathrm{USp}(2N_m)/[\mathrm{USp}(2N_0) \times \mathrm{USp}(2(N_m - N_0))], \end{cases} \tag{5}$$

for $\beta = 1, 2, 4$, respectively. The non-zero singular values of the rectangular product matrix (1) are identical to the singular values of the square product matrix $Y_M = X_M X_{M-1} \cdots X_1$ with Y_M and X_m, $m = 1, \ldots, M$, defined above. For this reason, we can safely replace the random matrix model containing rectangular matrices with a random matrix model containing square matrices, only. In terms of the new variables we get for the partition function, in analogy to [32] for $M = 1$,

$$Z_\beta^M \propto \prod_{m=1}^{M} \int |DX_m| \det{}^{\beta\nu_m/2}(X_m X_m^\dagger) \exp[-\operatorname{Tr} X_m X_m^\dagger], \tag{6}$$

where $\nu_m \equiv N_m - N_0 \geq 0$. A more general version of this result will be derived in [24]. In the partition function (6) and in most of this section we neglect an overall normalisation constant, which is irrelevant for the computations. We reintroduce the normalisation in equation (16) and give the explicit value in equation (21).

The Gaussian weight times a determinantal prefactor is sometimes referred to as the induced weight. For $M = 1$ its complex eigenvalues have been studied in [32].

In order to derive the joint probability density function for the singular values of the product matrix Y_M and thereby of equation (1), we follow the idea in [23], and reformulate the partition function (6) in terms of the product matrices $Y_m = X_m Y_{m-1} = X_m X_{m-1} \cdots X_1$, for $m = 1, \ldots, M$. In the following we assume that the product matrices, Y_m, are invertible (note that this restriction only removes a set of measure zero). We then know that [23]

$$\prod_{m=1}^{M} |DX_m| = |DY_1| \prod_{m=2}^{M} |DY_m| \det{}^{-\beta N_0/2}(Y_{m-1} Y_{m-1}^\dagger). \tag{7}$$

Changing variables from X_m to Y_m in the partition func-

tion equation (6) results in

$$Z_\beta^M \propto \Big[\prod_{m=1}^{M}\int |DY_m|\Big] \det{}^{\beta\nu_M/2}(Y_M Y_M^\dagger) \exp\Big[-\operatorname{Tr} Y_1 Y_1^\dagger\Big] \times \Bigg[\prod_{i=2}^{M} \det{}^{\beta(\nu_{i-1}-\nu_i-N_0)/2}(Y_{i-1}Y_{i-1}^\dagger) \times \exp\Big[-\operatorname{Tr} Y_i Y_i^\dagger (Y_{i-1}Y_{i-1}^\dagger)^{-1}\Big]\Bigg]. \quad (8)$$

With this expression for the partition function we can express everything in terms of the singular values and a family of unitary matrices. We employ for each matrix Y_i a singular value decomposition [23] to write the product matrices as

$$Y_i = U_i \Sigma_i V_i^{-1}, \quad (9)$$

where $\Sigma_i = \operatorname{diag}\{\sigma_1^i, \sigma_2^i, \ldots, \sigma_{N_0}^i\}$ are positive definite diagonal matrices; the diagonal elements are the singular values of Y_i (for $\beta = 4$ the singular values show Kramer's degeneracy). The unitary matrices, U_i and V_i, belong to

$$U_i \in \begin{cases} \mathrm{O}(N_0), \\ \mathrm{U}(N_0), \\ \mathrm{USp}(2N_0), \end{cases} \qquad V_i \in \begin{cases} \mathrm{O}(N_0), \\ \mathrm{U}(N_0)/\mathrm{U}(1)^{N_0}, \\ \mathrm{USp}(2N_0)/\mathrm{U}(1)^{N_0}, \end{cases} \quad (10)$$

for $\beta = 1, 2, 4$, respectively. It is well-known that this change of variables yields the new measure

$$|DY_i| = |DU_i||DV_i| \prod_{k=1}^{N_0} d\sigma_k^i (\sigma_k^i)^{\beta-1} |\Delta_{N_0}((\sigma^i)^2)|^\beta, \quad (11)$$

where $|DU_i|$ and $|DV_i|$ are the Haar measures for their corresponding groups and

$$\Delta_N(x) = \prod_{1\le a<b\le N}(x_a - x_b) = \det_{1\le a,b\le N}[x_a^{N-b}] \quad (12)$$

denotes the Vandermonde determinant. Inserting this parametrisation into the partition function (8) and performing the shift $U_{\ell-1}^{-1}U_\ell \to U_\ell$ for $\ell = 2, \ldots, M$, we obtain

$$Z_\beta^M \propto \Bigg[\prod_{k=1}^{N_0}\Big[\prod_{m=1}^{M}\int_0^\infty d\sigma_k^m\Big](\sigma_k^M)^{\beta(\nu_M+1)-1} e^{-(\sigma_k^1)^2} \times \prod_{i=2}^{M}(\sigma_k^{i-1})^{\beta(\nu_{i-1}-\nu_i-N_0+1)-1}\Bigg] \prod_{j=1}^{M}|\Delta_{N_0}((\sigma^j)^2)|^\beta \times \prod_{\ell=2}^{M}\int |DU_\ell||DV_\ell| \exp\Big[-\operatorname{Tr} U_\ell \Sigma_\ell^2 U_\ell^{-1}\Sigma_{\ell-1}^{-2}\Big]. \quad (13)$$

The integrations over V_ℓ are trivial and only contribute to the normalisation constant; the integration over U_ℓ is however more complicated. For $\beta = 2$, the integrals over U_ℓ are Harish-Chandra–Itzykson–Zuber integrals [30, 31], while the integrals for $\beta = 1$ and $\beta = 4$ are still unknown in closed form. For this reason, we will restrict ourselves to the complex case ($\beta = 2$), where we can carry out all integrals explicitly, and obtain an analytical expression for the joint probability density function. Recall that the complex ($\beta = 2$) product matrix is exactly the channel matrix used in wireless telecommunication to model MIMO channels with multiple scattering.

With the restriction to the $\beta = 2$ case, U_ℓ should be integrated over the unitary group, which yields [30, 31]

$$\int_{\mathrm{U}(N_0)} |DU_\ell| \exp\Big[-\operatorname{Tr} U_\ell \Sigma_\ell^2 U_\ell^{-1}\Sigma_{\ell-1}^{-2}\Big] \propto \frac{\prod_{k=1}^{N_0}(\sigma_k^{\ell-1})^{2(N_0-1)}}{\Delta_{N_0}((\sigma^\ell)^2)\Delta_{N_0}((\sigma^{\ell-1})^2)} \det_{1\le a,b\le N_0}\Big[e^{-(\sigma_a^\ell)^2/(\sigma_b^{\ell-1})^2}\Big], \quad (14)$$

for $\ell = 2, \ldots, M$. Inserting this into the partition function (13) with $\beta = 2$ gives an expression for the partition function solely in terms of the singular values of the product matrices Y_i,

$$Z^M \equiv Z_{\beta=2}^M \propto \Bigg[\prod_{k=1}^{N_0}\int_0^\infty d\sigma_k^M (\sigma_k^M)^{2\nu_M+1}\Bigg] \Delta_{N_0}((\sigma^M)^2) \times \Bigg[\prod_{i=1}^{M-1}\Big[\prod_{\ell=1}^{N_0}\int_0^\infty d\sigma_\ell^i (\sigma_\ell^i)^{2(\nu_i-\nu_{i+1})-1}\Big] \times \det_{1\le a,b\le N_0}\Big[e^{-(\sigma_a^{i+1})^2/(\sigma_b^i)^2}\Big]\Bigg] \Big[\prod_{k=1}^{N_0} e^{-(\sigma_k^1)^2}\Big] \times \Delta_{N_0}((\sigma^1)^2). \quad (15)$$

For notational simplicity we will change variables from the singular values to $s_a^i = (\sigma_a^i)^2$, i.e. the singular values (and eigenvalues) of the Wishart matrices $Y_i Y_i^\dagger$ (the singular values of $Y_M Y_M^\dagger$ will simply be denoted by $s_a = s_a^M$). Furthermore, due to symmetrisation we can replace the determinants of the exponentials by their diagonals, which will only change the partition function by a factor $(N_0!)^{M-1}$. Exploiting this, the partition function becomes

$$Z^M = C_M^{-1}\Bigg[\prod_{b=1}^{N_0}\int_0^\infty ds_b\,(s_b)^{\nu_M}\Bigg]\,\Delta_{N_0}(s) \times \Bigg[\prod_{a=1}^{N_0}\Big[\prod_{i=1}^{M-1}\int_0^\infty \frac{ds_a^i}{s_a^i}\,(s_a^i)^{\nu_i-\nu_{i+1}}\, e^{-s_a^{i+1}/s_a^i}\Big] e^{-s_a^1}\Bigg] \times \Delta_{N_0}(s^1), \quad (16)$$

where C_M is a normalisation constant.

The integrations over $s_a^1, \ldots, s_a^{M-1}$ have a similar structure. Hence, we can perform all these integrals in a similar fashion. We write the first exponential containing s_a^1 as a Meijer G-function using equation (A10), i.e.

$$\Delta_{N_0}(s^1)\prod_{a=1}^{N_0} e^{-s_a^1} = \det_{1\le a,b\le N_0}\Big[G_{0,1}^{1,0}\Big(\begin{matrix}-\\ b-1\end{matrix}\,\Big|\, s_a^1\Big)\Big]. \quad (17)$$

After a change of variables all the integrals can be performed inductively using the identities (A7) and (A5). These integrations finally give the joint probability density function, $\mathcal{P}_{\rm jpdf}$, for the singular values $s_1,\dots,s_{N_0}$ of the Wishart matrix $Y_M Y_M^\dagger$,

$$\mathcal{P}^M_{\rm jpdf}(s_1,\dots,s_{N_0}) = C_M^{-1}\Delta_{N_0}(s) \times \det_{1\le a,b\le N_0}\Big[G^{M,0}_{0,M}\Big(\begin{matrix}-\\ \nu_M,\nu_{M-1},\dots,\nu_2,\nu_1+b-1\end{matrix}\,\Big|\,s_a\Big)\Big]. \quad (18)$$

The partition function is thus given by

$$Z^M = \prod_{a=1}^{N_0}\int_0^\infty ds_a\,\mathcal{P}^M_{\rm jpdf}(s_1,\dots,s_{N_0}). \quad (19)$$

This generalises the joint probability density function for the product of square matrices from the Wishart ensemble given in [23] to the case of rectangular matrices. In principle all k-point correlation functions for the singular values, $R^M_k(s_1,\dots,s_k)$, can be calculated from the joint probability density function (18) as

$$R^M_k(s_1,\dots,s_k) = \frac{N_0!}{(N_0-k)!}\prod_{a=k+1}^{N_0}\int_0^\infty ds_a\,\mathcal{P}^M_{\rm jpdf}(s_1,\dots,s_{N_0}). \quad (20)$$

Due to the Meijer G-function inside the determinant (18) this is a non-trivial computation for $M\ge 2$. In complete analogy to the square case [23], it turns out that the correlation functions are more easily obtained using a two-matrix model and the method of bi-orthogonal polynomials. We will discuss this in section III, including other methods of derivation.

The normalisation constant in equations (15) and (18) is

$$C_M = N_0!\prod_{n=1}^{N_0}\prod_{m=0}^{M}\Gamma[n+\nu_m], \quad (21)$$

such that the partition function is equal to unity, which is straightforward to check using the Andréief integration formula. The one-point correlation function (or density) is normalised to the number of singular values,

$$\int_0^\infty ds\,R^M_1(s) = N_0, \quad (22)$$

which becomes evident in the following section.

III. TWO-MATRIX MODEL AND BI-ORTHOGONAL POLYNOMIALS

The purpose of this section is to find an explicit expression for the k-point correlation functions (20). We will follow the idea in [23] and rewrite our problem as a two-matrix model by keeping the integrals over the s^1_a's and s^M_a's in Eq. (16) while integrating over the remaining variables. Within this model we will exploit the method of bi-orthogonal polynomials to achieve our goal. First, we use the identity (A5) for the Meijer G-function to write the partition function (19) with $M\ge 2$ as

$$Z^M = \prod_{a=1}^{N_0}\int_0^\infty ds_a\prod_{i=1}^{N_0}\int_0^\infty dt_i\,\widetilde{P}^M_{\rm jpdf}(s\,;t\,), \quad (23)$$

where the joint probability density function is given by

$$\widetilde{P}^M_{\rm jpdf}(s\,;t\,) = C_M\Delta_{N_0}(s)\Delta_{N_0}(t)\det_{1\le k,\ell,\le N_0}\big[w^M_\nu(s_k,t_\ell)\big], \quad (24)$$

$s_a\equiv s^M_a$ and $t_a\equiv s^1_a$, and the weight function depending on all indices ν_m collectively denoted by ν reads

$$w^M_\nu(s,t) = t^{\nu_1-1}e^{-t}G^{M-1,0}_{0,M-1}\Big(\begin{matrix}-\\ \nu_M,\nu_{M-1},\dots,\nu_2\end{matrix}\,\Big|\,\frac{s}{t}\Big). \quad (25)$$

The structure of the joint probability density function (24) is similar to that of the two-matrix model discussed in [33]. Although the focus in [33] is on a multi-matrix model with an Itzykson–Zuber interaction, the argument given is completely general and applies to our situation as well. The (k,ℓ)-point correlation functions for this two-matrix model are defined as

$$R^M_{k,\ell}(s\,;t\,) = \frac{(N_0!)^2}{(N_0-k)!(N_0-\ell)!}\times\prod_{a=k+1}^{N_0}\int_0^\infty ds_a\prod_{i=\ell+1}^{N_0}\int_0^\infty dt_i\,\widetilde{P}^M_{\rm jpdf}(s;t). \quad (26)$$

Obviously, we can obtain the k-point correlation functions (20) by integrating out all t_i's, i.e. setting $\ell=0$.

The benefit of the two-matrix model is that we can exploit the method of bi-orthogonal polynomials as in [33]. We choose a family of monic polynomials $q^M_j(t)=t^j+\cdots$ and $p^M_j(s)=s^j+\cdots$, which are bi-orthogonal with respect to the weight (25),

$$\int_0^\infty ds\int_0^\infty dt\,w^M_\nu(s,t)q^M_i(t)p^M_j(s) = h^M_j\delta_{ij}, \quad (27)$$

where h^M_j are constants. Furthermore, we introduce the functions $\psi^M_j(t)$ and $\varphi^M_j(s)$ defined as integral transforms of the bi-orthogonal polynomials,

$$\psi^M_j(t) \equiv \int_0^\infty ds\,w^M_\nu(s,t)p^M_j(s), \quad (28)$$

$$\varphi^M_j(s) \equiv \int_0^\infty dt\,w^M_\nu(s,t)q^M_j(t). \quad (29)$$

Note that $\psi^M_j(t)$ and $\varphi^M_j(s)$ are not necessarily polynomials. It is evident from the bi-orthogonality of the polynomials (27) that we have the orthogonality relations

$$\int_0^\infty dt\,q^M_i(t)\psi^M_j(t) = \int_0^\infty ds\,p^M_i(s)\varphi^M_j(s) = h^M_j\delta_{ij}. \quad (30)$$

Moreover, it follows from the discussion in [33] that the (k,ℓ)-point correlation functions are given by a determinantal point process

$$R^M_{k,\ell}(s\,;t\,) = \det_{\substack{1\le a,b\le k\\ 1\le i,j\le \ell}} \begin{bmatrix} K^M_{11}(s_a,s_b) & K^M_{12}(s_a,t_j) \\ K^M_{21}(t_i,s_b) & K^M_{22}(t_i,t_j)\end{bmatrix}, \quad (31)$$

where the four sub-kernels are defined in terms of the bi-orthogonal polynomials and the weight function as

$$\begin{aligned} K^M_{11}(s_a,s_b) &= \sum_{n=0}^{N_0-1} \frac{p^M_n(s_a)\varphi^M_n(s_b)}{h^M_n}, \\ K^M_{12}(s_a,t_j) &= \sum_{n=0}^{N_0-1} \frac{p^M_n(s_a)q^M_n(t_j)}{h^M_n}, \\ K^M_{21}(t_i,s_b) &= \sum_{n=0}^{N_0-1} \frac{\psi^M_n(t_i)\varphi^M_n(s_b)}{h^M_n} - w^M_\nu(s_b,t_i), \\ K^M_{22}(t_i,t_j) &= \sum_{n=0}^{N_0-1} \frac{\psi^M_n(t_i)q^M_n(t_j)}{h^M_n}. \end{aligned} \quad (32)$$

In particular we have that the k-point correlation functions (20) for the singular values of the product matrix $Y_M Y_M^\dagger$ are given by

$$R^M_k(s_1,\ldots,s_k) = \det_{1\le a,b\le k}\big[K^M_{11}(s_a,s_b)\big]. \quad (33)$$

The goal is to find the bi-orthogonal polynomials, $q^M_j(t)$ and $p^M_j(s)$, and the norms, h^M_j, and thereby all correlation functions for the singular values of the product matrix, $\mathbb{Y}_M$. Note that we use a slightly different notation for the sub-kernels than in [23]; the notation in this paper is chosen to emphasise the fact that all the statistical properties of the singular values are determined by the bi-orthogonal polynomials, $q^M_j(t)$ and $p^M_j(s)$, and the weight function, $w^M_\nu(s,t)$.

In order to find the bi-orthogonal polynomials we follow the approach in [23] and start by computing the bimoments

$$I^M_{ij} \equiv \int_0^\infty ds \int_0^\infty dt\, w^M_\nu(s,t) s^i\, t^j = (i+j+\nu_1)! \prod_{m=2}^{M} (i+\nu_m)! \quad (34)$$

for $M \ge 2$. Here the integration has been performed using integral identities for the Meijer G-function, see equations (A4) and (A5). Using Cramer's rule, the bi-orthogonal polynomials as well as the norms can be expressed in terms of the bimoments as [34, 35],

$$\begin{aligned} q^M_n(t) &= \frac{1}{D^M_{n-1}} \det \begin{bmatrix} I^M_{00} & I^M_{10} & \cdots & I^M_{(n-1)0} & 1 \\ I^M_{01} & I^M_{11} & \cdots & I^M_{(n-1)1} & t \\ \vdots & \vdots & & \vdots & \vdots \\ I^M_{0n} & I^M_{1n} & \cdots & I^M_{(n-1)n} & t^n \end{bmatrix}, \\ p^M_n(s) &= \frac{1}{D^M_{n-1}} \det \begin{bmatrix} I^M_{00} & I^M_{01} & \cdots & I^M_{0(n-1)} & 1 \\ I^M_{10} & I^M_{11} & \cdots & I^M_{1(n-1)} & s \\ \vdots & \vdots & & \vdots & \vdots \\ I^M_{n0} & I^M_{n1} & \cdots & I^M_{n(n-1)} & s^n \end{bmatrix}, \end{aligned} \quad (35)$$

where

$$D^M_n \equiv \det_{0\le i,j\le n}[I^M_{ij}] = \prod_{i=0}^{n}\prod_{m=0}^{M}(i+\nu_m)!. \quad (36)$$

The norms can be expressed as

$$h^M_n = D^M_n / D^M_{n-1} = \prod_{m=0}^{M}(n+\nu_m)!. \quad (37)$$

Recall that $\nu_i \equiv N_i - N_0 \ge 0$ are non-negative integers by definition ($\nu_0 = 0$).

In order to get more explicit expressions for the bi-orthogonal polynomials, we define the bimoment matrix (34) for $M = 1$ as the bimoments with respect to the Laguerre weight,

$$I^{M=1}_{ij} \equiv \int_0^\infty ds\, e^{-s} s^{\nu_1+i+j} = (i+j+\nu_1)!. \quad (38)$$

It follows that the polynomials (35) for $M = 1$ are the Laguerre polynomials in monic normalisation,

$$p^{M=1}_n(s) = q^{M=1}_n(s) = \widetilde{L}^{\nu_1}_n(s) \equiv (-1)^n n! L^{\nu_1}_n(s), \quad (39)$$

where $L^{\nu_1}_n(s)$ are the associated Laguerre polynomials. We recall that the Laguerre polynomials are defined as

$$\widetilde{L}^{\nu_1}_n(s) = \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!}\frac{(n+\nu_1)!}{(k+\nu_1)!}\frac{n!}{k!} s^k \quad (40)$$

and satisfy the orthogonality relation

$$\int_0^\infty ds\, e^{-s} s^{\nu_1} \widetilde{L}^{\nu_1}_k(s) \widetilde{L}^{\nu_1}_\ell(s) = h^{M=1}_k \delta_{k\ell} \quad (41)$$

with $h^{M=1}_k = k!(k+\nu_1)!$.

The bimoment matrix, $[I^M_{ij}]_{0\le i,j\le n}$, with $M \ge 2$ given by equation (34) differs from the bimoment matrix, $[I^1_{ij}]_{0\le i,j\le n}$, given by equation (38) by multiplication of a diagonal matrix. It directly follows from this fact that the polynomials $q^M_n(t)$ are related to the Laguerre polynomials as

$$q^M_n(t) = \prod_{i=0}^{n-1}\prod_{m=2}^{M}(i+\nu_m)! \frac{D^1_{n-1}}{D^M_{n-1}} \widetilde{L}^{\nu_1}_n(t) = \widetilde{L}^{\nu_1}_n(t). \quad (42)$$

The evaluation of the polynomials $p_n^M(s)$ is slightly more complicated. For the polynomials $q_n^M(t)$, the factorisation is the same for all powers of t, but for the polynomials $p_n^M(s)$ we have to treat the powers differently; in particular we substitute $s^k \to s^k / \prod_{m=2}^M (k+\nu_m)!$. Using the explicit expression for the Laguerre polynomials (40) we find

$$p_n^M(s) = \sum_{k=0}^{n} \frac{(-1)^{n+k} n!}{(n-k)!} \left(\prod_{m=1}^{M} \frac{(n+\nu_m)!}{(k+\nu_m)!} \right) \frac{s^k}{k!}, \quad (43)$$

which is a generalised hypergeometric polynomial (see equation (A2) in appendix A)

$$p_n^M(s) = (-1)^n \prod_{m=1}^{M} \frac{(n+\nu_m)!}{\nu_m!} \, {}_1F_M\Big(\begin{smallmatrix} -n \\ 1+\nu_M, \dots, 1+\nu_1 \end{smallmatrix} \Big| \, s \Big). \quad (44)$$

For $\nu_M = \cdots = \nu_1 = 0$ this polynomial reduces to the result presented in [23], while the monic Laguerre polynomials are reobtained by setting $M = 1$. Alternatively we may write $p_n^M(s)$ as a Meijer G-function,

$$p_n^M(s) = (-1)^n \prod_{m=0}^{M} (n+\nu_m)! \, G^{1,\,0}_{1,\,M+1}\Big(\begin{smallmatrix} n+1 \\ 0, -\nu_M, \dots, -\nu_1 \end{smallmatrix} \Big| \, s \Big). \quad (45)$$

This expression will be particularly useful in section IV, where we discuss the asymptotic behaviour of the endpoints of support of the spectral density. In equation (45) we have used the relation (A9) between generalised hypergeometric polynomials and Meijer G-functions. It might not be immediately clear that the Meijer G-function in (45) is a polynomial. To see this, one writes the Meijer G-function as a contour integral using its definition (A3). The integrand has exactly n simple poles and the contour is closed such that these poles are encircled. The residue for each pole gives a monomial, such that the complete contour integral yields a polynomial.

With the explicit expressions for the bi-orthogonal polynomials (42) and (44), we are ready to compute the functions $\psi_n^M(t)$ and $\varphi_n^M(s)$ defined in equation (29), and thereby implicitly find all the sub-kernels (32). The functions $\psi_n^M(t)$ turn out to be polynomials, too,

$$\psi_n^M(t) = \prod_{m=2}^{M} (n+\nu_m)! \, t \widetilde{L}_n^{\nu_1}(t), \quad (46)$$

which can be directly obtained from the definition (29) using the integral identity (A4).

Likewise, we can obtain an explicit expression for the functions $\varphi_n^M(s)$ by inserting the polynomial (42) into the definition (29). It follows from the integral identity (A5) that

$$\varphi_n^M(s) = \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(n+\nu_1)!}{(k+\nu_1)!} \frac{n!}{k!} \times G^{M,\,0}_{0,\,M}\Big(\begin{smallmatrix} - \\ \nu_M, \dots, \nu_2, \nu_1+k \end{smallmatrix} \Big| \, s \Big). \quad (47)$$

However, it is possible to get a more compact expression. Recall that the Laguerre polynomials can be expressed using Rodrigues' formula,

$$\widetilde{L}_n^{\nu_1}(t) = (-1)^n t^{-\nu_1} e^t \frac{d^n}{dt^n} \big(t^{n+\nu_1} e^{-t} \big). \quad (48)$$

We insert Rodrigues' formula into the definition for $\varphi_n^M(s)$, see equation (29). The differentiation in equation (48) can easily be changed to a differentiation of the Meijer G-function (stemming from the weight function) using integration by parts, since all boundary terms are zero. Then the differentiation can be computed using equation (A8), while the final integration over t can be performed using the identity (A5). This finally leads to

$$\varphi_n^M(s) = (-1)^n G^{M,\,1}_{1,\,M+1}\Big(\begin{smallmatrix} -n \\ \nu_M, \nu_{M-1}, \dots, \nu_1, 0 \end{smallmatrix} \Big| \, s \Big). \quad (49)$$

In addition to the fact that equation (49) is a more compact expression than the representation (47), we is also immediate that $\varphi_n^M(s)$ is symmetric in all the indices ν_m, which is far from obvious in equation (47).

Now we have explicit expressions for all components contained in the formula for the (k,ℓ)-point correlation functions (31), which completes the derivation. In particular combining equations (37), (44), and (49) the subkernel $K_{11}^M(s_a, s_b)$ is given by

$$K_{11}^M(s_a, s_b) = \sum_{n=0}^{N_0-1} \frac{1}{n!} \prod_{m=1}^{M} \frac{1}{\nu_m!} \, {}_1F_M\Big(\begin{smallmatrix} -n \\ 1+\nu_M, \dots, 1+\nu_1 \end{smallmatrix} \Big| \, s_a \Big) \times G^{M,\,1}_{1,\,M+1}\Big(\begin{smallmatrix} -n \\ \nu_M, \dots, \nu_1, 0 \end{smallmatrix} \Big| \, s_b \Big). \quad (50)$$

It provides a direct generalisation of the formula given in [23] for square matrices to the case of rectangular matrices. If we use the alternative formula (45) for $p_n^M(s)$ we obtain

$$K_{11}^M(s_a, s_b) = \sum_{n=0}^{N_0-1} G^{1,\,0}_{1,\,M+1}\Big(\begin{smallmatrix} n+1 \\ 0, -\nu_M, \dots, -\nu_1 \end{smallmatrix} \Big| \, s_a \Big) \times G^{M,\,1}_{1,\,M+1}\Big(\begin{smallmatrix} -n \\ \nu_M, \dots, \nu_1, 0 \end{smallmatrix} \Big| \, s_b \Big). \quad (51)$$

The k-point correlation functions for the singular values are immediately found from equation (33). Note that the kernel and thereby all k-point correlation functions are symmetric in all the indices ν_m. This symmetry reflects the invariance of the singular values of the product matrix, $Y_M = X_M \cdots X_1$, under reordering of the matrices X_m which we prove in a more general setting in [24]. The normalisation of the spectral density (22) is immediately clear from the orthogonality relation (30).

Finally we would like to mention an alternative derivation for the correlation functions (20) in terms of the kernel K_{11}^M. Given the orthogonality relation (30) of the polynomials p_i^M (43) and the functions φ_j^M (47) we can generate these by adding columns in the two determinants in the joint probability density function (18) and

then proceed with the standard Dyson theorem. This is in complete analogy as described in [23]. Alternatively, the kernel can be derived by using bi-orthogonal functions and explicitly inverting the bimoment matrix [36]. Furthermore, a construction using multiple orthogonal polynomials exist [37, 38], too.

IV. MOMENTS AND ASYMPTOTICS

In this section we take a closer look at the spectral density. First we will use the density to find an explicit expression for the moments. Second we will discuss the macroscopic large-N_0 limit of the density.

We know from the previous section that the density, or one-point correlation function, is given as a sum over Meijer G-functions,

$$R_1^M(s) = \sum_{n=0}^{N_0-1} G^{1,\,0}_{1,\,M+1}\Big({}^{\,n+1}_{0,\,-\nu_M,\,\ldots\,,\,-\nu_1} \,\Big|\, s\Big) \times G^{M,\,1}_{1,\,M+1}\Big({}^{\,-n}_{\nu_M,\,\ldots\,,\,\nu_1,0} \,\Big|\, s\Big), \qquad (52)$$

which is normalised to the number of singular values, N_0. Figure 1 shows a comparison between the analytical expression and numerical simulations for an example. The expectation value for the singular values is defined in terms of the density (52) as

$$\mathbb{E}\{f(s)\} \equiv \frac{1}{N_0}\int_0^\infty ds\, R_1^M(s)\, f(s), \qquad (53)$$

where the factor $1/N_0$ is included since the density (52) is normalised to the number of singular values.

We will first look at the moments, $\mathbb{E}\{s^\ell\}$. Note that we do not assume that ℓ is an integer, and that the half-integer values of ℓ are interesting, too, since the singular values, σ_a, of the product matrix, $\mathbb{Y}_M$, are given by the square roots of the eigenvalues of the Wishart matrix, i.e. $\sigma_a = \sqrt{s_a}$. In order to calculate the moments, we explicitly write the first Meijer G-function in equation (52) as a polynomial, see equations (43) and (45), and rewrite the moments as

$$N_0\mathbb{E}\{s^\ell\} = \sum_{n=0}^{N_0-1}\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!}\prod_{m=0}^{M}\frac{1}{(k+\nu_m)!} \times \int_0^\infty ds\, s^{\ell+k} G^{M,\,1}_{1,\,M+1}\Big({}^{\,-n}_{\nu_M,\,\ldots\,,\,\nu_1,0} \,\Big|\, s\Big). \qquad (54)$$

The integral over s can be performed using an identity for the Meijer G-function (A4). After reordering the sums and applying Euler's reflection formula for the gamma-function we get

$$N_0\mathbb{E}\{s^\ell\} = \sum_{k=0}^{N_0-1}\prod_{m=0}^{M}\frac{\Gamma[\ell+k+\nu_m+1]}{(k+\nu_m)!} \times \sum_{n=0}^{N_0-k-1}\frac{(-1)^n}{n!\,\Gamma[\ell-n+1]}, \qquad (55)$$

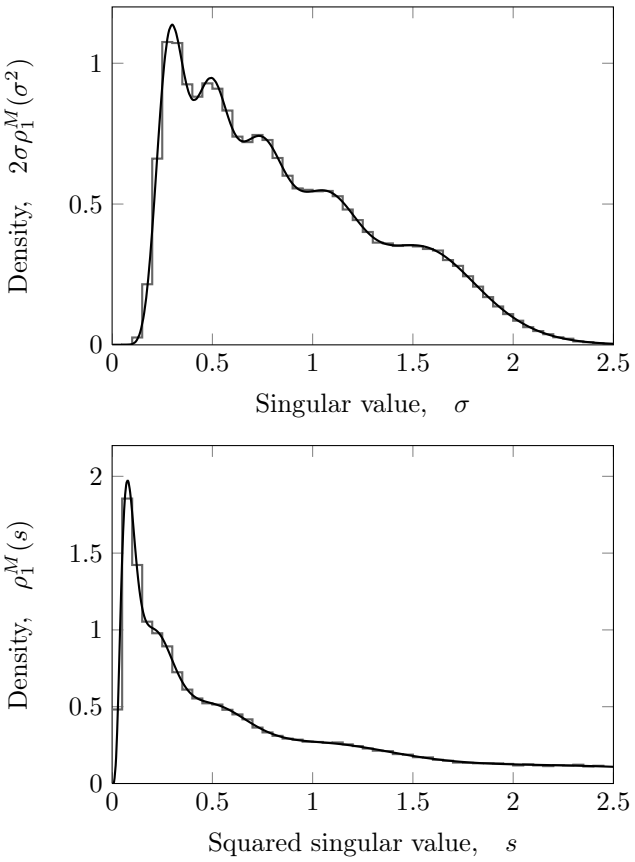

Figure 1. The histograms (bin width is 0.05) show the distributions of singular values (top) and squared singular values (bottom) for 50 000 realisations of the product matrix $\mathbb{Y}_3 = \mathbb{X}_3\mathbb{X}_2\mathbb{X}_1$ for $M = 3$, with $\nu_1 = 5$, $\nu_2 = 10$, $\nu_3 = 15$ and $N_0 = 5$. The solid curves are the analytical predictions for the rescaled densities of singular values, $2\sigma\rho_1^3(\hat{\sigma}^2)$, and of squared singular values, $\rho_1^3(\hat{s})$, respectively, cf. equation (61).

where ℓ may also take non-integer values. For integer values of ℓ some of the terms will vanish due to the poles of the gamma-function. Note that the moments are divergent whenever $\ell \leq -\nu_{\min} - 1$ is an integer ($\nu_{\min} \equiv \min\{\nu_1, \ldots, \nu_M\}$), but well-defined for all other values of ℓ. The second sum in equation (55) can be evaluated by a relation for the (generalised) binomial series

$$\sum_{n=0}^{N}(-1)^n\binom{z}{n} = (-1)^N\binom{z-1}{N}, \quad z\in\mathbb{C}. \qquad (56)$$

We write the first sum in equation (55) in reverse order ($k \to N_0 - k - 1$) and perform the second sum using the identity (56) yielding

$$N_0\mathbb{E}\{s^\ell\} = \sum_{k=0}^{N_0-1}\frac{(-1)^k}{k!\Gamma[\ell-k]\ell}\prod_{m=0}^{M}\frac{\Gamma[\ell+N_m-k]}{\Gamma[N_m-k]}. \qquad (57)$$

Alternatively, the moments can be written as

$$N_0\mathbb{E}\{s^\ell\} = \sum_{k=0}^{N_0-1} \frac{(-1)^{1+k}\prod_{j=0}^{N_0-1}(j-\ell-k)}{k!(N_0-1-k)!\ell} \times \prod_{m=1}^{M} \frac{\Gamma[\ell+\nu_m+k+1]}{\Gamma[\nu_m+k+1]} \quad (58)$$

which is useful when considering the limit of negative integer ℓ. Recall that N_m are the different matrix dimensions of the original product (1) and $\nu_m = N_m - N_0$.

For $\ell \to 0$ all terms in the sum are equal to one and we recover the normalisation. Simplifications also occur when ℓ is an integer; here most of the terms in the sum vanish, due to the gamma-function in the denominator. In particular, the first positive moment and the first negative moment are given by

$$\mathbb{E}\{s\} \equiv \mathcal{N}_M = \prod_{m=1}^{M} N_m \quad \text{and} \quad \mathbb{E}\{s^{-1}\} = \prod_{m=1}^{M} \frac{1}{\nu_m}. \quad (59)$$

The second moment is slightly more complicated,

$$\mathbb{E}\{s^2\} = \frac{1}{2}\prod_{m=1}^{M} N_m \Big[\prod_{m=0}^{M} (N_m+1) - \prod_{m=0}^{M} (N_m-1) \Big]. \quad (60)$$

When $M = 1$ these formulae reduce to the well-known results for the Wishart–Laguerre ensemble (e.g. see [11]), while we get the result [23] for square matrices by setting $N_0 = \cdots = N_M$. Note that any negative moment is divergent if $\nu_m = 0$ for any $1 \le m \le M$.

The first moment, $\mathcal{N}_M$, provides us with a natural scaling of the spectral density,

$$\rho_1^M(\hat{s}) \equiv \frac{\mathcal{N}_M}{N_0} R_1^M(\hat{s}\,\mathcal{N}_M), \quad (61)$$

such that the rescaled density has a finite first moment of unity also in the large-N_0 limit. In equation (61) and the following, we use a hat '$\hat{}$' to denote rescaled variables.

The expectation value with respect to the rescaled density (61) is related to the definition (53) by a simple scaling of the variable,

$$\hat{\mathbb{E}}\{f(\hat{s})\} \equiv \int_0^\infty d\hat{s}\, \rho_1^M(\hat{s}) f(\hat{s}) = \mathbb{E}\left\{ f\left(\frac{\hat{s}}{\mathcal{N}_M} \right) \right\}, \quad (62)$$

for any observable $f(\hat{s})$. The rescaling ensures that we have a well-defined probability density with compact support in the large-N_0 limit; in particular the density $\rho_1^1(\hat{s})$ for a single matrix $M = 1$ reduces to the celebrated Marčenko–Pastur density for $N_0 \to \infty$.

An algebraic way to obtain the macroscopic behaviour of the spectral density (61) for arbitrary M was provided in [14], using the resolvent also known as the Stieltjes transform, $G^M(\hat{z})$, defined as

$$G^M(\hat{z}) \equiv \int_0^\infty d\hat{s} \lim_{N_0\to\infty} \frac{\rho_1^M(\hat{s})}{\hat{z}-\hat{s}}, \quad (63)$$

with $\hat{z}$ outside the limiting support of ρ_1^M. It was shown that in the large-N_0 limit the resolvent satisfies a polynomial equation [14],

$$\hat{z}\, G^M(\hat{z}) \prod_{m=1}^{M} \frac{\hat{z}\, G^M(\hat{z}) + \hat{\nu}_m}{\hat{\nu}_m + 1} = \hat{z}(\hat{z}\, G^M(\hat{z}) - 1), \quad (64)$$

where $\hat{z}$ lies outside the support of the singular values and $\hat{\nu}_m$ denotes the rescaled differences in matrix dimensions, i.e. $\hat{\nu}_m \equiv \nu_m/N_0$ for $m = 1, \ldots, M$. In general one needs to solve an $(M+1)$-st order equation in order to find the resolvent, $G^M(\hat{z})$. It is clear, that such an equation can generically only be solved analytically for $M \le 3$ (see also the discussions in [37, 39]).

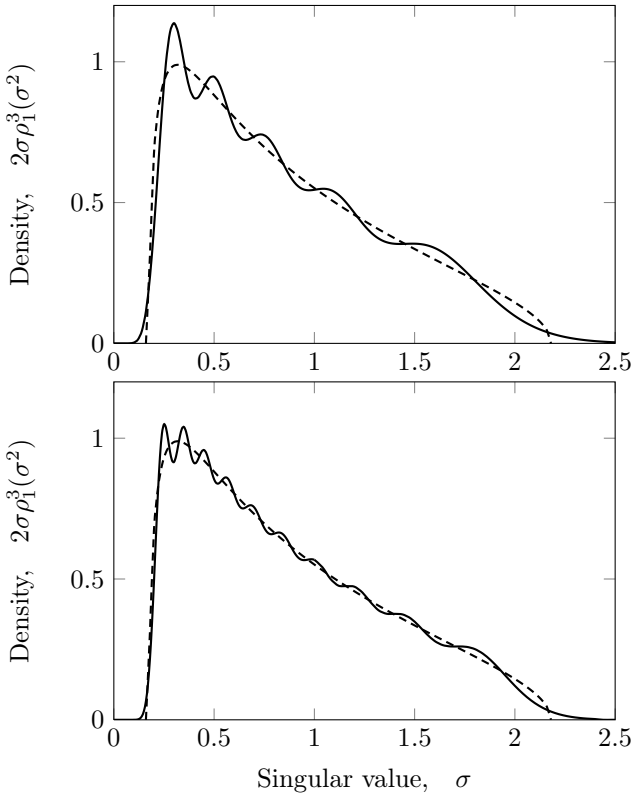

Figure 2. The solid lines show the $M = 3$ rescaled spectral densities for the singular values for $N_0 = 5$ (top) and $N_0 = 10$ (bottom) both with $\hat{\nu}_1 = 1$, $\hat{\nu}_2 = 2$, $\hat{\nu}_3 = 3$. The dashed curves indicate the corresponding macroscopic limit [14].

The correct resolvent is chosen by its asymptotic behaviour, $\hat{z}G^M(\hat{z}) \to 1$ for $\hat{z} \to \infty$. When an expression for the resolvent is known, then the spectral density can be directly obtained from the resolvent using

$$\rho_1^{M,\infty}(\hat{s}) \equiv \lim_{N_0\to\infty} \rho_1^M(\hat{s}) = \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \operatorname{Im} G^M(\hat{s} - \imath\varepsilon). \quad (65)$$

In figure 2 we compare this macroscopic limit with the rescaled density (52) at finite N_m.

For the case $M = 1$ one can readily derive the well-known Marčenko–Pastur law. Another particular case

in which the spectral density $\rho_1^{M,\infty}$ can be directly calculated is $M = 2$ with $\hat{\nu}_1$ and $\hat{\nu}_2$ arbitrary. This case plays an important role when studying cross correlation matrices of two different sets of time series as it appears in forecasting models [10, 40] where time-lagged correlation matrices are non-symmetric. Our random matrix model then corresponds to the case of two time series which are uncorrelated. Despite the independence of the distribution of the matrix elements correlations among the singular values of the cross correlation matrix follow. The solution of equation (64) yields the level density

$$\rho_1^{M,\infty}(\hat{s}) = \frac{\sqrt{3(\hat{\nu}_1+1)(\hat{\nu}_2+1)\hat{s}+\hat{\nu}_1^2-\hat{\nu}_1\hat{\nu}_2+\hat{\nu}_2^2}}{3\pi\hat{s}} \times \mathrm{Im}\left[A^{-1/3}(f(\hat{s}))+A^{1/3}(f(\hat{s}))\right] \tag{66}$$

with

$$f\left(\frac{z}{(\hat{\nu}_1+1)(\hat{\nu}_2+1)}\right) = 3\frac{\left[3z+\hat{\nu}_1^2-\hat{\nu}_1\hat{\nu}_2+\hat{\nu}_2^2\right]^3}{\left[3(3+\hat{\nu}_1+\hat{\nu}_2)z+\hat{\nu}_1^3-(\hat{\nu}_1+\hat{\nu}_2)^3/3+\hat{\nu}_2^3\right]^2} \tag{67}$$

and

$$A(z) = \sqrt{\frac{27}{4z}-1}-\sqrt{\frac{27}{4z}}. \tag{68}$$

Indeed the special case $\hat{\nu}_1 = \hat{\nu}_2 = 0$ agrees with the result derived in [23, 26, 37] because $f(\hat{s})|_{\hat{\nu}_1=\hat{\nu}_2=0} = \hat{s}$.

It is also desirable to know where the endpoints of support of the macroscopic spectrum are located. These edges can be found from the algebraic formula for the resolvent (64) using a simple trick. We assume that the resolvent behaves as $|G^M(\hat{z})| \sim |\hat{z}-\hat{s}_\pm|^{\alpha_\pm}$ with $\alpha_\pm < 1$ and $\alpha_\pm \neq 0$ in the vicinity of the edges, $\hat{s}_\pm$. This edge behaviour of the resolvent is known to hold in certain cases, e.g. $M = 1$ yields $\alpha_\pm = 1/2 < 1$ (except when the inner edge is zero, $\hat{s}_- = 0$, then $\alpha_- = -1/2 < 1$). Due to known universality results for random matrices, it is expected that $\alpha_\pm < 1$ and $\alpha_\pm \neq 0$ in general. With this particular edge behaviour, it is clear that $|d\hat{G}^M/d\hat{z}| \to \infty$ for $\hat{z} \to \hat{s}_\pm$, or equivalently $d\hat{z}/d\hat{G}^M \to 0$ for $\hat{z} \to \hat{s}_\pm$. Differentiating both sides of equation (64) with respect to G^M and evaluating them at $d\hat{z}/d\hat{G}^M = 0$ yields an equation for the extrema of $\hat{z}$,

$$\hat{z}_0 = \left(1+\sum_{j=1}^{M}\frac{\hat{z}_0 G^M(\hat{z}_0)}{\hat{z}_0 G^M(\hat{z}_0)+\hat{\nu}_j}\right)\prod_{m=1}^{M}\frac{\hat{z}_0 G^M(\hat{z}_0)+\hat{\nu}_m}{\hat{\nu}_m+1}. \tag{69}$$

Two of these extrema are the inner edge, $\hat{z}_0 = \hat{s}_-$, and the outer edge, $\hat{z}_0 = \hat{s}_+$. The edges, $\hat{s}_\pm$, also satisfy equation (64). Combining both equations, we get an expression for the edges

$$\hat{s}_\pm = \frac{\hat{u}_0}{1+\hat{u}_0}\prod_{m=1}^{M}\frac{\hat{\nu}_m-\hat{u}_0}{\hat{\nu}_m+1}, \tag{70}$$

in terms of $\hat{u}_0 \equiv -\hat{z}_0 G^M(\hat{z}_0)$ which is given by

$$\sum_{m=1}^{M}\frac{\hat{u}_0(\hat{u}_0+1)}{\hat{\nu}_m-\hat{u}_0} = 1. \tag{71}$$

This equation is equivalent to a polynomial equation of $(M+1)$'st order as it is the case for the resolvent, see equation (64). However, in certain cases equation (71) simplifies. In particular, equation (71) reduces to an M-th order equation if $\hat{\nu}_i = \hat{\nu}_j$ for $i \neq j$, if $\hat{\nu}_i \to 0$ or if $\hat{\nu}_i \to \infty$. The latter means that $N_i \gg N_0$ meaning that the matrix dimension N_i decouples from the macroscopic theory.

In general the set of equations (70) and (71) yields $(M+1)$ solutions of which two correspond to the inner and outer edge of the spectral density. In the special case where $\hat{\nu} \equiv \hat{\nu}_1 = \cdots = \hat{\nu}_M$, there are only two solutions (see figure 3)

$$\hat{s}_\pm(\hat{\nu}) = \frac{M+1+2\hat{\nu}\pm\sqrt{(M+1)^2+4M\hat{\nu}}}{2(\hat{\nu}+1)} \times\left(\frac{M+1+2M\hat{\nu}\pm\sqrt{(M+1)^2+4M\hat{\nu}}}{2M+2M\hat{\nu}}\right)^M. \tag{72}$$

Note that for $M = 1$ this result reduces to the known values for the edges of the Marčenko–Pastur density (e.g. see [11]), while the limit $\hat{s}_\pm(\hat{\nu} \to 0)$ reproduces the result for the product of square matrices, see [23, 26, 37]. It is easy to numerically verify that the result holds in general.

Looking at the equations (70) and (71), an obvious question is: Which solutions correspond to the edges of the spectrum? In order to answer this question, we will derive the same equations through a different route. The rescaled spectral density (61) serves as the starting point, and the locations of the edges are determined using a saddle point approximation for large N_0. This also illustrates the point that the finite N_m expression discussed in this paper is equivalent to the result presented in [14] in the macroscopic limit.

In the large-N_0 limit we may approximate the sum over n, see equation (52), by an integral. Moreover, we write the Meijer G-functions as contour integrals (A3) and approximate the gamma-functions using Stirling's formula. The rescaled density (61) becomes

$$\rho_1^M(\hat{s}) \approx \frac{\mathcal{N}_M}{N_0}\int_0^1 d\hat{n}\,\frac{N_0}{2\pi\imath}\int_{L_1} d\hat{v}\,e^{-N_0 S(-\hat{v},\hat{n})} \times\frac{N_0}{2\pi\imath}\int_{L_2} d\hat{u}\,e^{N_0 S(\hat{u},\hat{n})}, \tag{73}$$

where the action, S, is given by

$$S(\hat{u},\hat{n}) = \hat{u}\ln\mathcal{N}_M\hat{s}+\sum_{m=1}^{M}(\hat{\nu}_m-\hat{u})(\ln N_0(\hat{\nu}_m-\hat{u})-1) +(\hat{n}+\hat{u})(\ln N_0(\hat{n}+\hat{u})-1) -\hat{u}\,(\ln N_0\hat{u}-1) \tag{74}$$

with $\hat{n} = n/N_0$, $\hat{u} = u/N_0$ and $\hat{\nu}_m = \nu_m/N_0$. It is important to note that the integrand in the definition of the Meijer G-function (A3) contains poles which lie on the real axis. The contours L_1 and L_2 encircle the poles of the original Meijer G-functions in accordance to definition (A3). In the large-N_0 limit these poles condense into cuts, such that the complex $\hat{u}$-plane has a cut on the interval $(\hat{\nu}_{\min}, \infty)$ and the complex $(-\hat{v})$-plane has a cut on the interval $(-1, 0)$. The contours L_1 and L_2 encircle these cuts in the $\hat{v}$-plane and the $\hat{u}$-plane, respectively. Both contour integrals can be evaluated by a saddle point approximation. Furthermore, variation with respect to $\hat{n}$ yields $\hat{u} = -\hat{v}$ at the saddle point and due to the symmetry between the two saddle point equations we can restrict our attention to one of them. The saddle point equation for $\hat{u}$ yields

$$\hat{s} = \frac{\hat{u}_0}{\hat{n} + \hat{u}_0} \prod_{m=1}^{M} \frac{\hat{\nu}_m - \hat{u}_0}{\hat{\nu}_m + 1}, \qquad 0 \leq \hat{n} \leq 1. \tag{75}$$

Equation (75) gives the saddle points, $\hat{u}_0$, for any given $\hat{s}$. In order to find the saddle points for the edges of the spectrum, we have to find the values of $\hat{n}$ and $\hat{u}_0$ which give the extremal values of $\hat{s}$.

Optimising with respect to $\hat{n}$, we see that $\hat{n}$ has no optimal value within the interval $(0, 1)$, hence $\hat{n}$ must lie on the boundary due to the Laplace approximation (saddle point approximation on a real support). The only non-trivial result comes from $\hat{n} = 1$. Inserting this condition into the saddle point equation (75) we reproduce formula (70). The condition for $\hat{u}_0$ is given by differentiating the left hand side of the saddle point equation (75) and setting this result equal to zero,

$$\frac{d}{d\hat{u}_0} \left[\frac{\hat{u}_0}{1 + \hat{u}_0} \prod_{m=1}^{M} \frac{\hat{\nu}_m - \hat{u}_0}{\hat{\nu}_m + 1} \right] = 0. \tag{76}$$

This condition is identical to formula (71). Hence the saddle point method reproduces the result obtained from the algebraic equation (64) for the resolvent.

The saddle points, which satisfy equation (76), are the extrema of the function within the square brackets. This function has a pole at -1 and goes to $+\infty$ for $\hat{u}_0 \to -\infty$ such that there is exactly one minimum to the left of the pole, see figure 3. On the right of the pole the function oscillates such that it has zeros at $0, \hat{\nu}_1, \ldots, \hat{\nu}_M$. Since the rational function on the right hand side of equation (75) is continuous it has extrema between neighbouring zeros, see figure 3, yielding M additional extrema. It follows that the optimisation problem (76) has $M + 1$ solutions for $\hat{u}_0$, which are all real: One solution $\hat{u}_0^+ < -1$ which gives the outer edge of the spectrum $\hat{s}_+$, one solution $0 \leq \hat{u}_0^- \leq \hat{\nu}_{\min}$ which gives the inner edge of the spectrum $\hat{s}_-$, and $M-1$ solutions $\hat{u}_0 \geq \nu_{\min}$ which must be disregarded due to the cut in the complex $\hat{u}$-plane mentioned above. It is clear that equation (76) cannot have more than $M+1$ solutions implying that we have found all solutions. With this result we know how to choose the correct solution of equation (71), which was what we wanted to establish.

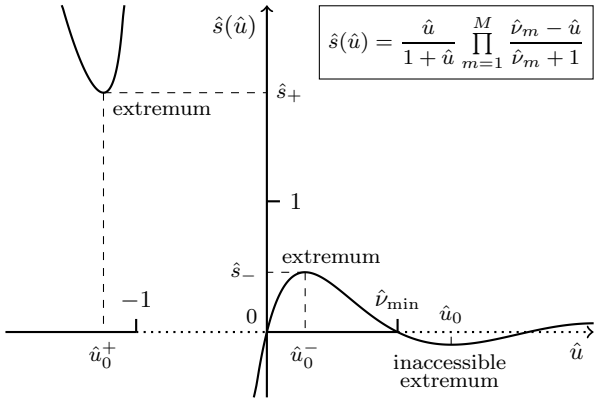

Figure 3. Illustration of the optimisation problem given by equation (76). Extrema within the intervals $(-1, 0)$ and $(\hat{\nu}_{\min}, \infty)$ must be disregarded due to the cuts in the complex $(-\hat{v})$-plane and complex $\hat{u}$-plane, respectively. This leaves only two valid extrema which correspond to the inner edge and the outer edge, respectively. Note that the solutions for the inner edge and the outer edge are separated by the pole at -1.

Before ending the discussion about the edges of the spectral density, it is worth noting that equation (71) is an $(M + 1)$-st order equation, and the general case can for this reason not be solved analytically. However, it is possible to set up some analytical bounds for the edges. The starting point are the conditions $0 \leq \hat{u}_0^- \leq \hat{\nu}_{\min}$ and $-\infty < \hat{u}_0^+ < -1$ for the saddle points. We will analyse step by step first the bounds on the inner edge, $\hat{s}_-$, and then on the outer edge, $\hat{s}_+$.

Let us consider the inner edge, $\hat{s}_-$. Since $0 \leq \hat{\nu}_{\min} \leq \hat{\nu}_m$, $m = 1, \ldots, M$, we can readily estimate

$$\min\left\{ \frac{\hat{\nu}_m}{\hat{\nu}_m + 1}, \frac{\hat{\nu}_{\max} - \hat{u}_0}{\hat{\nu}_{\max} + 1} \right\} \geq \frac{\hat{\nu}_m - \hat{u}_0}{\hat{\nu}_m + 1} \geq \frac{\hat{\nu}_{\min} - \hat{u}_0}{\hat{\nu}_{\min} + 1} \tag{77}$$

for any $\hat{u}_0 \geq 0$. Note that these bounds hold since the rational function, $(\hat{\nu}_m - \hat{u}_0)/(\hat{\nu}_m + 1)$, is strictly monotonously increasing in $\hat{\nu}_m$ for $\hat{u}_0 \geq 0$. We plug equation (77) into equation (70) and extremise the lower and upper bound which yields

$$0 \leq \hat{s}_-(\hat{\nu}_{\min}) \leq \hat{s}_- \leq \min\left\{ \prod_{m=1}^{M} \frac{\hat{\nu}_m}{\hat{\nu}_m + 1}, \hat{s}_-(\hat{\nu}_{\max}) \right\} < 1, \tag{78}$$

where we made use of the result (72) for the case when all $\hat{\nu}$ are equal to $\hat{\nu}_{\min}$ or to $\hat{\nu}_{\max}$. The bounds (78) are not at all optimal. However they immediately reflect the fact that the inner edge vanishes if and only if $\hat{\nu}_{\min}$ vanishes.

For the outer edge we have to employ the condition $\hat{u}_0 < -1$ which yields the estimates

$$\frac{\hat{\nu}_{\min} - \hat{u}_0}{\hat{\nu}_{\min} + 1} \geq \frac{\hat{\nu}_m - \hat{u}_0}{\hat{\nu}_m + 1} \geq \frac{\hat{\nu}_{\max} - \hat{u}_0}{\hat{\nu}_{\max} + 1}. \tag{79}$$

Hereby we used the fact that the rational function, $(\hat{\nu}_m - \hat{u}_0)/(\hat{\nu}_m + 1)$, is monotonously decreasing in $\hat{\nu}_m$ in the

considered regime. Employing the result (72) we find the bounds

$$1 < \hat{s}_+(\hat{\nu}_{\max}) \leq \hat{s}_+ \leq \hat{s}_+(\hat{\nu}_{\min}) \leq \frac{(M+1)^{M+1}}{M^M} < \infty. \tag{80}$$

Again the bounds can certainly be improved but they give a good picture what the relation is between the case of degenerate $\hat{\nu}$, cf. equation (72), and the general case, $\hat{\nu}_j \neq \hat{\nu}_i$ for $j \neq i$.

V. MUTUAL INFORMATION FOR PROGRESSIVE SCATTERING

We will now turn to a brief discussion of the mutual information, which is an important quantity in wireless telecommunication. We look at a MIMO communication channel with multi-fold scattering as mentioned in section I. The communication link is described by a channel matrix given by a product of complex ($\beta = 2$) matrices from the Wishart ensemble as in equation (1). The mutual information is defined as

$$\begin{aligned}\mathcal{I}(\gamma, s) &= \log_2 \det\Big[\mathbf{1}_{N_0} + \gamma \frac{\mathbb{Y}_M \mathbb{Y}_M^\dagger}{\mathcal{N}_M}\Big] \\ &= \sum_{a=1}^{N_0} \log_2\Big(1 + \gamma \frac{s_a}{\mathcal{N}_M}\Big),\end{aligned} \tag{81}$$

where γ is the constant signal-to-noise ratio at the transmitter and s_a are the singular values distributed according to the density (52). The mutual information measures an upper bound for the spectral efficiency in bits per time per bandwidth (bit/s/Hz).

In order to evaluate the expectation value of the mutual information, the so-called ergodic mutual information, we rewrite the logarithm as a Meijer G-function, see equation (A10). We use the expression (47) for the functions $\varphi_n^M(s)$, while we write $p_n^M(s)$ in polynomial form (43). The integration over the product of two Meijer G-functions can be performed using equation (A6), which finally yields

$$\begin{aligned}\hat{\mathbb{E}}\{\mathcal{I}(\gamma,\hat{s})\} = \frac{1}{\ln 2} \sum_{n=0}^{N_0-1} \sum_{k,\ell=0}^{n} \frac{(-1)^{k+\ell}}{(n-k)!(n-\ell)!} \\ \times \frac{n!}{k!\ell!} \frac{(n+\nu_1)!}{(\ell+\nu_1)!} \prod_{m=1}^{M} \frac{1}{(k+\nu_m)!} \\ \times G_{2,\,M+2}^{M+2,\,1}\Big({}^{0,\,1}_{k+1+\nu_M,\,\ldots,\,k+\ell+1+\nu_1,\,0,\,0} \,\Big|\, \gamma^{-1}\Big)\end{aligned} \tag{82}$$

For square matrices, i.e. $\nu_i = 0$ for all $i = 1 \ldots M$, this triple sum was derived in [23]. Although it is not obvious from this formulation, the mutual information is also independent of the ordering of ν_m. This is reflected after simplifying the expression (82) with help of a combination of the equations (40), (48), (A5), and (A8) to

$$\begin{aligned}\hat{\mathbb{E}}\{\mathcal{I}(\gamma,\hat{s})\} = \frac{1}{\ln 2} \sum_{n=0}^{N_0-1} \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} \prod_{m=1}^{M} \frac{1}{(k+\nu_m)!} \\ \times G_{3,\,M+3}^{M+2,\,2}\Big({}^{k-n+1,\,0,\,1}_{k+1+\nu_M,\,\ldots,\,k+1+\nu_1,\,0,\,0,\,k+1} \,\Big|\, \gamma^{-1}\Big)\end{aligned} \tag{83}$$

Hence, the channel matrix does not depend on the ordering of the scattering objects as long as the signal passes through all scatterers.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have studied the correlations of the singular values of the product of M rectangular complex matrices from independent Wishart ensembles. This generalises the classical result for the so-called Wishart–Laguerre unitary ensemble (or chiral unitary ensemble) at $M = 1$, and is a direct extension of a recent result for the product of square matrices [23]. We have seen that the problem of determining the statistical properties of the product of rectangular matrices can be equivalently formulated as a problem with the product of quadratic matrices and a modified, also called induced measure, see [24] for a general derivation. The expense of this reformulation of the problem is the introduction of additional determinants in the partition function.

We have shown that the joint probability density function for the singular values can be expressed in terms of Meijer-G functions. The approach which we have used relies on an integration formula for the Meijer-G function as well as on the Harish-Chandra–Itzykson–Zuber integration formula. Due to the latter this method is limited to the complex case ($\beta = 2$). Furthermore, it has been shown, using a two-matrix model and the method of bi-orthogonal polynomials, that all correlation functions can be expressed as a determinantal point process containing Meijer-G functions. From the explicit expressions we derived it follows that all correlation functions are independent of the ordering of the matrix dimensions.

The level density (or one-point correlation function) was discussed in detail. We used the spectral density to calculate all moments and derived its macroscopic limit. In particular, we analysed the location of the end points of the spectrum in the macroscopic limit for arbitrary M and derived some narrow bounds for the location of these edges.

As an application we briefly discussed the ergodic mutual information, and how the singular values of products of random matrices are related to progressive scattering in MIMO communication channels.

The results presented in this work concern matrices of finite size, while previous results for the product of rectangular random matrices were only derived in the macroscopic large-N_0 limit. The explicit expressions for

all correlation functions at finite size make it possible to also discuss microscopic properties, such as the local correlations in the bulk and at the edges. Due to known universality results for random matrices it is expected that such an analysis should reproduce the universal sine and Airy kernel in the bulk and at the soft edge(s), respectively, after an appropriate unfolding. Close to the origin the level statistics will crucially depend on whether or not the difference of the individual matrix dimensions to the smallest one, $\nu_m = N_m - N_0$, scales with N_0. If it does this will lead to a soft edge. Else it is expected, that the microscopic behaviour at the origin will be sensitive to M and ν_m. For a single matrix with $M = 1$ (the Wishart–Laguerre ensemble), it is already known that this limit yields different Bessel universality classes labelled by ν_1.

Furthermore, the determinantal structure of the correlation functions make it possible to study the distribution of individual singular values, which is an intriguing problem in its own right.

It has been pointed out in [37], that for the product of two square matrices, $M = 2$ and $\nu_1 = 0$, the bi-orthogonal polynomials in question are special cases of multiple orthogonal polynomials associated with the modified Bessel function of the second kind. It is an intriguing task to see whether this approach can be extended to the more general case with $M \geq 2$ and rectangular matrices. Progress in this direction has already been made [38].

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Appendix A: Special Functions and some of their Identities

In this appendix we collect some definitions and identities for the generalised hypergeometric function and for the Meijer G-function, which are used in this paper.

The generalised hypergeometric function is defined by a power series in its region of convergence [41],

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) \equiv \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_k}{\prod_{i=1}^{q} (b_i)_k} \frac{z^k}{k!}, \tag{A1} $$

where the Pochhammer symbol is defined by $(a)_0 = 1$ and $(a)_n \equiv (a+n-1)(a)_{n-1} = a(a+1)\cdots(a+n-1)$ for $n \geq 1$. It is clear that the hypergeometric series (A1) terminates if any of the a_i's is a negative integer. In particular, if n is a positive integer then

$$ {}_{p+1}F_q\left(\begin{matrix} -n, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{\prod_{i=1}^{p} (a_i)_k}{\prod_{i=1}^{q} (b_i)_k} \frac{z^k}{k!}, \tag{A2} $$

which is a polynomial of degree n or less.

The Meijer G-function can be considered as a generalisation of the generalised hypergeometric function. It is usually defined by a contour integral in the complex plane [41],

$$ G_{p,\,q}^{m,\,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) \equiv \frac{1}{2\pi\imath} \int_L du\, z^u \frac{\prod_{i=1}^{m} \Gamma[b_i - u] \prod_{i=1}^{n} \Gamma[1 - a_i + u]}{\prod_{i=n+1}^{p} \Gamma[a_i - u] \prod_{i=m+1}^{q} \Gamma[1 - b_i + u]}. \tag{A3} $$

The contour runs from $-\imath\infty$ to $+\imath\infty$ and is chosen such that it separates the poles stemming from $\Gamma[b_i - u]$ and the poles stemming from $\Gamma[1 - a_i + u]$. Furthermore this contour can be considered as an inverse Mellin transform. For an extensive discussion of the integration path L and the requirements for convergence see [42].

It follows that the Mellin transform of a Meijer G-function is given by [41]

$$ \int_0^{\infty} ds\, s^{u-1} G_{p,\,q}^{m,\,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| sz \right) = z^{-u} \frac{\prod_{i=1}^{m} \Gamma[b_i + u] \prod_{i=1}^{n} \Gamma[1 - a_i - u]}{\prod_{i=n+1}^{p} \Gamma[a_i + u] \prod_{i=m+1}^{q} \Gamma[1 - b_i - u]}, \tag{A4} $$

which is results from the definition of the Meijer G-function (A3). In combination with the definition of the gamma-function we have another identity

$$ \int_0^{\infty} dt\, e^{-t} t^{b_0 - 1} G_{p,\,q}^{m,\,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{s}{t} \right) = G_{p,\,q+1}^{m+1,\,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_0, \dots, b_q \end{matrix} \middle| s \right). \tag{A5} $$

Both of these integral identities are used throughout this paper. Another integral identity, which is used in section V, allows us to integrate over the product of two Meijer G-functions [43],

$$ \int_0^{\infty} ds\, G_{p,\,q}^{m,\,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \eta s \right) G_{\sigma,\,\tau}^{\mu,\,\nu}\left(\begin{matrix} c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau \end{matrix} \middle| \omega s \right) = \frac{1}{\omega} G_{p+\tau,\,q+\sigma}^{m+\nu,\,n+\mu}\left(\begin{matrix} a_1, \dots, a_n, -d_1, \dots, -d_\tau, a_{n+1}, \dots, a_p \\ b_1, \dots, b_m, -c_1, \dots, -c_\sigma, b_{m+1}, \dots, b_q \end{matrix} \middle| \frac{\eta}{\omega} \right). \tag{A6} $$

The full set of restrictions on the indices for this integration formula can be found in [43].

In addition to the integral identities given above, we need some other identities for the Meijer G-function. We employ several times that it is possible to absorb powers

of the argument into the Meijer G-function, by making a shift in the arguments [41],

$$z^{\rho} G_{p,\,q}^{m,\,n}\Big({a_1,\ldots,a_p \atop b_1,\ldots,b_q} \,\Big|\, z\Big) = G_{p,\,q}^{m,\,n}\Big({a_1+\rho,\ldots,a_p+\rho \atop b_1+\rho,\ldots,b_q+\rho} \,\Big|\, z\Big). \quad \text{(A7)}$$

For computing the function $\varphi_n^M(s)$ in section III, we need the differential identity [43]

$$z^n \frac{d^n}{dz^n} G_{p,\,q}^{m,\,n}\Big({a_1,\ldots,a_p \atop b_1,\ldots,b_q} \,\Big|\, \frac{1}{z}\Big) = (-1)^n G_{p+1,\,q+1}^{m,\,n+1}\Big({1-n,\,a_1,\ldots,a_p \atop b_1,\ldots,b_q,\,1} \,\Big|\, \frac{1}{z}\Big). \quad \text{(A8)}$$

We also use that the generalised hypergeometric polynomial is related to the Meijer G-function by

$${}_1F_q\Big({-n \atop b_1,\ldots,b_q} \,\Big|\, z\Big) = n! \prod_{i=1}^{q} \Gamma[b_i]\, G_{1,\,M+1}^{1,\,0}\Big({n+1 \atop 0,\,1-b_1,\ldots,1-b_q} \,\Big|\, z\Big), \quad \text{(A9)}$$

in order to write the polynomial $p_n^M(s)$ as a Meijer G-function in section III.

As a last remark of this appendix, it should be mentioned that the Meijer G-function contains a vast number of elementary and special functions as special cases (e.g. see [44]). We mention that

$$G_{0,\,1}^{1,\,0}\Big({- \atop b} \,\Big|\, z\Big) = z^b e^{-z} \quad \text{and} \quad G_{2,\,2}^{1,\,2}\Big({1,\,1 \atop 1,\,0} \,\Big|\, z\Big) = \ln(1+z), \quad \text{(A10)}$$

which becomes useful in sections II and V, respectively.

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Weak Commutation Relations and Eigenvalue Statistics for Products of Rectangular Random Matrices

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We study the joint probability density of the eigenvalues of a product of rectangular real, complex or quaternion random matrices in a unified way. The random matrices are distributed according to arbitrary probability densities, whose only restriction is the invariance under left and right multiplication by orthogonal, unitary or unitary symplectic matrices, respectively. We show that a product of rectangular matrices is statistically equivalent to a product of square matrices. Hereby we prove a weak commutation relation of the random matrices at finite matrix sizes, which previously have been discussed for infinite matrix size. Moreover we derive the joint probability densities of the eigenvalues. To illustrate our results we apply them to a product of random matrices drawn from Ginibre ensembles and Jacobi ensembles as well as a mixed version thereof. For these weights we show that the product of complex random matrices yield a determinantal point process, while the real and quaternion matrix ensembles correspond to Pfaffian point processes. Our results are visualized by numerical simulations. Furthermore, we present an application to a transport on a closed, disordered chain coupled to a particle bath.

I. INTRODUCTION

Recently, products of random matrices have experienced a revival due to new mathematical insights about the statistics of the eigen- and singular values for finite as well as infinite matrix dimensions. Recent progress in the field has made it possible to study a product of an arbitrary number of random matrices of arbitrary size for certain matrix ensembles. The fact that the number of matrices and their size can be chosen freely, allows discussions of various limits. This includes macroscopic as well as microscopic structures for infinite matrix dimension, but also the limit where the number of matrices goes to infinite is available. Analogous to the study of individual random matrices, products of random matrices show a rich mathematical structure and various limits have revealed new universality classes, which are important in the physical sciences as well as in mathematics and beyond.

Products of random matrices have been applied to a broad spectrum of the physical sciences. To name only a few of them: Transport in disordered and chaotic systems [1], matrix-valued diffusions [2, 3], quantum chromodynamics at finite chemical potential [4, 5], Yang–Mills theory [6], and percolation theory (introduction of [7]). Furthermore, results about products of random matrices have been applied to fields beyond physics, such as the study of wireless telecommunication [8] and finance [9] as well as directions within mathematics, e.g. free probability [10]. In this work an example of chaotic transport (cf. sec. II) will be our main motivation, but we discuss the mathematical structure in a completely general setting. Hence the results presented in this paper can be directly applied to other situations as well.

Note that, even though certain symmetries of random matrices might be conserved under matrix multiplication (such as unitarity), in general a product matrix possesses less symmetry than the individual matrices. In particular, a product of Hermitian matrices will not generally be Hermitian itself, and the eigenvalues will spread into the complex plane. This loss of symmetry has let to a particular interest in products of non-Hermitian matrices, especially drawn from Gaussian ensembles [11–15]. These random matrix ensembles are also known as Ginibre ensembles or Wishart ensembles. The discussion of products of Ginibre matrices at finite matrix dimension can be considered as an extension of previous results related to the product of two matrices motivated by applications to quantum chromodynamics at finite chemical potential [4, 16, 17]. In our work we will go beyond the restriction to Ginibre ensembles and study a set of general weights only restricted by the invariance under left- and right-multiplication of unitary matrices. Furthermore, we will discuss all three Dyson classes in a unified way. We illustrate the underlying structure with the two particular ensembles of Ginibre and Jacobi (truncated unitary) matrices. These matrices are directly related to a transport on a closed one-dimensional chaotic chain in an environment as it is shown in Sec. II.

In Sec. III we will show that products of random matrices invariant under left- and right-multiplication of unitary matrices satisfy a weak commutation relation. This commutation relation holds even for finite matrix dimension and not only for infinite dimension as discussed in [18]. Moreover it has important physical consequences. In the specific example of a closed one-dimensional chaotic chain in an environment it reflects the invariance under reordering of potential wells as long as we do not consider cross correlations. Note that the weak commutation relation presented here is completely general and represents a general physical property of reordering invariance. For instance the same mecha-

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nism implies that communication channels with progressive scattering in wireless telecommunication are invariant under reordering of the clusters of scatterers [15]. A similar commutation relation has previously been discussed in the context of disordered wires with obstacles citeBM:1994.

In Secs. IV and V we explicitly discuss two explicit realizations of matrix ensembles, namely products of Ginibre and Jacobi matrices, as well as an intermix of these and apply this result to derive an expression of the Lyapunov exponent for the disordered chain proposed in Sec. II. Section VI is devoted to conclusions and outlook, while some technical details are presented in the appendix.

II. MOTIVATION: CLOSED ONE-DIMENSIONAL CHAOTIC CHAIN

In this section we consider a model of a unitary evolution. It differs from a similar quantum evolution discussed in Ref. [2, 20] by the idea that there is a coupling to an environment. Thus it is more in the spirit of one-dimensional quantum transport in a disordered system [21–23]. The unitary evolution matrix is taken to be time independent, hence we do not model a diffusive system but a chaotic quantum system, which scatters the particles into an environment. Here the environment is also modelled as a chaotic quantum system. Due to the coupling to the environment the evolution operator acts non-unitarily on the studied subsystem.

We consider a chain of M Hilbert-spaces of dimension $N_1, \ldots, N_M$ arranged along a ring. The Hilbert-spaces can be constructed by isolated potential wells. The particles in these wells can jump from one well to a consecutive one only in one direction and cannot stay in a well. Thus we have a totally asymmetric quantum transport. The system is constructed such that each well is coupled to a joint particle bath (environment) which can absorb the particles while the total amount of particles (on the chain and in the bath) is fixed. Moreover the Hamiltonian shall be time independent such that the transfer matrix is given by the unitary matrix

$$U = \begin{pmatrix} X\widehat{T} & V_1 \\ V_2 & V_3 \end{pmatrix}. \tag{1}$$

The translation matrix, $\widehat{T}$, is defined as

$$\widehat{T} = \begin{bmatrix} 0 & \mathbb{1}_{N_M} & 0 & & \cdots & 0 \\ & 0 & \mathbb{1}_{N_1} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & & & \\ 0 & & \cdots & & 0 & \mathbb{1}_{N_{M-2}} \\ \mathbb{1}_{N_{M-1}} & 0 & & \cdots & & 0 \end{bmatrix}, \tag{2}$$

such that particles can jump between consecutive wells in one direction only. The transport along the chain is determined by the block diagonal matrix

$$X = \mathrm{diag}(X_1, \ldots, X_M) \tag{3}$$

where the block matrices, X_m, are rectangular matrices with real ($\beta = 1$), complex ($\beta = 2$) or quaternion ($\beta = 4$) entries chosen according to Dyson's three-fold way [7, 24],

$$X_j \in \mathrm{gl}_\beta(N_j, N_{j-1}) \equiv \begin{cases} \mathbb{R}^{N_j \times N_{j-1}}, & \beta = 1, \\ \mathbb{C}^{N_j \times N_{j-1}}, & \beta = 2, \\ \mathbb{H}^{N_j \times N_{j-1}}, & \beta = 4. \end{cases} \tag{4}$$

We set $N_0 = N_M$, since we consider a closed chain. The other three block matrices, V_i, in Eq. (1) are chosen such that U is orthogonal ($\beta = 1$), unitary ($\beta = 2$), or unitary symplectic ($\beta = 4$), respectively.

Throughout this paper we will use the standard 2×2 matrix representation for the quaternion number field, $\mathbb{H}$, see e.g. [7].

Let N_{bath} be the dimension of the Hilbert space of the bath and $N_{\mathrm{chain}} = \sum_{j=1}^{M} N_j$ be the one of the chain. We assume that the jumping as well as the coupling to the bath is a stochastic process. Since we do not assume any additional symmetry breaking condition apart from the Dyson classification [7, 24] and the totally asymmetric process, the measure for U is given by the Haar-measure of $\mathrm{O}(N_{\mathrm{chain}} + N_{\mathrm{bath}})$, $\mathrm{U}(N_{\mathrm{chain}} + N_{\mathrm{bath}})$ and $\mathrm{USp}(2N_{\mathrm{chain}} + 2N_{\mathrm{bath}})$ for $\beta = 1, 2, 4$, respectively.

For this purpose we briefly rederive the measure for a rectangular, truncated unitary matrix $X \in \mathrm{gl}_\beta(N_1, N_2)$ resulting from

$$U = \begin{bmatrix} X & V_1 \\ V_2 & V_3 \end{bmatrix} \in \begin{cases} \mathrm{O}(L), & \beta = 1, \\ \mathrm{U}(L), & \beta = 2, \\ \mathrm{USp}(2L), & \beta = 4, \end{cases} \tag{5}$$

with $N_1, N_2 < L$. In order to deal with all three Dyson classes in a unified way, we introduce the variable

$$\gamma = \begin{cases} 1, & \beta = 1, 2, \\ 2, & \beta = 4. \end{cases} \tag{6}$$

The measure for the truncated matrix X is given by [34]

$$d\nu^{(\mathrm{J})}(X) \propto d[X] \int d[V_1] d[V_2] d[V_3]\, \delta(UU^\dagger - \mathbb{1}_{\gamma L}), \tag{7}$$

where $d[X]$ and $d[V_j]$ denote the product of all independent differentials (there are β real independent degrees of freedom per matrix entry). The Dirac δ-function for matrices is defined by the product of all Dirac δ-functions of all independent real entries. The Dirac δ-function ensures that U is orthogonal, unitary or unitary symplectic, respectively. It is straightforward to integrate out the irrelevant degrees of of freedom, V_i, which yields [34, 36]

$$d\nu^{(\mathrm{J})}(X) \propto d[X] \Theta(\mathbb{1}_{\gamma N_1} - XX^\dagger) \det{}^{\kappa}(\mathbb{1}_{\gamma N_1} - XX^\dagger), \tag{8}$$

which is known as the Jacobi ensemble [7, 25, 26] and is labelled by a superscript (J). The Heaviside Θ-function for matrices is equal to unity for positive definite matrices and zero otherwise; and the power of the determinant, κ,

is a constant given by $\kappa = \beta(L-N_1-N_2+1-2/\beta)/(2\gamma)$. The measure (8) plays an important role in rest of this paper.

The discussion of the truncated unitary matrix, discussed in precious paragraph, can immediately be applied to the case described by Eq. (1). Integrating over V_i in Eq. (1), we find

$$d\nu(X) \propto d[X]\Theta(\mathbb{1}_{\gamma N_{\rm chain}} - XX^\dagger)\det{}^\kappa(\mathbb{1}_{\gamma N_{\rm chain}} - XX^\dagger) \tag{9}$$

with $\kappa = \beta(N_{\rm bath} - N_{\rm chain} + 1 - 2/\beta)/(2\gamma)$. The size of the bath, $N_{\rm bath}$, and the size of the chain, $N_{\rm chain}$, are independent quantities and one might be interested in the limit where the size of the bath goes to infinite while the size of the chain is kept fixed. The matrix X will be of order $1/\sqrt{N_{\rm bath}}$, hence we rescale $X \to X/\sqrt{N_{\rm bath}}$. Thus for $N_{\rm bath} \gg N_{\rm chain}$ the measure of X equals a Gaussian distribution,

$$d\nu\left(\frac{X}{\sqrt{N_{\rm bath}}}\right) \propto \exp\left[-\frac{\beta}{2\gamma}\,{\rm Tr}\, XX^\dagger\right], \tag{10}$$

which follows from taking the limit in Eq. (9) without further restrictions. In particular, we do not need the central limit theorem, since the degrees of freedom are independent of $N_{\rm bath}$. As a consequence the sub-matrices, see Eq. (3), are also Gaussian distributed,

$$d\nu^{\rm (G)}(X_j) \propto \exp\left[-\frac{\beta}{2\gamma}\,{\rm Tr}\, X_jX_j^\dagger\right]. \tag{11}$$

This is also known as the Ginibre ensemble [7, 25, 26] and is denoted by a superscript (G). Depending on how large each well is, compared to the environment, one can also consider a mixed product of Ginibre and Jacobi matrices.

The distributions (9) and (10) enable us to calculate the spectral statistics of U in the sector of the Hilbert-space representing the chain. In particular, we can consider the spectral statistics of $X\widehat{T}$. Note that the eigenvalues of $X\widehat{T}$ are intimately related to $(X\widehat{T})^M$ or, equivalently, to the product matrix $X^{(M)} \equiv X_MX_{M-1}\cdots X_2X_1$. Also the Lyapunov exponents defined as the logarithm of either the eigenvalues of $X^{(M)}$ or of the singular values (eigenvalues of $X^{(M)}X^{(M)\,\dagger}$) are widely used, see for example Refs. [22, 23, 27]. They measure the difference of a vector transported once around the chain by the non-unitary evolution

III. EQUIVALENCE OF DIFFERENT PRODUCTS OF RANDOM MATRICES

An important property of products of rectangular random matrices is their relation to products of identically sized square matrices with deformed weights. The deformations are essentially prefactors of determinants of the random matrices and induce a repulsion from the origin, see Refs. [12, 15, 28–30] for particular examples of Gaussian weights. In this section we study this relation for all three Dyson classes in a unified way. We emphasize that we do not specify a particular probability density for the random matrices in this section. The only assumption we enforce on the independent weights is invariance under one of the three groups O(N), U(N), and USp($2N$) for $\beta = 1, 2, 4$, respectively.

We consider the product of M rectangular matrices,

$$X^{(M)} = X_MX_{M-1}\cdots X_2X_1, \tag{12}$$

where the individual matrices $X_j \in {\rm gl}_\beta(N_j, N_{j-1})$ have real, complex or quaternion entries according to the Dyson index, β. The rank of the product matrix is equal to $N_{\rm min} \equiv \min_{j=0,\ldots,M} N_j$. The product matrix is distributed according to the independent weights of the individual matrices,

$$d\nu_{P_1,P_2,\ldots,P_M}(X^{(M)}) = \prod_{j=1}^M P_j(X_j)d[X_j], \tag{13}$$
$$d[X_j] \equiv \prod_{a=1}^{N_j}\prod_{b=1}^{N_{j-1}}\prod_{\alpha=1}^{\beta} dX_{ab}^{(j,\alpha)},$$

where the product over α runs over all real degrees of freedom of a single matrix entry. The only assumptions about the weights, P_j, is the invariance under left and right rotations,

$$P_j(X_j) = P_j(VX_jU), \tag{14}$$

for all transformations $V \otimes U$ in

$$\mathrm{U}_\beta(N_j, N_{j-1}) \equiv \begin{cases} \mathrm{O}(N_j)\otimes\mathrm{O}(N_{j-1}), & \beta = 1, \\ \mathrm{U}(N_j)\otimes\mathrm{U}(N_{j-1}), & \beta = 2, \\ \mathrm{USp}(2N_j)\otimes\mathrm{USp}(2N_{j-1}), & \beta = 4. \end{cases} \tag{15}$$

These probability densities were referred to as isotropic weights in Ref. [31]. Particular examples are: Gaussian weights (Ginibre ensembles) [11, 12, 14, 15, 27, 28, 30, 32, 33],

$$d\nu^{\rm (G)}(X^{(M)}) = \prod_{j=1}^M P^{\rm (G)}(X_j) \propto \prod_{j=1}^M \exp\left[-\,{\rm Tr}\, X_jX_j^\dagger\right], \tag{16}$$

and weights which are the induced Haar measure of truncated unitary matrices (Jacobi ensembles) [30, 34–36]

$$d\nu_\kappa^{\rm (J)}(X^{(M)}) = \prod_{j=1}^M P_{\kappa_j}^{\rm (J)}(X_j) \tag{17}$$
$$\propto \prod_{j=1}^M \det{}^{\kappa_j}(\mathbb{1}_{\gamma N_j} - X_jX_j^\dagger)\Theta(\mathbb{1}_{\gamma N_j} - X_jX_j^\dagger)d[X_j],$$

where $\kappa_j + \beta\min\{N_j, N_{j-1}\}/2 + (\beta-2)/2 > 0$, compare with Eq. (8). Note that we deal with all three Dyson indices in a unified way. Hence the number of real independent degrees of freedom of a single matrix entry, $X_{ab}^{(j)}$, is equal to the Dyson index, β.

As will be discussed in detail below, a product of rectangular random matrices, $X^{(M)}$, can be expressed in terms of a product of square $\gamma N_{\min} \times \gamma N_{\min}$ random matrices and two truncated unitary matrices,

$$X^{(M)} = U_{\rm L} \widetilde{X}^{(M)} U_{\rm R} = U_{\rm L} \widetilde{X}_M \widetilde{X}_{M-1} \cdots \widetilde{X}_2 \widetilde{X}_1 U_{\rm R}. \quad (18)$$

Here $\widetilde{X}_j \in {\rm gl}_\beta(N_{\min}, N_{\min})$ are square matrices, while the $\gamma N_M \times \gamma N_{\min}$ matrix $U_{\rm L}$ consists of the first $\gamma N_{\min}$ columns of an element in the coset

$$\mathbb{G}_\beta(N_M, N_{\min}) \equiv \begin{cases} {\rm O}(N_M)/[{\rm O}(N_{\min}) \times {\rm O}(\nu_M)], & \beta = 1, \\ {\rm U}(N_M)/[{\rm U}(N_{\min}) \times {\rm U}(\nu_M)], & \beta = 2, \\ {\rm USp}(2N_M)/[{\rm USp}(2N_{\min}) \times {\rm USp}(2\nu_M)], & \beta = 4. \end{cases} \quad (19)$$

Likewise, the $\gamma N_{\min} \times \gamma N_0$ matrix $U_{\rm R}$ is equal to the first $\gamma N_{\min}$ rows of an element in the coset $\mathbb{G}_\beta(N_0, N_{\min})$. In (19) we have introduced the notation $\nu_j \equiv N_j - N_{\min}$. Note that the product matrix $\widetilde{X}^{(M)}$ is a square matrix. It is immediate from Eq. (18) that the nonzero singular values of $\widetilde{X}^{(M)}$ is identical to those of the original product matrix, $X^{(M)}$. Furthermore, if $X^{(M)}$ is a square matrix, then it turns out that also the eigenvalues will agree, except for $N_0 - N_{\min}$ additional eigenvalues, which are all equal to zero. We will refer to the square product matrix, $\widetilde{X}^{(M)}$, as the induced product matrix and it can be considered as a generalization of the induced ensemble discussed in Ref. [29].

Our main goal in this section is to derive the measure for the random matrix $X^{(M)}$ or equivalently the induced measures for the matrices $\widetilde{X}_j$ and $U_{\rm L}$ and $U_{\rm R}$. Furthermore, we establish a weak commutation relation for the square matrices, $\widetilde{X}_j$.

In subsection III A, we present the two simplest cases, namely when one of the "end-points" of the chain of dimensions encountered in the product matrix, $X^{(M)}$, is equal to $\gamma N_{\min}$, i.e. $N_0 = N_{\min}$ or $N_M = N_{\min}$. In these cases the resulting measure for $X^{(M)}$ can be readily derived. There seems to be an ambiguity of the resulting measure on the level of the individual random matrix measures. However, we show that commutativity of square random matrices does not only hold in the large N-limit, see Ref. [18], but also at finite matrix size, see Sec. III B. See also Ref. [19] In Sec. III C we also derive a weak commutation relation between Jacobi matrices and an arbitrary isotropic random matrix which marginally changes the original weights. In subsection III D, we discuss the general setting of arbitrary dimensions N_j of the rectangular matrices in an arbitrary order.

A. Two simple cases

We will first consider the case where $N_0 = N_{\min}$, which implies that $U_{\rm R}$ is equal to unity. The matrix X_1 can easily be rotated to the $\gamma N_{\min} \times \gamma N_{\min}$ matrix $\widetilde{X}_1$ by a unitary transformation $U_1 \in \mathbb{G}_\beta(N_1, N_{\min})$ from the left, which gives a block structure

$$X_1 = U_1 \begin{bmatrix} \widetilde{X}_1 \\ 0 \end{bmatrix}. \quad (20)$$

Here 0 represents a $\gamma\nu_1 \times \gamma N_{\min}$ matrix with all entries equal to zero (recall that $\nu_j = N_j - N_{\min}$). The unitary matrix U_1 can be absorbed due to the ${\rm U}_\beta(N_2, N_1)$ invariance of the measure P_2, cf. (14). Since the matrix U_1 is distributed according to the Haar measure on the coset $\mathbb{G}_\beta(N_1, N_{\min})$, the integral over U_1 completely factorizes from the rest and yields a constant. The change of coordinates (20) yields the measure

$$\widetilde{P}_1^{({\rm L})}(\widetilde{X}_1) \propto \det{}^{\beta\nu_1/(2\gamma)}(\widetilde{X}_1 \widetilde{X}_1^\dagger) P_1 \left(\begin{bmatrix} \widetilde{X}_1 \\ 0 \end{bmatrix} \right) \quad (21)$$

for $\widetilde{X}_1$. The superscript (L) denotes that we decompose X_1 via a left block QR-decomposition, see Eq. (20). The determinantal prefactor comes from the change of coordinates and enforces an additional repulsion of the singular values as well as eigenvalues from the origin, cf. Refs. [12, 15, 28–30].

The smaller dimension of $\widetilde{X}_1$ projects the dimension on the right side of X_2 down to $\gamma N_{\min}$ in the product matrix $X^{(M)}$. Hence, we can again perform the same procedure as before for X_1. We bring X_2 into a block structure using a left block QR-decomposition

$$X_2 = U_2 \left[\begin{array}{c|c} \widetilde{X}_2 & \\ 0 & X_2' \end{array} \right], \quad (22)$$

hence X_2 is decomposed into a unitary matrix times a block matrix consisting of two rectangular blocks. The unitary matrix $U_2 \in \mathbb{G}_\beta(N_2, N_{\min})$ is absorbed in the measure P_3. The integration over the rectangular matrix $X_2' \in {\rm gl}_\beta(N_2, \nu_1)$ is comprised in the definition of the reduced measure

$$\widetilde{P}_2^{({\rm L})}(\widetilde{X}_2) \propto \det{}^{\beta\nu_2/(2\gamma)}(\widetilde{X}_2 \widetilde{X}_2^\dagger) \int d[X_2'] P_2 \left(\left[\begin{array}{c|c} \widetilde{X}_2 & \\ 0 & X_2' \end{array} \right] \right). \quad (23)$$

Again the determinantal prefactor is the result of the degrees of freedom decoupling via the unitary matrix U_2. We repeat the same procedure for $X_3, X_4 \ldots X_M$, and the new measures for the matrices $\widetilde{X}_j$, $2 \le j \le M$, defined by the choice of coordinates

$$X_j = U_j \left[\begin{array}{c|c} \widetilde{X}_j & \\ 0 & X_j' \end{array} \right] \quad (24)$$

are up to normalization constants

$$\widetilde{P}_j^{({\rm L})}(\widetilde{X}_j) \propto \det{}^{\beta\nu_j/(2\gamma)}(\widetilde{X}_j \widetilde{X}_j^\dagger) \int d[X_j'] P_j \left(\left[\begin{array}{c|c} \widetilde{X}_j & \\ 0 & X_j' \end{array} \right] \right), \quad (25)$$

where we integrate over the rectangular matrix $X'_j \in \mathrm{gl}_\beta(N_j, \nu_{j-1})$. Apart from $U_M \in \mathbb{G}_\beta(N_M, N_{\min})$ all unitary matrices can be absorbed in the measures P_j because of their group invariance. Due to projection, U_{L} is a matrix consisting of the first $\gamma N_{\min}$ rows of the unitary matrix U_M. Thus the measure of the product matrix, $X^{(M)}$ is given by

$$d\nu_{\widetilde{P}_1^{(\mathrm{L})}\cdots\widetilde{P}_M^{(\mathrm{L})}}(X^{(M)}) = d\mu(U_{\mathrm{L}}) \prod_{j=1}^{M} \widetilde{P}_j^{(\mathrm{L})}(\widetilde{X}_j) d[\widetilde{X}_j]. \quad (26)$$

Here $d\mu(U)$ denotes the Haar measure on $\mathbb{G}_\beta(N_M, N_{\min})$. Note that also the new weights, $\widetilde{P}_j^{(\mathrm{L})}$, are invariant under left and right multiplication of group elements in $\mathrm{O}(N_{\min})$, $\mathrm{U}(N_{\min})$, and $\mathrm{USp}(2N_{\min})$ for $\beta = 1, 2, 4$, respectively.

Let us state the Ginibre and Jacobi ensemble as explicit examples of the new measure. In the Gaussian case, the integrals over X'_j factorize and we find

$$d\nu_{I_{\mathrm{L}}}^{(\mathrm{G,L})}(X^{(M)}) \propto \quad (27)$$
$$d\mu(U_{\mathrm{L}}) \prod_{j=1}^{M} \exp\left[-\operatorname{Tr} \widetilde{X}_j \widetilde{X}_j^\dagger\right] \det{}^{\beta\nu_j/(2\gamma)}(\widetilde{X}_j \widetilde{X}_j^\dagger) d[\widetilde{X}_j],$$

where the multi-index $I_{\mathrm{L}} = (\nu_1, \ldots, \nu_M)$ encodes the ordering of the exponents. Recall that $\nu_j = N_j - N_{\min}$. This measure was studied in Ref. [15, 30] for complex matrices ($\beta = 2$) and in Ref. [12] for quaternion matrices ($\beta = 4$).

For the Jacobi ensemble the integral (25) is more involved. The Wishart matrix of the matrix X_j has the form

$$X_j X_j^\dagger = U_j \left(\begin{bmatrix} \widetilde{X}_j \widetilde{X}_j^\dagger & 0 \\ 0 & 0 \end{bmatrix} + X'_j X_j'^\dagger \right) U_j^\dagger \quad (28)$$

The matrix $(\mathbb{1}_{\gamma N_{\min}} - \widetilde{X}_j \widetilde{X}_j^\dagger)$ has to be positive definite, too. Therefore the transformation

$$X'_j \to \begin{bmatrix} (\mathbb{1}_{\gamma N_{\min}} - \widetilde{X}_j \widetilde{X}_j^\dagger)^{1/2} & 0 \\ 0 & \mathbb{1}_{\gamma(N_j - N_{\min})} \end{bmatrix} X'_j \quad (29)$$

is well-defined via the spectral decomposition theorem. Now, the integration over the rectangular matrices $X'_j \in \mathrm{gl}_\beta(N_j, \nu_{j-1})$ factorizes, and we end up with the new measure for the truncated unitary matrices

$$d\nu_{\kappa, I_{\mathrm{L}}}^{(\mathrm{J,L})}(X^{(M)}) \propto d\mu(U_{\mathrm{L}}) \quad (30)$$
$$\times \prod_{j=1}^{M} \Big[\det{}^{\kappa_j + \beta\nu_{j-1}/(2\gamma)}(\mathbb{1}_{\gamma N_{\min}} - \widetilde{X}_j \widetilde{X}_j^\dagger)$$
$$\times \det{}^{\beta\nu_j/(2\gamma)}(\widetilde{X}_j \widetilde{X}_j^\dagger) \Theta(\mathbb{1}_{\gamma N_{\min}} - \widetilde{X}_j \widetilde{X}_j^\dagger) d[\widetilde{X}_j] \Big].$$

Again, the subscripts denote $\kappa = (\kappa_1, \ldots, \kappa_M)$ and $I_{\mathrm{L}} = (\nu_1, \ldots, \nu_M)$, respectively. Notice that the new measure has a different exponent of the determinant $\det(\mathbb{1}_{\gamma N_{\min}} - \widetilde{X}_j \widetilde{X}_j^\dagger)$ due to the integration over the rectangular matrices X'_j. This measure was studied in Ref. [36] for complex matrices.

Let us return to general weights but now we consider the case $N_M = N_{\min}$. The matrix U_{L} is equal to unity. This time we start with X_M and rotate it to the $\gamma N_{\min} \times \gamma N_{\min}$ matrix $\widetilde{X}_M$. Again we get a determinantal prefactor, which is now $\det^{\beta\nu_{M-1}/(2\gamma)}(\widetilde{X}_M \widetilde{X}_M^\dagger)$. We repeat the same procedure as described in the paragraphs above only starting from the left and ending at the right. We use a right block QR-decomposition (RQ-decomposition)

$$X_j = \left[\begin{array}{c|c} \widetilde{X}_j & 0 \\ \hline X'_j & \end{array} \right] U_j. \quad (31)$$

The resulting measure is

$$d\nu_{\widetilde{P}_1^{(\mathrm{R})}\cdots\widetilde{P}_M^{(\mathrm{R})}}(X^{(M)}) = d\mu(U_{\mathrm{R}}) \prod_{j=1}^{M} \widetilde{P}_j^{(\mathrm{R})}(\widetilde{X}_j) d[\widetilde{X}_j]. \quad (32)$$

with the individual weights given by

$$\widetilde{P}_j^{(\mathrm{R})}(\widetilde{X}_j) \propto \det{}^{\beta\nu_{j-1}/(2\gamma)}(\widetilde{X}_j \widetilde{X}_j^\dagger) \int d[X'_j] P_j \left(\left[\begin{array}{c|c} \widetilde{X}_j & 0 \\ \hline X'_j & \end{array} \right] \right). \quad (33)$$

Here we integrate over the rectangular matrix $X'_j \in \mathrm{gl}_\beta(\nu_j, N_{j-1})$ and the superscript (R) refers to the right block QR-decomposition (31). In particular, we have

$$d\nu_{I_{\mathrm{R}}}^{(\mathrm{G,R})}(X^{(M)}) \propto \quad (34)$$
$$d\mu(U_{\mathrm{R}}) \prod_{j=1}^{M} \exp\left[-\operatorname{Tr} \widetilde{X}_j \widetilde{X}_j^\dagger\right] \det{}^{\beta\nu_{j-1}/(2\gamma)}(\widetilde{X}_j \widetilde{X}_j^\dagger) d[\widetilde{X}_j],$$

for the Ginibre ensemble and

$$d\nu_{\kappa, I_R}^{(\mathrm{J,R})}(X^{(M)}) \propto d\mu(U_{\mathrm{R}}) \quad (35)$$
$$\times \prod_{j=1}^{M} \Big[\det{}^{\kappa_j + \beta\nu_j/(2\gamma)}(\mathbb{1}_{\gamma N_{\min}} - \widetilde{X}_j \widetilde{X}_j^\dagger)$$
$$\times \det{}^{\beta\nu_{j-1}/(2\gamma)}(\widetilde{X}_j \widetilde{X}_j^\dagger) \Theta(\mathbb{1}_{\gamma N_{\min}} - X_j X_j^\dagger) d[\widetilde{X}_j] \Big].$$

for the Jacobi ensemble, where the multi-index $I_{\mathrm{R}} = (\nu_0, \ldots, \nu_{M-1})$ encodes the order of the exponents.

Note that the "left" and "right" measures (23) and (33) and therefore also Eqs. (26) and (32) differs by a replacement $\nu_j \leftrightarrow \nu_{j-1}$. How can we explain this discrepancy? And more importantly, does the case $N_M = N_0 = N_{\min}$ yields a conflict, since both measures apply in this case? This problem can be easily resolved for Gaussian weights, see subsection III B. In subsection III C, we show a neat weak commutation relation between an induced weight of one random matrix reduced to a square matrix and the weight of a truncated unitary matrix (drawn from one of the three Jacobi ensembles). These two weak commutation relations are everything we need to understand the discrepancy between the measures (26) and (32).

B. A weak commutation relation of square random matrices of finite size

In this section we show that any ordering of the exponents of the determinantal prefactors in the Gaussian case yields the same statistics for the product matrix $X^{(M)}$. We also make this statement stronger and show that any square random matrices distributed by probability densities invariant under the corresponding group commute inside the average. This weak commutation relation was already proven for matrices with infinite large matrix size [18], but is also exact at finite matrix size as we will show. Moreover it holds for all three Dyson classes. A weak commutation relation has also been discussed in the context of disordered wires with obstacles [19].

Let f be an integrable test function for the random matrix $Y = Y_2 Y_1$ given as the product of two square random matrices $Y_{1,2} \in \mathrm{gl}_\beta(N_{\min}, N_{\min})$ and distributed according to

$$d\nu(Y) = p_1(Y_1) d[Y_1] p_2(Y_2) d[Y_2] \tag{36}$$

where the p_j are probability densities invariant under $\mathrm{U}_\beta(N_{\min}, N_{\min})$, i.e.

$$\int p_j(Y_j) d[Y_j] = 1 \quad \text{and} \quad p_j(U Y_j V) = p_j(Y_j). \tag{37}$$

for all $U \otimes V \in \mathrm{U}_\beta(N_{\min}, N_{\min})$. It follows that the integral over f is invariant under $\mathrm{U}_\beta(N_{\min}, N_{\min})$.

$$\int f(Y) d\nu(Y) = \int f(U_1 Y U_2) d\nu(Y), \tag{38}$$

for all $U_1 \otimes U_2 \in \mathrm{U}_\beta(N_{\min}, N_{\min})$. This is clear, since we can absorb U_1 and U_2 in the measures for Y_2 and Y_1, respectively. Integrating over U_1 and U_2 with respect to the normalized Haar measure on $\mathrm{U}_\beta(N_{\min}, N_{\min})$, we define the function

$$g(Y) = \int f(U_1 Y U_2) d\mu(U_1) d\mu(U_2). \tag{39}$$

Indeed this auxiliary function only depends on the singular values of Y due to the invariance under $\mathrm{U}_\beta(N_{\min}, N_{\min})$. Additionally Y and $Y^\dagger$ share the same singular values and lie in the same matrix space, namely $\mathrm{gl}_\beta(N_{\min}, N_{\min})$. Hence, g has the same functional dependence on Y as on $Y^\dagger$. Thus the following relation holds

$$\begin{aligned} \int f(Y_1 Y_2) d\nu(Y) &= \int g(Y) d\nu(Y) = \int g(Y^\dagger) d\nu(Y) \\ &= \int f(Y_2^\dagger Y_1^\dagger) d\nu(Y). \end{aligned} \tag{40}$$

Finally we use the invariance of the measure $d\nu$ under the interchange $Y_j \leftrightarrow Y_j^\dagger$, which yields the main result of this section,

$$\int f(Y_1 Y_2) d\nu(Y) = \int f(Y_2 Y_1) d\nu(Y). \tag{41}$$

Thus the two random matrices commute in a weak sense.

For example, let the measures be deformed Gaussians,

$$P_j(Y_j) = \frac{1}{Z_j} \det{}^{\nu_j}(Y_j Y_j^\dagger) \exp[-\operatorname{Tr} Y_j Y_j^\dagger], \tag{42}$$

where Z_j is a normalization constant. Then the measure of $Y = Y_1 Y_2$ is invariant under the interchange of the two exponents ν_1 and ν_2 in the measure of Y_1 and Y_2. Applying this knowledge to the measure (27) we can show that any two neighbouring matrices can be interchanged. These interchanges are the generators of the permutation group $\mathbb{S}(M)$ of M elements. It follows that

$$\int f(X^{(M)}) d\nu^{(\mathrm{G})}_{I_\mathrm{L}}(X^{(M)}) = \int f(X^{(M)}) d\nu^{(\mathrm{G})}_{\omega(I_\mathrm{L})}(X^{(M)}), \tag{43}$$

for all $\omega \in \mathbb{S}(M)$ and any integrable test-function f of the random matrix $X^{(M)}$. Therefore the ordering of the multi-index I_L is irrelevant. The equivalence relation will be indeed reflected in the discussion of the eigenvalue statistics in Sec. IV B.

A naïve generalization of the weak commutation relation (41) to any two rectangular random matrices does not work since in general the matrix dimensions does not close, meaning that two matrices might be multipliable like $Y_1 Y_2$ but not like $Y_2 Y_1$.

C. A weak commutation relation of square random matrices with an induced measure

Let us consider a second weak commutation relation since the former commutation relation only solves the problem of an ambiguity of the resulting weight of $X^{(M)}$ when the integrals over X'_j factorize from the rest, as for Gaussian weights. What happens with other random matrix ensembles? To answer this question we consider the rectangular random matrix $Y \in \mathrm{gl}_\beta(N_1, N_0)$ distributed by the weight P. Furthermore, we assume that f is an arbitrary integrable function on the set $\mathrm{gl}_\beta(N_{\min}, N_{\min})$ with $N_0, N_1 \geq N_{\min}$.

We consider the integral

$$I[f] = \int f\left(X_\mathrm{L} Y_\mathrm{L}\right) P_\mathrm{L}(Y_\mathrm{L}) d[Y_\mathrm{L}] d\nu^{(\mathrm{J})}_{\kappa_1}(X_\mathrm{L}), \tag{44}$$

where $\kappa_1 = \beta(N_1 - 2N_{\min} + 1 - 2/\beta)/2\gamma$, cf. Eq. (8). The truncated unitary matrix $X_\mathrm{L} \in \mathrm{gl}_\beta(N_{\min}, N_{\min})$ is distributed according to the Jacobi measure

$$\begin{aligned} d\nu^{(\mathrm{J})}_{\kappa_1}(X_\mathrm{L}) \propto \Theta(\mathbb{1}_{\gamma N_{\min}} - X_\mathrm{L} X_\mathrm{L}^\dagger) \\ \times \det{}^{\kappa_1}(\mathbb{1}_{\gamma N_{\min}} - X_\mathrm{L} X_\mathrm{L}^\dagger) d[X_\mathrm{L}], \end{aligned} \tag{45}$$

cf. Eq. (8). The measure P_L is the induced measure

$$P_\mathrm{L}(Y_\mathrm{L}) \propto \det{}^{\beta\nu_1/(2\gamma)}(Y_\mathrm{L} Y_\mathrm{L}^\dagger) \int d[Y'] P\left(\left[\begin{array}{c|c} Y_\mathrm{L} & Y' \\ 0 & \end{array} \right] \right), \tag{46}$$

for the sub-block $Y_{\rm L} \in {\rm gl}_\beta(N_{\rm min}, N_{\rm min})$, where we integrate over the rectangular matrix $Y' \in {\rm gl}_\beta(N_1, \nu_0)$, cf. Eq. (25).

The random matrix $X_{\rm L}$ can be written as a product of three matrices. Two of them are projections which restrict the group element $U_{\rm L} \in \mathbb{G}_\beta(N_1, N_{\rm min})$ to its first $\gamma N_{\rm min}$ rows and columns,

$$X_{\rm L} = \left[\begin{array}{cc} \mathbb{1}_{\gamma N_{\rm min}} & 0 \end{array} \right] U_{\rm L} \left[\begin{array}{c} \mathbb{1}_{\gamma N_{\rm min}} \\ 0 \end{array} \right], \tag{47}$$

where $U_{\rm L}$ is weighted with respect to the Haar measure on $\mathbb{G}_\beta(N_1, N_{\rm min})$. Employing the inverse decomposition, see Eq. (24),

$$Y = U_{\rm L} \left[\begin{array}{c|c} Y_{\rm L} & Y' \\ 0 & \end{array} \right]. \tag{48}$$

We can rewrite the integral (44) in terms of an integral over Y distributed according to the density P,

$$\begin{aligned} I[f] &= \int f \left(\left[\begin{array}{cc} \mathbb{1}_{\gamma N_{\rm min}} & 0 \end{array} \right] U_{\rm L} \left[\begin{array}{c} Y_{\rm L} \\ 0 \end{array} \right] \right) P_{\rm L}(Y_{\rm L}) d[Y_{\rm L}] d\mu(U_{\rm L}) \\ &= \int f \left(\left[\begin{array}{cc} \mathbb{1}_{\gamma N_{\rm min}} & 0 \end{array} \right] Y \left[\begin{array}{c} \mathbb{1}_{\gamma N_{\rm min}} \\ 0 \end{array} \right] \right) P(Y) d[Y]. \end{aligned} \tag{49}$$

This procedure can be inverted only with the difference that we do it to the right, i.e. we consider the decomposition

$$Y = \left[\begin{array}{c} Y_{\rm R} \;\; 0 \\ \hline Y'' \end{array} \right] U_{\rm R}. \tag{50}$$

with $Y_{\rm R} \in {\rm gl}_\beta(N_{\rm min}, N_{\rm min})$ and the truncated matrix

$$X_{\rm R} = \left[\begin{array}{cc} \mathbb{1}_{\gamma N_{\rm min}} & 0 \end{array} \right] U_{\rm R} \left[\begin{array}{c} \mathbb{1}_{\gamma N_{\rm min}} \\ 0 \end{array} \right] \in {\rm gl}_\beta(N_{\rm min}, N_{\rm min}), \tag{51}$$

which induces the measures

$$\begin{aligned} d\nu^{(\rm J)}_{\kappa_0}(X_{\rm R}) \propto \Theta(\mathbb{1}_{\gamma N_{\rm min}} - X_{\rm R} X_{\rm R}^\dagger) \\ \times \det{}^{\kappa_0}(\mathbb{1}_{\gamma N_{\rm min}} - X_{\rm R} X_{\rm R}^\dagger) d[X_{\rm R}] \end{aligned} \tag{52}$$

with $\kappa_0 = \beta(N_0 - 2N_{\rm min} + 1 - 2/\beta)/2\gamma$ and

$$P_{\rm R}(Y_{\rm R}) \propto \det{}^{\beta\nu_0/(2\gamma)}(Y_{\rm R} Y_{\rm R}^\dagger) \int d[Y''] P \left(\left[\begin{array}{c} Y_{\rm R} \;\; 0 \\ \hline Y'' \end{array} \right] \right), \tag{53}$$

where we integrate over the rectangular matrix $Y'' \in {\rm gl}_\beta(\nu_1, N_0)$. Thus we find the following identity

$$\begin{aligned} I[f] &= \int f\left(X_{\rm L} Y_{\rm L}\right) P_{\rm L}(Y_{\rm L}) d[Y_{\rm L}] d\nu^{(\rm J)}_{\kappa_1}(X_{\rm L}) \\ &= \int f\left(Y_{\rm R} X_{\rm R}\right) P_{\rm R}(Y_{\rm R}) d[Y_{\rm R}] d\nu^{(\rm J)}_{\kappa_0}(X_{\rm R}) \end{aligned} \tag{54}$$

which is the main result of this section. This identity can be considered as some kind of weak commutation relation for induced square matrices with truncated unitary random matrices. It is the missing link between the effective weights (26) and (32).

D. The general case

Let us consider the general product matrix, $X^{(M)} = X_M \cdots X_1$ where $N_J = N_{\rm min}$ for some $J \in \{0 \ldots M\}$. It is irrelevant whether J is unique or not, due to the weak commutation relations discussed in subsections III B and III C. If there is more than one J such that $N_J = N_{\rm min}$, then one can take any of these. In particular the identity (54) allows us to transform the resulting measure for $X^{(M)}$ to equivalent weights.

We define the sub-product matrices

$$\begin{aligned} X^{(M,{\rm L})} &= X_M X_{M-1} \cdots X_{J+2} X_{J+1} \in {\rm gl}_\beta(N_M, N_{\rm min}), \\ X^{(M,{\rm R})} &= X_J X_{J-1} \cdots X_2 X_1 \in {\rm gl}_\beta(N_{\rm min}, N_0), \end{aligned} \tag{55}$$

such that $X^{(M)} = X^{(M,{\rm L})} X^{(M,{\rm R})}$. We apply the procedure for deriving the measure (26) on the matrix $X^{(M,{\rm L})}$ and the procedure for the measure (32) on the matrix $X^{(M,{\rm R})}$. Then we obtain the measure and the main result of this section

$$d\nu_{\widetilde{P}_1 \ldots \widetilde{P}_M}(X^{(M)}) = d\mu(U_{\rm L}) d\mu(U_{\rm R}) \prod_{j=1}^{M} \widetilde{P}_j(\widetilde{X}_j) d[\widetilde{X}_j], \tag{56}$$

where $\widetilde{P}_j$ and $\widetilde{X}_j$ are given by Eqs. (25) and (24) for $j > J$ and by Eqs. (33) and (31) for $j \leq J$.

For the Gaussian case we obtain

$$\begin{aligned} & d\nu^{(\rm G)}_I(X^{(M)}) \\ & \propto d\mu(U_{\rm L}) \prod_{j=J+1}^{M} \exp\left[- {\rm Tr}\, \widetilde{X}_j \widetilde{X}_j^\dagger \right] \det{}^{\beta\nu_j/(2\gamma)}(\widetilde{X}_j \widetilde{X}_j^\dagger) d[\widetilde{X}_j] \\ & \times d\mu(U_{\rm R}) \prod_{j=1}^{J} \exp\left[- {\rm Tr}\, \widetilde{X}_j \widetilde{X}_j^\dagger \right] \det{}^{\beta\nu_{j-1}/(2\gamma)}(\widetilde{X}_j \widetilde{X}_j^\dagger) d[\widetilde{X}_j] \end{aligned} \tag{57}$$

with the multi index $I = (\nu_0, \ldots, \nu_{J-1}, \nu_{J+1}, \ldots, \nu_M)$. Indeed we can again apply the weak commutation relation (43) to the product matrix $X^{(M)}$ which tells us that we can take any permutation of the multi-index I and, hence, of the exponents of the determinantal prefactors.

The effective measure for a product of matrices drawn from Jacobi ensembles yields a result similar to Eq. (57). We have only to plug in the measures (30) for $X^{(M,{\rm L})}$ and (35) for $X^{(M,{\rm R})}$ instead of the deformed Gaussians.

We emphasize that the effective measure (56) does not only apply for the discussion of the eigenvalue or singular value statistics. One can also apply this reduction to a product of square matrices to correlations of

the eigenvectors as well as to cross-correlations of eigenvectors and eigenvalues. Moreover the permutation invariance due to the commutation relation (41) and the choice of J, if more than one N_j is equal to the minimal dimension $N_{\min}$, has to be obviously reflected in the statistics of the eigenvalues and singular values (c.f. Ref. [15]) of $\widetilde{X}^{(M)}$ and, thus, of $X^{(M)}$. This has crucial physical implications. Consider the chaotic quantum system from Sec. II, the permutation symmetry tells us that the order of the potential wells is completely irrelevant, if we consider only the statistics of the product matrix $X^{(M)}$ involved in these problems. Note that the cross-correlations between particular positions of the chain are still affected by the order of the potential wells.

IV. EIGENVALUE STATISTICS FOR PRODUCTS OF MATRICES

In this section we will discuss the eigenvalues of $X^{(M)} = X_M \cdots X_1$, which are of general interest in a broad spectrum of applications. Due to the discussion of Sec. III we can restrict ourselves to the case of square matrices, $N_0 = \ldots = N_M = N_{\min} = N$, without loss of generality. The matrices X_j are square matrices with different weights, which can be chosen to imitate a product of rectangular matrices. We are interested in the statistical properties of the eigenvalues of the product matrix $X^{(M)} \in \mathrm{gl}_\beta(N,N)$, see Eq. (12). Thus we are looking for the zeros of the characteristic polynomial

$$\det(X^{(M)} - z\mathbb{1}_{\gamma N}) = \det(X_M \cdots X_2 X_1 - z\mathbb{1}_{\gamma N}) = 0. \tag{58}$$

In subsection IV A we perform an eigenvalue decomposition for arbitrary weights of the matrices X_j. We specify this decomposition for the Gaussian case in subsection IV B and for the Jacobi ensemble in subsection IV C and calculate an explicit expression for the joint probability density function for both ensembles.

A. The eigenvalue decomposition

We pursue the idea of the Ginibre ensembles for the Dyson indices $\beta = 2, 4$ employed by the authors of Refs. [11, 12] and perform a generalized Schur decomposition. Let

$$B = X\widehat{T} \tag{59}$$

and $X = \mathrm{diag}(X_1, \ldots, X_M)$ a block-diagonal matrix and $\widehat{T}$ the constant matrix as in Eq. (2). Then B^M has the same eigenvalues as $X^{(M)}$ only that they are M times degenerate.

1. $\beta = 2$

We first consider the simplest case $\beta = 2$. In the first step we perform a simultaneous decomposition of B in a diagonal matrix $Z = \mathrm{diag}(Z_1, \ldots, Z_M)$ with complex eigenvalues $Z_j = \mathrm{diag}(z_1^{(j)}, \ldots, z_N^{(j)})$, an upper triangular matrix $\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_M)$ and a unitary matrix $U = \mathrm{diag}(U_1, \ldots, U_M)$. We have

$$B = X\widehat{T} = U(Z + \Delta)\widehat{T}U^\dagger. \tag{60}$$

The differential of B is given by

$$dB = dX\widehat{T} = U\left((dZ + d\Delta)\widehat{T} + [dA, (Z+\Delta)\widehat{T}]_-\right)U^\dagger, \tag{61}$$

where $dA = U^\dagger dU$. The differentials of Z and Δ completely factorize from the rest. Only the Z-dependent part in the commutator, $[\,\cdot\,,\,\cdot\,]_-$, contributes to the Jacobian. The upper triangular matrix Δ incorporates a recursive shift of dA which results in a upper triangular part of the Jacobian, as well. The variable $dA_{ab}^{(j)}$ denotes a matrix element of the j-th matrix $dA_j = U_j^\dagger dU_j$ which is a complex variable. Moreover $dA_{aa}^{(j)} = 0$ since these degrees of freedom are incorporated in Z, hence

$$\{[dA, Z\widehat{T}]_-\widehat{T}^\dagger\}_{ab}^{(j)} = dA_{ab}^{(j)} z_b^{(j)} - z_a^{(j)} dA_{ab}^{(j+1)}. \tag{62}$$

The Jacobian resulting from this transformation is a determinant with a diagonal part corresponding to dZ and $d\Delta$ and a part proportional to $\widehat{T}$ resulting from dA. Then we arrive at

$$\prod_{j=1}^{M} d[X_j] \propto |\Delta_N\,(Z_M \cdots Z_1)|^2 \prod_{j=1}^{M} d\mu(U_j) d[Z_j] d[\Delta_j]. \tag{63}$$

The differential for Z_j is $d[Z_j] = \prod_{a=1}^{N} d\,\mathrm{Re}\; z_a^{(j)} d\,\mathrm{Im}\; z_a^{(j)}$, while $\Delta_N(Z)$ denotes the Vandermonde determinant.

Let us return to the full measure where each matrix X_j is distributed via the probability density P_j. Due to the factorization of the differentials $d[\Delta_j]$ we define the reduced weights

$$\widehat{P}_j(Z_j) \equiv \int P_j(Z_j + \Delta^{(j)}) d[\Delta^{(j)}]. \tag{64}$$

The joint probability density of the eigenvalues of the product matrix $X^{(M)}$ reads in these new weights

$$p^{(\beta=2)}(Z^{(M)}) \propto |\Delta_N(Z^{(M)})|^2 \times \prod_{j=1}^{M} \int d[Z_j] \widehat{P}_j(Z_j) \prod_{a=1}^{N} \delta^{(2)}(z_a - z_a^{(M)} \cdots z_a^{(1)}) \tag{65}$$

where we use the Dirac δ-function for complex variables, $\delta^{(2)}(z) = \delta(\mathrm{Re}\; z)\delta(\mathrm{Im}\; z)$, and define the diagonal product matrix $Z^{(M)} = Z_1 \cdots Z_M$. Expression (65) is the farthest one can calculate for an arbitrary weight. If

one wants to have a more concrete result one has to specify the measures $\widehat{P}_j$. We will do this for the Gaussian measure in subsection IV B and for the Jacobi measure in subsection IV C and recover the results derived in Refs. [11, 30, 36].

2. $\beta = 4$

The next case we consider is $\beta = 4$. In this case we can again decompose B in an upper triangular matrix Δ, a unitary symplectic matrix $U \in \mathrm{USp}(2N)$ and a $2N \times 2N$ matrix $\widehat{Z} = \mathrm{diag}(Z, Z^*)$ where Z is the same complex, diagonal $N \times N$ matrix as in the case $\beta = 2$. We replace $Z \to \widehat{Z}$ in Eqs. (61 - 64). Moreover the complex matrix elements in the case $\beta = 2$, $\{\Delta_j\}_{ab}$ and $dA^{(j)}_{ab}$ with $a \neq b$, are now 2×2 quaternion matrix blocks with four real independent elements. Each of the diagonal 2×2 blocks $dA^{(j)}_{aa}$ only contain one off-diagonal complex variable. Hence the analogue to Eq. (62) is

$$\{[dA, Z\widehat{T}]_- \widehat{T}^\dagger\}^{(j)}_{ab} = dA^{(j)}_{ab} \begin{pmatrix} z^{(j)}_b & 0 \\ 0 & z^{(j)*}_b \end{pmatrix} - \begin{pmatrix} z^{(j)}_a & 0 \\ 0 & z^{(j)*}_a \end{pmatrix} dA^{(j+1)}_{ab} \quad (66)$$

The computation of the Jacobian works exactly the same as in the case $\beta = 2$ and we find the joint probability density of the eigenvalues

$$p^{(\beta=4)}(Z^{(M)}) \propto \Delta_{2N}(Z^{(M)}, {Z^{(M)}}^*) \prod_{a=1}^{N} (z_a - z_a^*) \times \prod_{j=1}^{M} \int d[Z_j] \widehat{P}_j(Z_j) \prod_{b=1}^{N} \delta^{(2)}(z_b - z^{(M)}_b \cdots z^{(1)}_b) \quad (67)$$

with

$$\widehat{P}_j(Z_j) \propto \int P_j(\mathrm{diag}(Z_j, Z_j^*) + \Delta^{(j)}) d[\Delta^{(j)}]. \quad (68)$$

Again one needs specific weights P_j to calculate further. For the Gaussian case the resulting measure was studied by one of the authors in Ref. [12].

3. $\beta = 1$

Finally let us consider the case $\beta = 1$. We have to distinguish between odd and even matrix dimensions. For this reason we introduce the notation $N = 2\widetilde{N} + \chi$ with $\chi = 0$ or $\chi = 1$. Unlike the complex Schur decomposition, the real Schur decomposition will not generally trace B back to a triangular form; instead similarity transformations with orthogonal matrices $U \in \mathrm{O}(2\widetilde{N} + \chi)$ bring B to a block diagonal matrix $\widehat{Z} = \mathrm{diag}(\widehat{Z}^{(1)}_1, \widehat{Z}^{(1)}_2, \ldots, \widehat{Z}^{(1)}_{\widetilde{N}+\chi}, \widehat{Z}^{(2)}_1, \ldots, \widehat{Z}^{(M)}_{\widetilde{N}+\chi})$ and an upper block triangular matrix Δ [49]. The blocks $\widehat{Z}^{(1)}_1, \ldots, \widehat{Z}^{(M)}_{\widetilde{N}}$ are 2×2 real matrices and $\widehat{Z}^{(j)}_{\widetilde{N}+1}$ (only for $\chi = 1$) is a real number. Thus the matrix elements of Δ and $dA^{(j)}_{ab}$, $a \neq b$ and $a, b \neq \widetilde{N}+1$, in the case $\beta = 2$ are again replaced by 2×2 matrix blocks with four real independent variables. The block diagonal elements $dA^{(j)}_{aa}$ are zero. For $\chi = 1$ we have two additional real variables arranged in a two dimensional vector $dA^{(j)}_{a,\widetilde{N}+1}$ for each $a = 1 \ldots \widetilde{N}$ and $j = 1 \ldots M$. In the case of an even dimensional matrix ($\chi = 0$) the differentials are

$$\{[dA, Z\widehat{T}]_- \widehat{T}^\dagger\}^{(j)}_{ab} = dA^{(j)}_{ab} \widehat{Z}^{(j)}_b - \widehat{Z}^{(j)}_a dA^{(j+1)}_{ab} \quad (69)$$

as in Eq. (62) but with 2×2 real matrices. For odd dimensional matrices ($\chi = 1$) one needs to treat the case where a or b are equal to $\widetilde{N} + 1$ separately, then the 2×2 real matrices $dA^{(j)}_{ab}$ in Eq. (69) have to be replaced by a chain of two dimensional real vectors, $dA^{(j)}_{a,\widetilde{N}+1}$ or $dA^{(j)}_{\widetilde{N}+1,b}$, which is in the spirit of Ref. [37].

Let $\widetilde{Z}^{(M)}_a = \widehat{Z}^{(M)}_a \cdots \widehat{Z}^{(1)}_a$ and $\widetilde{Z}^{(M)} = \widehat{Z}^{(M)} \cdots \widehat{Z}^{(1)}$. Then the resulting joint probability density of the matrix blocks $\widetilde{Z}^{(M)}_a$ is

$$\widehat{p}^{(\beta=1)}(\widetilde{Z}^{(M)}) \propto \prod_{1 \leq a < b \leq \widetilde{N}} \left| \det \left[\widetilde{Z}^{(M)}_a \otimes \mathbb{1}_2 - \mathbb{1}_2 \otimes \widetilde{Z}^{(M)}_b \right] \right| \prod_{j=1}^{M} \int d[\widehat{Z}^{(j)}] \widehat{P}_j(\widehat{Z}^{(j)}) \prod_{a=1}^{\widetilde{N}} \delta^{(4)} \left(\widetilde{Z}^{(M)}_a - \widehat{Z}^{(M)}_a \cdots \widehat{Z}^{(1)}_a \right)$$

for an even dimension N and

$$\widehat{p}^{(\beta=1)}(\widetilde{Z}^{(M)}) \propto \prod_{1\le a<b\le \widetilde{N}} \left|\det\left[\widetilde{Z}_a^{(M)}\otimes \mathbb{1}_2 - \mathbb{1}_2\otimes \widetilde{Z}_b^{(M)}\right]\right| \prod_{j=1}^{\widetilde{N}} \left|\det\left[\widetilde{Z}_j^{(M)} - \widetilde{Z}_{\widetilde{N}+1}^{(M)}\mathbb{1}_2\right]\right| \times \prod_{j=1}^{M}\int d[\widehat{Z}^{(j)}]\widehat{P}_j(\widehat{Z}^{(j)})\delta^{(2)}\left(\widetilde{Z}_{\widetilde{N}+1} - \widehat{Z}_{\widetilde{N}+1}^{(M)}\cdots\widehat{Z}_{\widetilde{N}+1}^{(1)}\right)\prod_{a=1}^{\widetilde{N}}\delta^{(4)}\left(Z_a^{(M)} - \widehat{Z}_a^{(M)}\cdots\widehat{Z}_a^{(1)}\right) \quad (70)$$

for an odd one. The first product of determinants incorporates the differences of pairs of 2×2 matrices. Therefore those determinants are over 4×4 matrices and are reminiscent of a Vandermonde determinant. The Dirac δ-function over a 2×2 real matrix is the product of the Dirac δ-functions of all four real independent variables. The reduced probability densities are defined as always,

$$\widehat{P}_j(\widehat{Z}^{(j)}) \equiv \int P_j(\widehat{Z}^{(j)} + \Delta^{(j)})d[\Delta^{(j)}]. \quad (71)$$

The only difference of this definition to Eqs. (64) and (68) is that we remain with a distribution for block diagonal matrices, $\widehat{Z}^{(j)}$, instead with diagonal ones.

The eigenvalues of the 2×2 real matrices $\widetilde{Z}_a^{(M)}$ are either a complex conjugate pair or two independent real eigenvalues. Since the probability densities $\widehat{P}_j$ are invariant under left and right multiplication of $\mathrm{O}(N)$ we can replace the argument $\widetilde{Z}^{(M)}$ of the recursive integral (for simplicity only shown for even dimension, $\chi = 0$)

$$I(\widetilde{Z}^{(M)}) = \prod_{j=1}^{M}\int d[\widehat{Z}^{(j)}]\widehat{P}_j(\widehat{Z}^{(j)})\prod_{a=1}^{\widetilde{N}}\delta^{(4)}(\widetilde{Z}_a^{(M)} - \widehat{Z}_a^{(M)}\cdots\widehat{Z}_a^{(1)}) \quad (72)$$

by the positive definite matrix $\sqrt{\widetilde{Z}^{(M)}\widetilde{Z}^{(M)\,\dagger}}$. This matrix is a block diagonal matrix which can be readily expressed by a singular value decomposition $\widetilde{Z}^{(M)} = U_{\mathrm{L}}\Lambda^{(M)}U_{\mathrm{R}}$ with $U_{\mathrm{L}}, U_{\mathrm{R}} \in \mathrm{O}(2)^{\widetilde{N}}$ and Λ a positive diagonal matrix. The idea is to calculate an integral representation of the joint probability distribution of the eigenvalues of $\widetilde{Z}^{(M)}$ in terms of the singular values in $\Lambda^{(M)}$. For this purpose it is quite helpful that we could reduce the whole problem to a 2×2 matrix problem.

In appendix A we derived a general relation between the eigenvalues and the singular values of a 2×2 real matrix. Employing the result of the calculations (A6 - A8) we find the joint probability density of the eigenvalues $Z^{(M)} = \mathrm{diag}(z_1,\ldots,z_N)$ of $X^{(M)}$,

$$p^{(1)}(Z^{(M)}) \propto |\Delta_{2\widetilde{N}}(Z^{(M)})|\prod_{j=1}^{M}\int d[\widehat{Z}^{(j)}]\widehat{P}_j(\widehat{Z}^{(j)})\Bigg[\prod_{a=1}^{\widetilde{N}}\left(\delta(\mathrm{Im}\ z_{2a-1})\delta(\mathrm{Im}\ z_{2a}) + 2\delta^{(2)}(z_{2a-1} - z_{2a}^*)\right) \times \int_\alpha^\infty d\alpha_a\delta^{(4)}(\Lambda(z_{2a-1}, z_{2a}, \alpha_a) - \widehat{Z}_a^{(M)}\cdots\widehat{Z}_a^{(1)})\Bigg] \quad (73)$$

for even dimension and

$$p^{(1)}(Z^{(M)}) \propto |\Delta_{2\widetilde{N}+1}(Z^{(M)})|\prod_{j=1}^{M}\int d[\widehat{Z}^{(j)}]\widehat{P}_j(\widehat{Z}^{(j)})\delta\left(\mathrm{Im}\ z_{2\widetilde{N}+1}\right)\delta\left(\mathrm{Re}\ z_{2\widetilde{N}+1} - \widehat{Z}_{2\widetilde{N}+1}^{(M)}\cdots\widehat{Z}_{2\widetilde{N}+1}^{(1)}\right) \times \left[\prod_{a=1}^{\widetilde{N}}\left(\delta(\mathrm{Im}\ z_{2a-1})\delta(\mathrm{Im}\ z_{2a}) + 2\delta^{(2)}(z_{2a-1} - z_{2a}^*)\right)\int_\alpha^\infty d\alpha_a\delta^{(4)}(\Lambda(z_{2a-1}, z_{2a}, \alpha_a) - \widehat{Z}_a^{(M)}\cdots\widehat{Z}_a^{(1)})\right] \quad (74)$$

for odd dimension, where $\alpha = |\,\mathrm{Im}(z_{2a-1} - z_{2a})|/2$. In both cases we employed the functional dependence of the singular values on the eigenvalues, i.e.

$$\Lambda(z_{2a-1}, z_{2a}, \alpha_a) = \begin{bmatrix}\lambda_+(z_{2a-1}, z_{2a}, \alpha_a) & 0\\ 0 & \lambda_-(z_{2a-1}, z_{2a}, \alpha_a)\end{bmatrix} = \sqrt{\alpha_a^2 + \frac{(z_{2a-1}+z_{2a})^2}{4}}\mathbb{1}_2 + \sqrt{\alpha_a^2 + \frac{(z_{2a-1}-z_{2a})^2}{4}}\sigma_3 \quad (75)$$

cf. Eq. (A9). The 2×2 matrix σ_3 is the third Pauli matrix. The integrals over α_a are reminiscent to the integrals found in the real Ginibre ensemble (equal to the case $M = 1$) generating the error function [37] and in the real chiral Ginibre ensemble (equal to the case $M = 2$) yielding an integral over a Bessel function [38]. Also the prefactor consisting of the Dirac δ-function is the same in both cases and is a universal factor reflecting the nature of the eigenvalues of arbitrary real matrices.

An important remark is in order. Assuming one of the pairs of eigenvalues is real, say (z_{2N-1}, z_{2N}), one can also approach an eigenvalue decomposition of the product of 2×2 matrices by a generalized Schur decomposition. Thus the following integral over $\widehat{Z}_N^{(j)}$ is equivalent

$$\delta(\text{Im } z_{2N-1})\delta(\text{Im } z_{2N}) \prod_{j=1}^{M} \int d[\widehat{Z}_N^{(j)}] \widehat{P}_j \left(\begin{bmatrix} \widehat{Z}_1^{(j)} & & 0 \\ & \ddots & \\ 0 & & \widehat{Z}_N^{(j)} \end{bmatrix} \right) \int_0^\infty d\alpha_N \delta^{(4)} \left(\Lambda(z_{2N-1}, z_{2N}, \alpha_N) - \widehat{Z}_N^{(M)} \cdots \widehat{Z}_N^{(1)} \right)$$

$$\propto \delta(\text{Im } z_{2N-1})\delta(\text{Im } z_{2N}) \prod_{j=1}^{M} \int dx_1^{(j)} dx_2^{(j)} dx_3^{(j)} \widehat{P}_j \left(\left[\begin{array}{ccc|cc} \widehat{Z}_1^{(j)} & & 0 & & \\ & \ddots & & 0 & \\ 0 & & \widehat{Z}_{N-1}^{(j)} & & \\ \hline & 0 & & x_1^{(j)} & x_2^{(j)} \\ & & & 0 & x_3^{(j)} \end{array} \right] \right)$$

$$\times \delta \left(\text{Re } z_{2N-1} - x_1^{(M)} \cdots x_1^{(1)} \right) \delta \left(\text{Re } z_{2N} - x_3^{(M)} \cdots x_3^{(1)} \right). \quad (76)$$

The integral on the right hand side was used quite recently in Ref. [33] to calculate the probability of a fixed number of real eigenvalues for a product of Ginibre matrices. Notice that the integral identity (76) is not at all trivial and we know only that it has to be in general true since both approaches are legitimized.

Also in the case of real matrices we need a specific measure to calculate any further. This is exactly what we do in the next two subsections and restrict our discussion to the Ginibre and Jacobi ensemble. We emphasize that the discussion so far have been for completely arbitrary probability weights and can be applied to a broad class of ensembles.

B. Products of Ginibre matrices

As discussed in Sec. III, a product of rectangular Ginibre matrices, $X^{(M)} = X_M \cdots X_1$, is closely related to a product of square matrices, see Eq. (27). Applying a Schur decomposition, the rotations are trivially integrated out and they contribute only to the normalization. Likewise, the triangular matrices from the Schur decomposition completely drops out in the determinantal prefactor and factorizes in the Gaussian part, such that also these integrals result in a constant.

Let us again restrict ourselves to complex matrices ($\beta = 2$) first. Starting from Eqs. (65) and (27), the joint probability density is

$$p_\nu^{(\text{G},\beta=2)}(Z^{(M)}) \propto |\Delta_{N_{\min}}(Z^{(M)})|^2 \prod_{a=1}^{N_{\min}} g_\nu^{(\text{G},M)}(z_a) \quad (77)$$

with the one-point weight

$$g_\nu^{(\text{G},M)}(z) = \prod_{j=1}^{M} \int_{\mathbb{C}} d^2 z^{(j)} |z^{(j)}|^{2\nu_j} e^{-|z^{(j)}|^2} \times \delta^{(2)}(z - z^{(M)} \cdots z^{(1)}). \quad (78)$$

We employ the abbreviation $\nu = (\nu_1, \nu_2, \ldots, \nu_M)$. The integral is equal to a representation of a Meijer G-function [39]

$$g_\nu^{(\text{G},M)}(z) = G_{M,\,0}^{0,\,M} \left(\begin{matrix} - \\ \nu_1, \ldots, \nu_M \end{matrix} \middle| |z|^2 \right) = \int_{\mathcal{C}} \frac{du}{2\pi\imath} |z|^{2u} \prod_{j=1}^{M} \Gamma(\nu_j - u) \quad (79)$$

The second line is a quite useful integral representation of the Meijer G-function, where the contour $\mathcal{C}$ runs around the poles of the gamma functions. Recall that the invariance under permutations of the indices, $\nu_j = N_j - N_{\min}$ reflects the weak commutation relation of probability densities, see Sec. III B. The result (77) agrees with the results derived in Refs. [11, 30].

For quaternion matrices ($\beta = 4$) everything works along the same lines as for $\beta = 2$. We combine Eqs. (67) and (27) and find

$$p_\nu^{(\text{G},\beta=4)}(Z^{(M)}) \propto \Delta_{2N}(Z^{(M)}, Z^{(M)^*}) \times \prod_{a=1}^{N_{\min}} (z_a - z_a^*) g_{2\nu}^{(\text{G},M)}(2^{M/2} z_a). \quad (80)$$

The one-point weight $g_{2\nu}^{(\text{G},M)}$ is apart from a replacement $\nu \to 2\nu$ exactly the same weight as for complex matrices,

see Eq. (79). This joint probability density was studied in Ref. [12].

Let $N_{\rm min} = 2\widetilde{N} + \chi$. The joint probability density for the real matrices is much more involved. Again the measures $\widehat{P}_j$ are deformed Gaussians, see Eq. (27), but their arguments are now 2×2 real random matrices instead of complex random variables, cf. Eqs. (73) and (74). Thus the joint probability density is

$$p_\nu^{(\mathrm{G},\beta=1)}(Z^{(M)}) \propto |\Delta_{2\widetilde{N}}(Z^{(M)})| \times \prod_{a=1}^{\widetilde{N}} \Big(\delta(\mathrm{Im}\ z_{2a-1})\delta(\mathrm{Im}\ z_{2a}) + 2\delta^{(2)}(z_{2a-1} - z_{2a}^*)\Big) \times h_\nu^{(\mathrm{G},M)}(z_{2a-1}, z_{2a}) \tag{81}$$

for even matrix dimension and

$$p_\nu^{(\mathrm{G},\beta=1)}(Z^{(M)}) \propto |\Delta_{2\widetilde{N}+1}(Z^{(M)})| \times g_{\nu/2}^{(\mathrm{G},M)}(\mathrm{Re}\ z_{2\widetilde{N}+1})\delta(\mathrm{Im}\ z_{2\widetilde{N}+1}) \times \prod_{a=1}^{\widetilde{N}} \Big(\delta(\mathrm{Im}\ z_{2a-1})\delta(\mathrm{Im}\ z_{2a}) + 2\delta^{(2)}(z_{2a-1} - z_{2a}^*)\Big) \times h_\nu^{(\mathrm{G},M)}(z_{2a-1}, z_{2a}) \tag{82}$$

for odd dimension. The one-point weight, $g^{(M)}_{(\nu-1)/2}$, is again the Meijer G-function (79) but now with the indices $\nu/2 = (\nu_1/2, \ldots, \nu_M/2)$. It becomes a Gaussian in the Ginibre case ($M = 1$, see Ref. [37]) and the Bessel function of the second kind in the chiral Ginibre case ($M = 2$, see Ref. [38]). The two point weight is

$$h_\nu^{(\mathrm{G},M)}(z_1, z_2) = \int_{|\,\mathrm{Im}(z_1-z_2)|/2}^{\infty} d\alpha\ \delta^{(4)}(\Lambda(z_1, z_2, \alpha) - \widehat{Z}^{(M)} \cdots \widehat{Z}^{(1)}) \times \prod_{j=1}^{M} \int d[\widehat{Z}^{(j)}]\, |\mathrm{det}\widehat{Z}^{(j)}|^{\nu_j} \exp[-\,\mathrm{Tr}\,\widehat{Z}^{(j)}\widehat{Z}^{(j)\,T}]. \tag{83}$$

This integral can be partially performed by first substituting $Y_1 = \widehat{Z}^{(1)}$ and $Y_j = \widehat{Z}^{(j)}Y_{j-1}$ and then evaluating the four dimensional Dirac δ-function such that

$$h_\nu^{(\mathrm{G},M)}(z_1, z_2) = |z_1 z_2|^{\nu_M} \int_{|\,\mathrm{Im}(z_1-z_2)|/2}^{\infty} d\alpha \left(\prod_{j=1}^{M-1} \int d[Y^{(j)}] |\mathrm{det}Y^{(j)}|^{\nu_j - \nu_{j+1} - 2}\right) \times \exp\left[-\,\mathrm{Tr}\,\Lambda^2(z_1, z_2, \alpha)(Y_{M-1}^T Y_{M-1})^{-1} - \sum_{i=2}^{M-1} \mathrm{Tr}\, Y_i^T Y_i (Y_{i-1}^T Y_{i-1})^{-1} - \mathrm{Tr}\, Y_1^T Y_1\right]. \tag{84}$$

Performing singular value decompositions for each of the matrices Y_j and integrating over the corresponding groups yields

$$h_\nu^{(\mathrm{G},M)}(z_1, z_2) \propto |z_1 z_2|^{\nu_M} \int_{|\,\mathrm{Im}(z_1-z_2)|/2}^{\infty} d\alpha \left(\prod_{j=1}^{M-1} \int_0^\infty dy_{1j} \int_0^\infty dy_{2j} |y_{1j}^2 - y_{2j}^2| |y_{1j} y_{2j}|^{\nu_j - \nu_{j+1} - 2}\right) \times \exp\left[-\frac{(4\alpha^2 + z_1^2 + z_2^2)(y_{1M-1}^2 + y_{2M-1}^2)}{2y_{1M-1}^2 y_{2M-1}^2} - \sum_{i=2}^{M-1} \frac{(y_{1i}^2 + y_{2i}^2)(y_{1i-1}^2 + y_{2i-1}^2)}{2y_{1i-1}^2 y_{2i-1}^2} - y_{11}^2 - y_{21}^2\right] \times I_0\left(\frac{\sqrt{4\alpha^2 + (z_1+z_2)^2}\sqrt{4\alpha^2 + (z_1-z_2)^2}(y_{2M-1}^2 - y_{1M-1}^2)}{2y_{1M-1}^2 y_{2M-1}^2}\right) \prod_{i=2}^{M-1} I_0\left(\frac{(y_{1i}^2 - y_{2i}^2)(y_{2i-1}^2 - y_{1i-1}^2)}{2y_{1i-1}^2 y_{2i-1}^2}\right), \tag{85}$$

where I_0 is the modified Bessel function of the first kind. For the Ginibre ensemble, i.e. $M = 1$ and $\nu_1 = 0$, we can easily deduce the error function in the imaginary part of the complex eigenvalue pair as it was found in Ref. [37]. The case $M = 2$, $\nu_1 = 0$ and $\nu_2 = \nu$ arbitrary is the chiral Ginibre ensemble discussed in Refs. [38]. For arbitrary M the integral (85) is a generalization of these two particular cases.

Notice that in the case of a real pair of eigenvalues the two-point weight reduces to a product of one point weights,

$$h_\nu^{(\mathrm{G},M)}(\mathrm{Re}\ z_1, \mathrm{Re}\ z_2) \propto g_{\nu/2}^{(\mathrm{G},M)}(\mathrm{Re}\ z_1) g_{\nu/2}^{(\mathrm{G},M)}(\mathrm{Re}\ z_2). \tag{86}$$

Although this is not immediately clear from the integral (85) it can be derived by a generalized Schur decomposition of the 2×2 blocks, see Ref. [33].

C. Products of Jacobi matrices

Here we consider random matrices drawn from Jacobi ensembles, where the integrals over the strictly upper triangular matrices $\Delta^{(j)}$ are more involved than in the Gaussian case. Let us briefly discuss how to perform these integrations for $\beta = 2$. The derivation for $\beta = 1, 4$ works in a similar way. Starting with the Jacobi measure (30), we perform a generalized Schur decomposition decomposition for the individual matrices,

$$\widetilde{X}_j = U_j^{-1} S^{(j)} U_{j-1} \quad \text{with} \quad S^{(j)} = Z_j + \Delta^{(j)}. \tag{87}$$

As usual the Z_j's denote the diagonal matrices, while $\Delta^{(j)}$ are strictly upper triangular matrices. The unitary matrices, U_k, are trivially absorbed due to the invariance of the measures. We want to integrate over $\Delta^{(j)}$ in Eq. (64),

$$\widehat{P}_j(Z_j) \propto \int \det{}^{\kappa_j+\nu_{j-1}}(\mathbb{1}_{N_{\min}} - S^{(j)} S^{(j)\,\dagger}) \times \det{}^{\nu_j}(S^{(j)} S^{(j)\,\dagger}) \Theta(\mathbb{1}_{N_{\min}} - S^{(j)} S^{(j)\,\dagger}) d[\Delta^{(j)}]. \tag{88}$$

Notice that the second determinant can be pushed out the integral since it only depends on Z_j. In the first step we split the $N_{\min} \times N_{\min}$ upper triangular matrix $S^{(j)}$ like

$$S^{(j)} = \begin{bmatrix} S'^{(j)} & v^{(j)} \\ 0 & z^{(j)}_{N_{\min}} \end{bmatrix}, \tag{89}$$

where $S'^{(j)}$ is a $(N_{\min}-1) \times (N_{\min}-1)$ upper triangular matrix and $v^{(j)}$ a $(N_{\min}-1)$-dimensional vector. Thus we have

$$\det(\mathbb{1}_{N_{\min}} - S^{(j)} S^{(j)\,\dagger}) = \det\left[1 - |z^{(j)}_{N_{\min}}|^2 - v^{(j)\,\dagger}\left(\mathbb{1}_{N_{\min}-1} - S'^{(j)} S'^{(j)\,\dagger}\right)^{-1} v^{(j)}\right] \times \det\left(\mathbb{1}_{N_{\min}-1} - S'^{(j)} S'^{(j)\,\dagger}\right). \tag{90}$$

Rescaling

$$v^{(j)} \to \sqrt{(1 - |z^{(j)}_{N_{\min}}|^2)\left(\mathbb{1}_{N_{\min}-1} - S'^{(j)} S'^{(j)\,\dagger}\right)}\, v^{(j)} \tag{91}$$

the integral over $v^{(j)}$ factorizes and yields a constant such that we get

$$\widehat{P}_j(Z_j) \propto (1 - |z^{(j)}_{N_{\min}}|^2)^{\kappa_j+N_{j-1}-1} \Theta(1 - |z^{(j)}_{N_{\min}}|^2) \times |\det Z^{(j)}|^{2\nu_j} \int d[\Delta'^{(j)}] \Theta(\mathbb{1}_{N_{\min}-1} - S'^{(j)} S'^{(j)\,\dagger}) \times \det{}^{\kappa_j+\nu_{j-1}+1}(\mathbb{1}_{N_{\min}-1} - S'^{(j)} S'^{(j)\,\dagger}), \tag{92}$$

where $\Delta'^{(j)}$ is the strictly upper triangular part of $S'^{(j)}$. This procedure can be iterated and we find the well-known induced probability density [25, 30, 35, 36]

$$\widehat{P}_j(Z_j) \propto |\det Z^{(j)}|^{2\nu_j} \det{}^{\kappa_j+N_{j-1}-1}(\mathbb{1}_{N_{\min}} - |Z^{(j)}|^2) \times \Theta(\mathbb{1}_{N_{\min}} - |Z^{(j)}|^2) \tag{93}$$

for $\beta = 2$. In the real and quaternion case one can readily extend this procedure and finds the induced probability densities

$$\widehat{P}_j(Z_j) \propto |\det Z^{(j)}|^{4\nu_j} \det{}^{2(\kappa_j+N_{j-1}-1)}(\mathbb{1}_{N_{\min}} - |Z^{(j)}|^2) \times \Theta(\mathbb{1}_{N_{\min}} - |Z^{(j)}|^2) \tag{94}$$

for $\beta = 4$ and

$$\widehat{P}_j(\widehat{Z}_j) \propto |\det \widehat{Z}^{(j)}|^{\nu_j} \det{}^{\kappa_j+N_{j-1}/2-1}(\mathbb{1}_{N_{\min}} - \widehat{Z}^{(j)} \widehat{Z}^{(j)\,T}) \times \Theta(\mathbb{1}_{N_{\min}} - \widehat{Z}^{(j)} \widehat{Z}^{(j)\,T}) \tag{95}$$

for $\beta = 1$ and even $N_{\min}$ and

$$\widehat{P}_j(\widehat{Z}_j) \propto |\det \widehat{Z}^{(j)}|^{\nu_j} \det{}^{\kappa_j+N_{j-1}/2-1}(\mathbb{1}_{N_{\min}} - \widehat{Z}^{(j)} \widehat{Z}^{(j)\,T}) \times \sqrt{1 - \widehat{z}^{(j)}_{N_{\min}}}\, \Theta(\mathbb{1}_{N_{\min}} - \widehat{Z}^{(j)} \widehat{Z}^{(j)\,T}) \tag{96}$$

for odd $N_{\min}$. Recall that we have a block diagonal structure of $\widehat{Z}$ consisting of 2×2 blocks in the real case.

The joint probability density of the product matrix $X^{(M)}$ can be readily read off for $\beta = 2, 4$ and is

$$p^{(\mathrm{J},\beta=2)}_{\nu,\mu}(Z^{(M)}) \propto |\Delta_{N_{\min}}(Z^{(M)})|^2 \prod_{a=1}^{N_{\min}} g^{(\mathrm{J},M)}_{\nu,\mu}(z_a) \tag{97}$$

for $\beta = 2$, cf. Refs. [30, 36], and

$$p^{(\mathrm{J},\beta=4)}_{\nu,\mu}(Z^{(M)}) \propto \Delta_{2N_{\min}}(Z^{(M)}, Z^{(M)\,*}) \times \prod_{a=1}^{N_{\min}} (z_a - z_a^*) g^{(\mathrm{J},M)}_{2\nu,2\mu-1}(z_a) \tag{98}$$

for $\beta = 4$. The one-point weight is this time

$$g^{(\mathrm{J},M)}_{\nu,\mu}(z) = \prod_{j=1}^{M} \int_{|z^{(j)}|=1} d^2 z^{(j)}\, \frac{|z^{(j)}|^{2\nu_j} (1 - |z^{(j)}|^2)^{\mu_j-\nu_j-1}}{\Gamma(\mu_j - \nu_j)} \times \delta^{(2)}(z - z^{(M)} \cdots z^{(1)}), \tag{99}$$

where ν and μ collectively denote the constants $\nu_i = N_i - N_{\min}$ and $\mu_i = \kappa_i + \nu_i + N_{i-1}$, respectively. Recall that the ordering of the indices is irrelevant due to the weak communication relation. The one point weight can be again expressed as a Meijer G-function [39],

$$g^{(\mathrm{J},M)}_{\nu,\mu}(z) = G^{M,\,0}_{M,\,M}\left(\begin{matrix}\mu_1, \ldots, \mu_M \\ \nu_1, \ldots, \nu_M\end{matrix}\,\middle|\, |z|^2\right) = \int_{\mathcal{C}} \frac{du}{2\pi\imath} |z|^{2u} \prod_{j=1}^{M} \frac{\Gamma(\nu_j - u)}{\Gamma(\mu_j - u)}. \tag{100}$$

The analogue of the joint probability densities (81) and (82) for a product of truncated orthogonal matrices is

$$p_\nu^{(\mathrm{J},\beta=1)}(Z^{(M)}) \propto |\Delta_{2\widetilde{N}}(Z^{(M)})| \times \prod_{a=1}^{\widetilde{N}} \Big(\delta(\mathrm{Im}\ z_{2a-1})\delta(\mathrm{Im}\ z_{2a}) + 2\delta^{(2)}(z_{2a-1}-z_{2a}^*)\Big) \times h_\nu^{(\mathrm{J},M)}(z_{2a-1},z_{2a}) \tag{101}$$

for an even matrix dimension and

$$p_\nu^{(\mathrm{J},\beta=1)}(Z^{(M)}) \propto |\Delta_{2\widetilde{N}+1}(Z^{(M)})| \times g_{\nu/2,\tilde{\mu}}^{(\mathrm{J},M)}(\mathrm{Re}\ z_{2\widetilde{N}+1})\delta(\mathrm{Im}\ z_{2\widetilde{N}+1}) \times \prod_{a=1}^{\widetilde{N}} \Big(\delta(\mathrm{Im}\ z_{2a-1})\delta(\mathrm{Im}\ z_{2a}) + 2\delta^{(2)}(z_{2a-1}-z_{2a}^*)\Big) \times h_\nu^{(\mathrm{J},M)}(z_{2a-1},z_{2a}) \tag{102}$$

for an odd dimension. Here $\tilde{\mu}$ collectively denotes the constants $\tilde{\mu}_i = \kappa_i + (\nu_i + N_i + 1)/2$. The two-point weight is in this case

$$h_{\nu,\tilde{\mu}}^{(\mathrm{J},M)}(z_1,z_2) = \int_{|\,\mathrm{Im}(z_1-z_2)|/2}^{\infty} d\alpha\ \delta^{(4)}(\Lambda(z_1,z_2,\alpha) - \widehat{Z}^{(M)}\cdots\widehat{Z}^{(1)}) \times \Bigg[\prod_{j=1}^{M}\int d[\widehat{Z}^{(j)}]\,|\mathrm{det}\widehat{Z}^{(j)}|^{\nu_j}\Theta(\mathbb{1}_{N_{\min}} - \widehat{Z}^{(j)}\widehat{Z}^{(j)\,T}) \times \det{}^{\tilde{\mu}_j-(\nu_j+3)/2}(\mathbb{1}_{N_{\min}} - \widehat{Z}^{(j)}\widehat{Z}^{(j)\,T})\Bigg]. \tag{103}$$

This weight can be also rephrased to something like Eq. (85) which we omit here since it looks quite complicated and does not yield new insights. Let us state, at least, what the weight for a real eigenvalue pair is

$$h_{\nu,\tilde{\mu}}^{(\mathrm{J},M)}(\mathrm{Re}\ z_1, \mathrm{Re}\ z_2) \ \propto\ g_{\nu/2,\tilde{\mu}}^{(\mathrm{J},M)}(\mathrm{Re}\ z_1) g_{\nu/2,\tilde{\mu}}^{(\mathrm{J},M)}(\mathrm{Re}\ z_2). \tag{104}$$

Again this can be derived by performing a generalized Schur decomposition of the 2×2 blocks along the idea of Ref. [33].

V. EIGENVALUE CORRELATION FUNCTIONS AND THE LYAPUNOV EXPONENT OF THE OPEN, CHAOTIC CHAIN

In this section we derive the eigenvalue correlation functions of products of Ginibre matrices, Jacobi matrices and an intermix of both kinds. Furthermore, we discuss the Lyapunov exponents of the eigenvalues of the product matrices. From the structure of the joint probability densities discussed in the previous section, we can immediately conclude that all eigenvalue correlations can be reduced to averages over one and two characteristic polynomials, which are thus the fundamental objects and determine the whole eigenvalue statistics. Moreover we can conclude that the k-point correlation functions as well as the averages over an arbitrary number of ratios of characteristic polynomials follow determinantal ($\beta = 2$) and Pfaffian ($\beta = 1, 4$) point processes. The reason is that the joint probability densities only depend on a product of a squared Vandermonde determinant and one-point weights ($\beta = 2$) corresponding to bi-orthogonal polynomials or on a Vandermonde determinant and a product of two-point weights ($\beta = 1, 4$) corresponding to skew-orthogonal polynomials. There is a whole scope of literature discussing such ensembles, see Refs. [7, 25, 26, 40–43] and references therein.

Indeed the determinantal and Pfaffian point processes carry over to a mixed product of Ginibre and Jacobi matrices due to the simple structure of both kinds of ensembles. This can be easily seen when considering the joint probability density of the eigenvalues of a product matrix $X^{(M_1+M_2)} = X_M X_{M-1}\cdots X_1$, where X_j, $j \in I_1 = \{j_1,\ldots,j_{M_1}\}$, are complex Ginibre matrices and X_i, $i \in I_2 = \{i_1,\ldots,i_{M_2}\}$, are truncated unitary matrices. The index sets I_1 and I_2 have an empty section, i.e. $I_1 \cap I_2 = \emptyset$, and a union equal to $I_1 \cup I_2 = \{1,\ldots,M_1+M_2 = M\}$.

One can consider eigenvalues (in a generalized sense) of rectangular matrices [45, 46], but here we restrict ourselves to square matrices, hence we choose $N_0 = N_M$. Equivalently one can consider the induced product matrix $\widetilde{X}^{(M)}$, cf. Eq. (18), which is a square matrix by definition. Then one can trivially combine the results (77) and (97) and finds

$$p_{\nu,\mu}^{(\beta=2)}(Z^{(M)}) \propto |\Delta_{N_{\min}}(Z^{(M)})|^2 \prod_{a=1}^{N_{\min}} g_{\nu,\mu}^{(M_1,M_2)}(z_a) \tag{105}$$

with the one point weight

$$g_{\nu,\mu}^{(M_1,M_2)}(z) \propto \prod_{j\in I_1}\int_{\mathbb{C}} d^2z^{(j)}|z^{(j)}|^{2\nu_j}\exp[-|z^{(j)}|^2] \times \prod_{j\in I_2}\int_{|z^{(j)}|=1} d^2z^{(j)}\ \frac{|z^{(j)}|^{2\nu_j}(1-|z^{(j)}|^2)^{\mu_j-\nu_j-1}}{\Gamma(\mu_j-\nu_j)} \times \delta^{(2)}(z - z^{(M)}\cdots z^{(1)}), \tag{106}$$

where ν and μ collectively denote the constants $\nu_i = N_i - N_{\min}$ and $\mu_{i_k} = \kappa_{i_k} + \nu_{i_k} + N_{i_k-1}$. Again, the one-point weight can be expressed as a Meijer G-function [39],

$$g_{\nu,\mu}^{(M_1,M_2)}(z) = G^{M_1+M_2,\,0}_{M_2,\,M_1+M_2}\left(\begin{matrix}\mu_{i_1},\ldots,\mu_{i_{M_2}}\\ \nu_1,\ldots,\nu_{M_1+M_2}\end{matrix}\,\middle|\,|z|^2\right) = \int_{\mathcal{C}}\frac{du}{2\pi\imath}|z|^{2u}\frac{\prod_{j=1}^{M_1+M_2}\Gamma(\nu_j-u)}{\prod_{i\in I_2}\Gamma(\mu_i-u)}. \tag{107}$$

Note that the weak commutation relation manifests itself in the weight through the invariance under permutations of the indices. The special cases where the product

consists solely of Ginibre or Jacobi matrices are deduced from this result by setting either M_1 or M_2 equal to zero.

Similar results can be obtained for the case of real and quaternion matrices. Here we will only state the quaternion case ($\beta = 4$),

$$p_{\nu,\mu}^{(\beta=4)}(Z^{(M)}) \propto \Delta_{2N_{\min}}(Z^{(M)}, Z^{(M)\,*}) \times \prod_{a=1}^{N_{\min}} (z_a - z_a^*) g_{2\nu,2\mu-1}^{(M_1,M_2)}(2^{M_1/2} z_a), \quad (108)$$

which have a structure closely related to the complex case (105).

In the ensuing two subsections we derive the eigenvalue densities of the complex and quaternion case. The discussion of the real case ($\beta = 1$) will be postponed to forthcoming publications. Moreover we will consider the more general case (105) and (108) of a mixed product of Ginibre and Jacobi matrices.

A. Complex matrices ($\beta = 2$)

Looking at the joint probability density (105) it is immediately clear that the corresponding orthogonal polynomials are the monomials z^a and $z^{*\,b}$, since the one-point weight is invariant under rotation in the complex phase. These monomials have the normalization [39]

$$\begin{aligned}\int_{\mathbb{C}} |z|^{2a} g_{\nu,\mu}^{(M_1,M_2)}(z) d^2 z &= \pi \int_0^\infty r^{a+1} G^{M_1+M_2,\,0}_{M_2,\,M_1+M_2}\left(\begin{matrix} \mu_{i_1}, \ldots, \mu_{i_{M_2}} \\ \nu_1, \ldots, \nu_{M_1+M_2} \end{matrix} \,\middle|\, r \right) \frac{dr}{r} \\ &= \pi \frac{\prod_{j=1}^{M_1+M_2} \Gamma(\nu_j + a + 1)}{\prod_{i \in I_2} \Gamma(\mu_i + a + 1)} \end{aligned} \quad (109)$$

with respect to the weight $g_{\nu,\mu}^{(M_1,M_2)}(z)$. Hence the joint probability density can be rewritten into the following determinantal structure,

$$p_{\nu,\mu}^{(\beta=2)}(Z^{(M)}) = \frac{1}{N_{\min}!} \det_{1 \le a,b \le N_{\min}} \left[K^{(N_{\min})}(z_a, z_b^*) \right], \quad (110)$$

see Refs. [7, 26, 40, 42] and references therein. The kernel is given by

$$K^{(N_{\min})}(z_a, z_b^*) = \frac{1}{\pi} \sqrt{g_{\nu,\mu}^{(M_1,M_2)}(z_a) g_{\nu,\mu}^{(M_1,M_2)}(z_b)} \times \sum_{l=0}^{N_{\min}-1} \frac{\prod_{i \in I_2} \Gamma(\mu_i + l + 1)}{\prod_{j=1}^{M_1+M_2} \Gamma(\nu_j + l + 1)} z_a^l z_b^{*\,l}. \quad (111)$$

It follows immediately that the level density is given by

$$\rho^{(N_{\min})}(z) = \frac{1}{\pi} g_{\nu,\mu}^{(M_1,M_2)}(z) \times \sum_{l=0}^{N_{\min}-1} \frac{\prod_{i \in I_2} \Gamma(\mu_i + l + 1)}{\prod_{j=1}^{M_1+M_2} \Gamma(\nu_j + l + 1)} |z|^{2l}, \quad (112)$$

where the density inherits the isotropic structure from the one-point weight. The normalization is chosen such that the integration over the density yields the generic number of non-zero eigenvalues, i.e. $\int \rho^{(N_{\min})}(z) d^2 z = N_{\min}$. If $N_0 = N_M > N_{\min} > 0$ then there are $N_M - N_{\min}$ generic zero modes. They will be reflected as additional Dirac δ-functions in the density (112).

The macroscopic limit, $N_{\min} \to \infty$ and $\hat{\mu}_i = \mu_i / N_{\min}$ and $\hat{\nu}_i = \nu_i / N_{\min}$ fixed, of the level density (112) can be obtained by the scaling $\hat{z} = N_{\min}^{M_1/2} z$. Notice that we do not scale with $N_{\min}^{(M_1+M_2)/2}$ since the spectrum of those matrices drawn from a truncation of unitary matrices is of order one while the spectrum of the Ginibre matrices is of order $\sqrt{N_{\min}}$. Then the macroscopic level density is

$$\rho(\hat{z}) = \lim_{N_{\min} \to \infty} \frac{1}{N_{\min}^{M_1+1}} \rho^{(N_{\min})} \left(\frac{\hat{z}}{N_{\min}^{M_1/2}} \right). \quad (113)$$

The easiest way to derive this level density is via the moments of this density

$$\begin{aligned} \langle |z|^{2k} \rangle_{\rho^{(N_{\min})}} &= \frac{1}{N_{\min}} \int_{\mathbb{C}} |z|^{2k} \rho^{(N_{\min})}(z) d\,\mathrm{Re}\; z\, d\,\mathrm{Im}\; z \\ &= \frac{1}{N_{\min}} \sum_{l=0}^{N_{\min}-1} \prod_{i \in I_2} \frac{\Gamma(\mu_i + l + 1)}{\Gamma(\mu_i + l + k + 1)} \\ &\quad \times \prod_{j=1}^{M_1+M_2} \frac{\Gamma(\nu_j + l + k + 1)}{\Gamma(\nu_j + l + 1)}. \end{aligned} \quad (114)$$

Employing the Stirling formula and approximating the sum by an integral we obtain

$$\begin{aligned} \langle |\hat{z}|^{2k} \rangle_\rho &= \lim_{N_{\min} \to \infty} \frac{1}{N_{\min}^{M_1 k}} \langle |z|^{2k} \rangle_{\rho^{(N_{\min})}} \\ &= \int_0^1 \left(\frac{\prod_{j=1}^{M_1+M_2} (\hat{\nu}_j + y)}{\prod_{i \in I_2} (\hat{\mu}_i + y)} \right)^k dy. \end{aligned} \quad (115)$$

The macroscopic level density can be read off and it is

$$\rho(\hat{z}) = \frac{1}{\pi} \int_0^1 \delta\left(R(y) - |\hat{z}|^2 \right) dy \quad (116)$$

with the rational function

$$R(y) = \frac{\prod_{j=1}^{M_1+M_2} (\hat{\nu}_j + y)}{\prod_{i \in I_2} (\hat{\mu}_i + y)} \ge 0, \quad \forall\, y \in [0,1]. \quad (117)$$

Usually the domain of the level density $\rho(\hat{z})$ is a centred annulus in the complex plane [47, 48]. To determine the inner and outer radius of the annulus it is quite convenient that $R(y)$ is strictly monotonous increasing on the interval $]0,1]$, i.e.

$$\frac{\partial}{\partial y} \ln R(y) = \sum_{j \in I_1} \frac{1}{\hat{\nu}_j + y} + \sum_{i \in I_2} \frac{\hat{\mu}_i - \hat{\nu}_i}{(\hat{\nu}_j + y)(\hat{\mu}_i + y)} > 0.$$

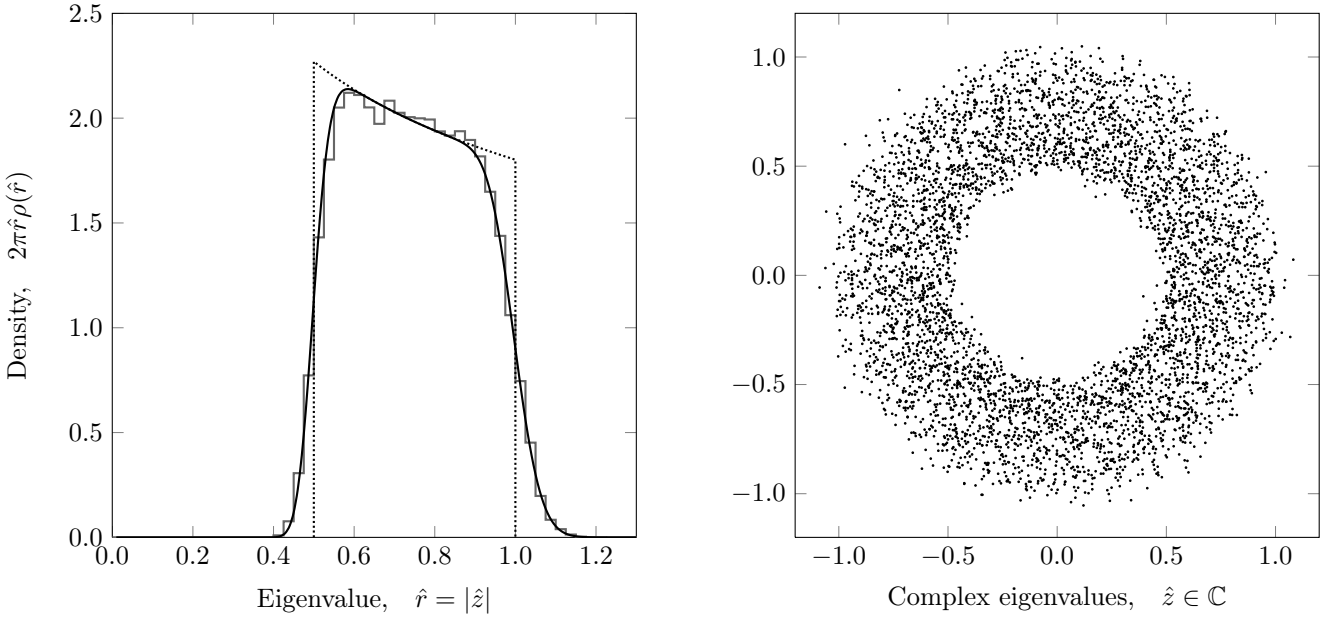

FIG. 1. Example of an induced product matrix, $\widetilde{X}^{(M)}$, with complex eigenvalues distributed within an annulus. The histogram depicted on the left panel shows the distribution of the absolute value of the eigenvalues for 500 realizations of a product matrix with $N_{\min} = 100$, $M_1 = 3$, $M_2 = 0$ and $\hat{\nu} = \{1, 2, 3\}$; the solid curve shows the corresponding analytical prediction, while the dotted curve indicate the macroscopic limit. The right panel shows a scatter plot of 50 out of the 500 realizations generating the histogram on the left panel. Note that the fact that $\hat{\nu}_j > 0$ for all j implies that the original product matrix, $X^{(M)} = X_M \cdots X_1 = U_L \widetilde{X}^{(M)} U_R$, is a rectangular matrix.

Notice that $\hat{\mu}_i > \hat{\nu}_i \geq 0$ for all $i \in I_2$; compare κ_i with the exponent of the determinant in Eq. (8). Therefore the inner and outer radius for the domain of $\rho(\hat{z})$ is

$$r_{\min} = R(0) = \frac{\prod_{j=1}^{M_1+M_2} \hat{\nu}_j}{\prod_{i\in I_2} \hat{\mu}_i} \quad \text{and} \tag{118}$$

$$r_{\max} = R(1) = \frac{\prod_{j=1}^{M_1+M_2} (\hat{\nu}_j + 1)}{\prod_{i\in I_2} (\hat{\mu}_i + 1)}, \tag{119}$$

respectively. Hence the inner radius vanishes if and only if one or more $\hat{\nu}_i$ vanish. If the inner radius vanishes the behaviour of the level density around the origin is $|\hat{z}|^{-2(\lambda-1)/\lambda}$ where λ is the number of indices with $\hat{\nu}_i = 0$. Note that if we are looking at a square product matrix $X^{(M)} = X_M \cdots X_1$, i.e. $N_0 = N_M$, then it immediately follows that at least one $\hat{\nu}_j$ is equal to zero, and therefore that the inner radius vanishes such that the eigenvalues are located within a disk rather than an annulus. When starting from the induced product matrix, $\widetilde{X}^{(M)} = \widetilde{X}_M \cdots \widetilde{X}_1$, the level density can be still located within an annulus, see Fig. 1. This mechanism is equivalent to that of induced Ginibre matrices [29].

For the Ginibre ensemble ($M_1 = 1$ and $M_2 = 0$) the density, $\rho(\hat{z})$, is the well-known complex unit disc with constant density [7]. For general M_1 and $M_2 = 0$ the macroscopic level density was indirectly given by a polynomial equation of the Green function,

$$G(z) = \int_{\mathbb{C}} \frac{\rho(\tilde{z}')d^2\tilde{z}}{\hat{z} - \tilde{z}} \;\Leftrightarrow\; \rho(z) = \frac{1}{\pi}\frac{\partial}{\partial \hat{z}^*} G(z), \tag{120}$$

in Ref. [31]. Due to the isotropy of the level density the result of Ref. [31] can be readily deduced by the relation

$$G(\hat{z}) = \frac{1}{\hat{z}} \int_0^1 \Theta\left(|\hat{z}|^2 - R(y)\right) dy = \frac{R^{-1}(|\hat{z}|^2)}{\hat{z}} \tag{121}$$

($|\hat{z}| \leq R(1)$) yielding the polynomial equation ($M_2 = 0$)

$$R(\hat{z}G(\hat{z})) = |\hat{z}|^2. \tag{122}$$

The case of a truncated unitary matrix ($M_1 = 0$ and $M_2 = 1$) was discussed in Ref. [34].

The macroscopic level density (116) can be easily numerically evaluated. The simplest way is to employ one of the many representations of the Dirac δ-function as a limiting function. In Figs. 1 and 2 we show the comparison of the macroscopic limit with numerical simulations for certain ensembles.

Let us return to the transport on the closed chain coupled to a particle bath, see Sec. II, one can easily calculate the average Lyapunov exponent $\mathrm{Lya} = \ln z = \ln r + \imath\varphi$, where we write the complex eigenvalues in polar coordinates, $z = re^{\imath\varphi}$. Potential wells with a size comparable to the bath are modelled by Jacobi ensembles, while potential wells with a size much smaller than the bath are modelled by Ginibre ensembles.

Since the level density is isotropic the angular part vanishes while the radial part yields the mean Lyapunov exponent

$$\langle \mathrm{Lya} \rangle = 2\pi \int_0^\infty \rho^{(N_{\min})}(r)\, r \ln r \, dr \tag{123}$$

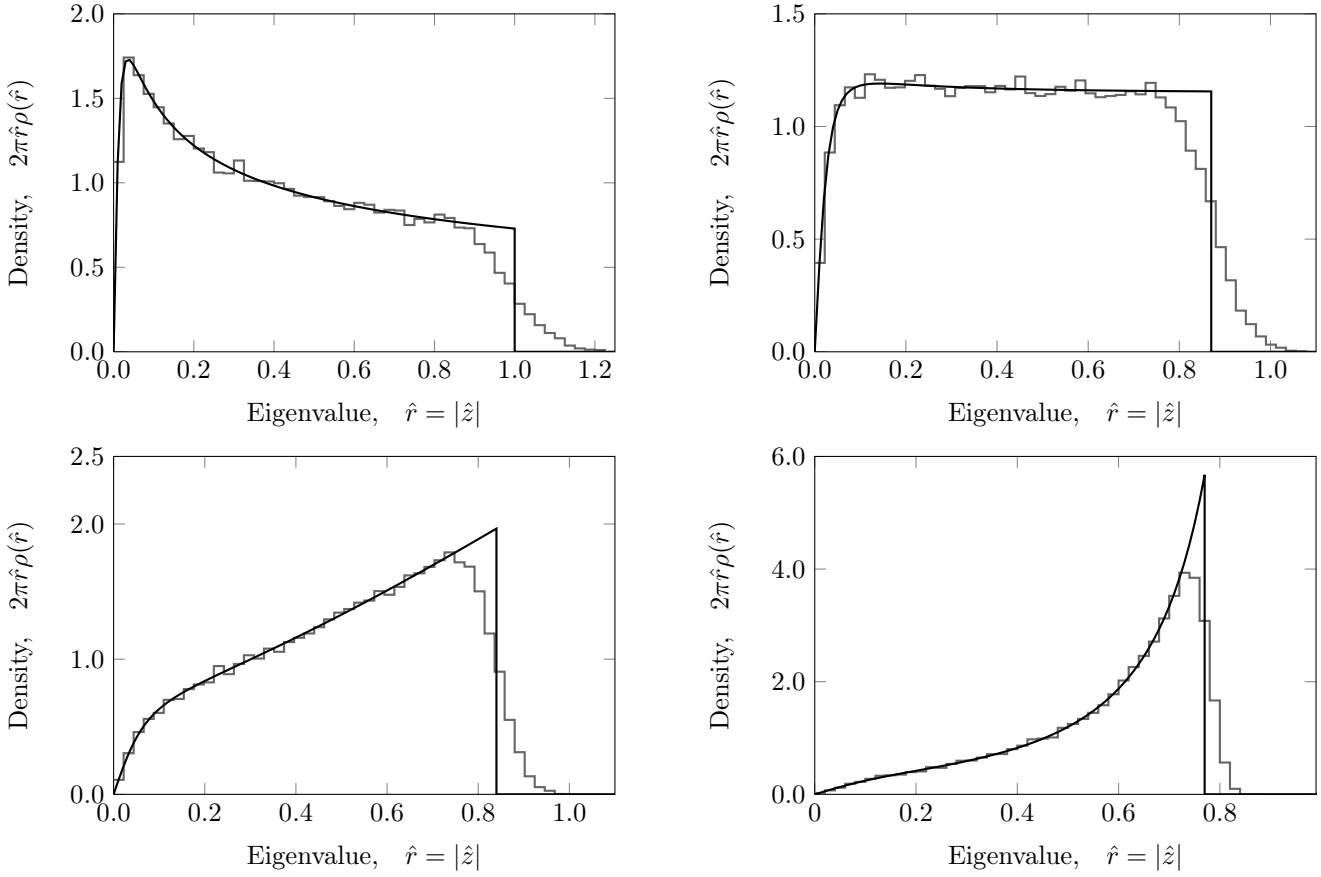

FIG. 2. Each histogram shows the distribution of the absolute value of the eigenvalues for 500 realizations of a product of three independent complex ($\beta = 2$) Ginibre and/or Jacobi random matrices with the smallest matrix dimension $N_{\min} = 100$. The top left histogram has $M_1 = 3$, $M_2 = 0$ and $\hat{\nu} = \{0, 1/10, 2/10\}$, the top right histogram has $M_1 = 2$, $M_2 = 1$, $\hat{\mu} = \{3/10\}$ and $\hat{\nu} = \{0, 1/10, 2/10\}$, the bottom left histogram has $M_1 = 1$, $M_2 = 2$, $\hat{\mu} = \{3/10, 3/10\}$ and $\hat{\nu} = \{0, 1/10, 2/10\}$, and the bottom right histogram has $M_1 = 0$, $M_2 = 3$, $\hat{\mu} = \{3/10, 3/10, 3/10\}$ and $\hat{\nu} = \{0, 1/10, 2/10\}$. The solid lines show the corresponding macroscopic limits, cf. (116). Note that the axes on the four plots have different scales.

at finite matrix dimension. This integral simplifies in the large $N_{\min}$ limit and we find

$$\langle \mathrm{Lya} \rangle = \frac{1}{2} \int_0^1 \ln R(y)\, dy - \frac{M_1}{2} \ln N_{\min}. \tag{124}$$

Interestingly the leading term, $\ln N_{\min}$, vanishes if the size of all potential wells is comparable to the bath, meaning that we have no random matrices drawn from the Ginibre ensemble. Any coupling of a Ginibre matrix with the bath implies that all particles will be sucked away after one round on the closed chain.

B. Quaternion matrices ($\beta = 4$)

We start from the joint probability density (108) with the one-point weight (107). Pursuing the calculation of the corresponding skew-orthogonal polynomials p_j in Ref. [12], i.e.

$$\begin{aligned} \langle p_a | p_b \rangle &= \int_{\mathbb{C}} d^2 z (z^* - z) g_{2\nu,2\mu-1}^{(M_1,M_2)}(2^{M_1/2} z) \\ &\qquad \times (p_a(z) p_b(z^*) - p_b(z) p_a(z^*)) \\ &= \begin{cases} h_l, & a = 2l+1, b = 2l, \\ -h_l, & a = 2l, b = 2l+1, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{125}$$

we find the polynomials

$$\begin{aligned} p_{2l}(z) &= \sum_{k=0}^{l} \left(\prod_{n=k+1}^{l} 2^{-M_1} \frac{\prod_{j=1}^{M_1+M_2} (2\nu_j + 2n)}{\prod_{i \in I_2} (2\mu_i + 2n - 1)} \right) z^{2k}, \\ p_{2l+1}(z) &= z^{2l+1} \end{aligned} \tag{126}$$

and the normalization

$$h_l = 2\pi\ 2^{-2M_1(l+1)} \frac{\prod_{j=1}^{M_1+M_2} \Gamma(2\nu_j + 2l + 2)}{\prod_{i \in I_2} \Gamma(2\mu_i + 2l + 1)} \tag{127}$$

in agreement with Ref. [12] which was for $M_2 = 0$ and $\nu_j = \nu$ for all $j = 1 \dots M_1$. This result directly follows from the isotropy of the one-point weight and the moments of this weight [39],

$$\int_0^\infty G^{M_1+M_2,\,0}_{M_2,\,M_1+M_2}\left(\begin{matrix}2\mu_{i_1}-1,\dots,2\mu_{i_{M_2}}-1\\ 2\nu_1,\dots,2\nu_{M_1+M_2}\end{matrix}\,\middle|\, 2^{M_1} r\right) r^l \frac{dr}{r} = 2^{-M_1 l}\frac{\prod_{j=1}^{M_1+M_2}\Gamma(2\nu_j+l)}{\prod_{i\in I_2}\Gamma(2\mu_i+l-1)}. \quad (128)$$

Thus the joint probability density can be written as a Pfaffian, see Refs. [7, 26, 43] and references therein,

$$p^{(\beta=4)}_{\nu,\mu}(Z^{(M)}) = \frac{1}{N_{\min}!}\operatorname*{Pf}_{1\le a,b\le N_{\min}}\begin{bmatrix}\widehat{K}^{(N_{\min})}(z_a,z_b) & \widehat{K}^{(N_{\min})}(z_a,z_b^*)\\ \widehat{K}^{(N_{\min})}(z_a^*,z_b) & \widehat{K}^{(N_{\min})}(z_a^*,z_b^*)\end{bmatrix} \times \prod_{j=1}^{N_{\min}}(z_j^*-z_j) g^{(M_1,M_2)}_{2\nu,2\mu-1}(2^{M_1/2} z_j) \quad (129)$$

with the pre-kernel

$$\widehat{K}^{(N_{\min})}(z_a,z_b) = \sum_{l=0}^{N_{\min}-1}\frac{p_{2l+1}(z_a)p_{2l}(z_b)-p_{2l}(z_a)p_{2l+1}(z_b)}{h_l}. \quad (130)$$

From Eq. (129) one can easily read off the level density which is

$$\rho^{(N_{\min},\beta=4)}(z) = (z^*-z) g^{(M_1,M_2)}_{2\nu,2\mu-1}(2^{M_1/2} z)\widehat{K}^{(N_{\min})}(z,z^*). \quad (131)$$

We have normalized the density to $N_{\min}$ again. Note that, despite the isotropic one-point weight, the level density (131) is not rotational symmetric. The reason is that the eigenvalues come in complex conjugate pairs, which results in a repulsion from the real axis.

The radial projection of the level density,

$$\rho^{(N_{\min},\beta=4)}_{\text{proj}}(r) = r\int_{-\pi}^{\pi}\rho^{(N_{\min},\beta=4)}(re^{\imath\varphi})d\varphi, \quad (132)$$

is an interesting quantity in many situations. For instance, hole probabilities and overcrowding at the origin only depend on the radial distribution. Moreover, the radial distribution is often useful for comparisons with numerical simulations because of the drastically improved statistics. The integral (132) yields

$$\rho^{(N_{\min},\beta=4)}_{\text{proj}}(r) = 2^{M_1/2+1} g^{(M_1,M_2)}_{2\nu,2\mu-1}(2^{M_1/2} r) \times \sum_{l=0}^{N_{\min}-1}\frac{\prod_{i\in I_2}\Gamma(2\mu_i+2l+1)}{\prod_{j=1}^{M_1+M_2}\Gamma(2\nu_j+2l+2)}(2^{M_1/2} r)^{4l+3} \quad (133)$$

and looks quite similar to the level density of $\beta = 2$, cf. Eq. (112). Actually, it yields the same macroscopic limit,

$$\lim_{N_{\min}\to\infty}\frac{1}{N_{\min}^{M_1/2+1}}\rho^{(N_{\min},\beta=4)}_{\text{proj}}\left(\frac{\hat{r}}{N_{\min}^{M_1/2}}\right) = 2\pi\hat{r}\rho(\hat{r}), \quad (134)$$

cf. Eq. (116). Thus the real part of the Lyapunov exponent of the transport model discussed in Sec. II has to be the same as for the symmetry class $\beta = 2$ in this particular limit. We emphasize that this is not true for finite $N_{\min}$.

VI. CONCLUSIONS

We studied some general properties of a product of M independent rectangular random matrices for all three Dyson classes in a unifying way. The only assumption on the weights of the individual matrices are their invariance under left- and right-multiplication of orthogonal, unitary and unitary symplectic matrices, respectively. These weights are also known as isotropic weights [18]. In this general context, we showed that a product of rectangular random matrices is equivalent to a product of square matrices with modified weights. More strikingly we proved that the individual matrices in the product matrix satisfy a weak commutation relation. This weak commutation relation tells us that a product of independently distributed square random matrices (they can also result from rectangular matrices) is independent of the order of the product when averaging over them. Note that this weak commutation relation has immediate consequences in physical systems. For the considered example of a closed one- dimensional chaotic chain in an environment, the ordering of the potential wells is irrelevant as long as we do not consider cross correlations. The same applies to telecommunications where the permutation of consecutive scatterers does not change the spectrum of the channel matrix, see [14, 15]. We underline that the weak commutation relation holds at finite matrix dimension and, thus, generalizes a known result for the macroscopic limit of the product of isotropic distributed matrices [18]. A weak commutation relation for products of random matrices has previously been discussed in the context of disordered wires with obstacles [19].

The weak commutation relation holds on the level of matrices and affects therefore many quantities of physical interest, such as eigen- and singular values, but also the eigenvectors. We focused on the spectral properties; especially the eigenvalue correlations. We derived the joint probability density functions for all three Dyson classes and for general weights. In particular, we showed that a product of Ginibre matrices, a product of Jacobi matrices or an intermix of both kinds of matrices yields a determinantal (for $\beta = 2$) or Pfaffian (for $\beta = 1, 4$) point process as it is well-known for many other ensembles, see [7, 26, 40, 42, 43] and references therein. We derived a representation of the one- and two-point weights in terms of a product of random variables for $\beta = 2, 4$ and in terms of a product of 2×2 random matrices for $\beta = 1$. For the one-point weight we explicitly integrated over the random variables and showed that they are equal to Meijer G-functions which were already shown for particular cases in [11, 12, 30, 36]. The numerical simulations performed

for the product matrices are in complete agreement with the analytical results.

We also considered the macroscopic limit of such an intermixing product of Ginibre and Jacobi matrices and derived an explicit representation of the level density in terms of a one-fold integral. This result agrees with the implicit polynomial equation derived for the corresponding Green function in [31]. We saw that the macroscopic level density either lives on an annulus or a complex disc centered around the origin, which is in agreement with the single ring theorem [47, 48]. After proper unfolding, universality should hold on a local scale as have been partially discussed in [11, 12, 36].

Finally, we briefly discussed the relation between matrix products and a closed one-dimensional chaotic chain in an environment. In particular, we calculate the Lyapunov exponents. We concluded that the ordering of the potential wells is irrelevant as long as we do not consider cross correlations, which directly follows from the weak commutation relation. Furthermore, we showed that if at least one potential well on the chain is small compared to the bath, then all particles disappear from the chain after a single revolution.

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Appendix A: Distributional relation of eigenvalues and singular values of a 2×2 real matrix

The given problem is the following. We have a joint probability distribution of the singular values $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ of a 2×2 real matrix $Z = V_\mathrm{L} \Lambda V_\mathrm{R}$ with $\lambda_1 \geq \lambda_2$, $Q(\lambda_1 + \lambda_2, \lambda_1 \lambda_2)|\lambda_1^2 - \lambda_2^2|$, and $V_{\mathrm{L/R}} \in \mathrm{O}(2)$ distributed by the Haar measure. Notice that we assume Q as a function of the trace and the modulus of the determinant of Z. What is the joint probability density of the eigenvalues of Z? To solve this question we pursue an idea similar to the calculations done in Refs. [37, 38].

We start from the zeros of the characteristic polynomials

$$\det(V_\mathrm{L} \Lambda V_\mathrm{R} - z \mathbb{1}_2) = 0. \tag{A1}$$

From this equation we notice that only one rotation angle (from now on denoted by $\varphi \in [0, \pi[$) parametrizing V_L and V_R plays a role. The other one cancels out. Let the sign of the determinant of Z be

$$s = \mathrm{sign}(\det Z) = \mathrm{sign}(\det V_\mathrm{L} \Lambda V_\mathrm{R}) = \mathrm{sign}(\det V_\mathrm{L}) \mathrm{sign}(\det V_\mathrm{R}). \tag{A2}$$

Then Eq. (A1) can be rewritten to

$$0 = \det \left(\frac{\lambda_1 - s\lambda_2}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{\lambda_1 + s\lambda_2}{2} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} - z \mathbb{1}_2 \right) = z^2 - (\lambda_1 + s\lambda_2) \cos\varphi \, z + s\lambda_1\lambda_2. \tag{A3}$$

Thus the eigenvalues are

$$z_\pm = \frac{\lambda_1 + s\lambda_2}{2} \cos\varphi \pm \sqrt{\frac{(\lambda_1 + s\lambda_2)^2}{4} \cos^2\varphi - s\lambda_1\lambda_2} = \frac{\lambda_1 + s\lambda_2}{2} \cos\varphi \pm \sqrt{\frac{(\lambda_1 - s\lambda_2)^2}{4} - \frac{(\lambda_1 + s\lambda_2)^2}{4} \sin^2\varphi}. \tag{A4}$$

We can only find a complex conjugate pair if $s = +1$ meaning $\det Z > 0$.

Let $\lambda_\pm = (\lambda_1 \pm \lambda_2)/2$. The joint probability density of the eigenvalues of Z is given by

$$\begin{aligned}
\widehat{Q}(z_1,z_2) &= \frac{8}{\pi}\int_0^\infty d\lambda_+ \int_0^{\lambda_+} d\lambda_- \int_0^\pi d\varphi Q(2\lambda_+, \lambda_+^2-\lambda_-^2)\lambda_+\lambda_-\delta^{(2)}(z_1-z_+)\delta^{(2)}(z_2-z_-). \qquad \text{(A5)}\\
&= \frac{2}{\pi}\sum_{s=\pm}\int_0^\infty d\lambda_+ \int_0^{\lambda_+} d\lambda_- \int_0^\pi d\varphi Q(2\lambda_+, \lambda_+^2-\lambda_-^2)\lambda_+\lambda_-\\
&\times \delta^{(2)}\left(\lambda_s\cos\varphi - \frac{z_1+z_2}{2}\right)\delta^{(2)}\left(\sqrt{\lambda_{-s}^2-\lambda_s^2\sin^2\varphi} - \frac{z_1-z_2}{2}\right)\\
&= \frac{4}{\pi}\delta\left(\mathrm{Im}\,(z_1+z_2)\right)\sum_{s=\pm}\Theta(sz_1z_2)\int_0^\infty d\lambda_+^2\int_0^{\lambda_+^2} d\lambda_-^2 Q(2\lambda_+,\lambda_+^2-\lambda_-^2)\\
&\times \frac{\Theta\left(\lambda_s - |\mathrm{Re}\,(z_1+z_2)|/2\right)}{\sqrt{\lambda_s^2-\mathrm{Re}^2(z_1+z_2)/4}}\delta^{(2)}\left(\sqrt{\lambda_{-s}^2-\lambda_s^2+\frac{\mathrm{Re}^2(z_1+z_2)}{4}} - \frac{z_1-z_2}{2}\right)\\
&= \widehat{Q}_{\mathrm{r},-}(z_1,z_2)+\widehat{Q}_{\mathrm{r},+}(z_1,z_2)+\widehat{Q}_{\mathrm{c}}(z_1,z_2).
\end{aligned}$$

This distribution splits into three terms. For $s=-1$ we have a distribution, $\widehat{Q}_{\mathrm{r},-}$, of two real eigenvalues where one is positive and the other one negative. For $s=+1$ we find a distribution, $\widehat{Q}_{\mathrm{r},+}$, of two real eigenvalues which are both positive or both negative (when $\lambda_-^2-\lambda_+^2+\mathrm{Re}^2(z_1+z_2)/4>0$) as well as one distribution, $\widehat{Q}_{\mathrm{c}}$, of a complex conjugate pair (only when $\lambda_-^2-\lambda_+^2+\mathrm{Re}^2(z_1+z_2)/4<0$).

First we concentrate on the case $s=-1$. We calculate

$$\begin{aligned}
\widehat{Q}_{\mathrm{r},-}(z_1,z_2) &= \frac{4}{\pi}\delta\left(\mathrm{Im}\ z_1\right)\delta\left(\mathrm{Im}\ z_2\right)\Theta(-\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2)\int_0^\infty d\lambda_+^2\int_0^{\lambda_+^2} d\lambda_-^2 Q(2\lambda_+,\lambda_+^2-\lambda_-^2) \qquad \text{(A6)}\\
&\times \frac{\Theta\left(\lambda_- - |\mathrm{Re}\,(z_1+z_2)|/2\right)}{\sqrt{\lambda_-^2-\mathrm{Re}^2(z_1+z_2)/4}}\delta\left(\sqrt{\lambda_+^2-\lambda_-^2+\frac{\mathrm{Re}^2(z_1+z_2)}{4}} - \frac{\mathrm{Re}\,(z_1-z_2)}{2}\right)\\
&\overset{(1)}{=} \frac{4|\,\mathrm{Re}\,(z_1-z_2)|}{\pi}\delta\left(\mathrm{Im}\ z_1\right)\delta\left(\mathrm{Im}\ z_2\right)\Theta(-\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2)\int_0^\infty d\lambda_+^2\int_0^1 d\widehat{\alpha}Q(2\lambda_+,\lambda_+^2(1-\widehat{\alpha}))\\
&\times \lambda_+^2\frac{\Theta\left(\lambda_+^2\widehat{\alpha}-\mathrm{Re}^2(z_1+z_2)/4\right)}{\sqrt{\lambda_+^2\widehat{\alpha}-\mathrm{Re}^2(z_1+z_2)/4}}\delta\left(\lambda_+^2(1-\widehat{\alpha})+\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2\right)\delta\left(\lambda_+^2(\widehat{\alpha}-1)+\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2\right)\\
&\overset{(2)}{=} \frac{4|\,\mathrm{Re}\,(z_1-z_2)|}{\pi}\delta\left(\mathrm{Im}\ z_1\right)\delta\left(\mathrm{Im}\ z_2\right)\Theta(-\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2)\int_{\mathrm{Re}^2(z_1-z_2)/4}^\infty d\lambda_+^2\frac{Q\left(2\lambda_+,-\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2\right)}{\sqrt{\lambda_+^2-\mathrm{Re}^2(z_1-z_2)/4}}\\
&\overset{(3)}{=} \frac{8|z_1-z_2|}{\pi}\delta\left(\mathrm{Im}\ z_1\right)\delta\left(\mathrm{Im}\ z_2\right)\Theta(-\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2)\int_0^\infty d\alpha Q\left(\sqrt{\mathrm{Re}^2(z_1-z_2)+4\alpha^2},-\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2\right)
\end{aligned}$$

We made the substitutions $\lambda_- = \lambda_+\sqrt{\widehat{\alpha}}$ in line (1) and $\alpha = \sqrt{\lambda_+^2-\mathrm{Re}^2(z_1-z_2)/4}$ in line (3). In line (2) we integrated

over $\widehat{\alpha}$. A similar calculation can be also performed in the case of real eigenvalues with $s = +1$,

$$\widehat{Q}_{\mathrm{r},+}(z_1,z_2) = \frac{4}{\pi}\delta\left(\mathrm{Im}\ z_1\right)\delta\left(\mathrm{Im}\ z_2\right)\Theta(\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2)\int_0^\infty d\lambda_+^2\int_0^{\lambda_+^2} d\lambda_-^2\, Q(2\lambda_+,\lambda_+^2-\lambda_-^2) \tag{A7}$$

$$\times\frac{\Theta\left(\lambda_+ - |\mathrm{Re}\left(z_1+z_2\right)|/2\right)}{\sqrt{\lambda_+^2-\mathrm{Re}^2(z_1+z_2)/4}}\Theta\left(\lambda_-^2-\lambda_+^2+\frac{\mathrm{Re}^2(z_1+z_2)}{4}\right)\delta\left(\sqrt{\lambda_-^2-\lambda_+^2+\frac{\mathrm{Re}^2(z_1+z_2)}{4}}-\frac{\mathrm{Re}\left(z_1-z_2\right)}{2}\right)$$

$$\overset{(1)}{=}\frac{4|\,\mathrm{Re}\left(z_1-z_2\right)|}{\pi}\delta\left(\mathrm{Im}\ z_1\right)\delta\left(\mathrm{Im}\ z_2\right)\Theta(\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2)$$

$$\times\int_0^\infty d\lambda_+^2\int_0^1 d\widehat{\alpha}Q(2\lambda_+,\lambda_+^2(1-\widehat{\alpha}))\lambda_+^2\frac{\Theta\left(\lambda_+-|\,\mathrm{Re}(z_1+z_2)|/2\right)}{\sqrt{\lambda_+^2-\mathrm{Re}^2(z_1+z_2)/4}}\delta\left(\lambda_+^2(\widehat{\alpha}-1)+\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2\right)$$

$$\overset{(2)}{=}\frac{4|\,\mathrm{Re}\left(z_1-z_2\right)|}{\pi}\delta\left(\mathrm{Im}\ z_1\right)\delta\left(\mathrm{Im}\ z_2\right)\Theta(\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2)\times\int_{\mathrm{Re}^2(z_1+z_2)/4}^\infty d\lambda_+^2\frac{Q\left(2\lambda_+,\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2\right)}{\sqrt{\lambda_+^2-\mathrm{Re}^2(z_1+z_2)/4}}$$

$$\overset{(3)}{=}\frac{8|z_1-z_2|}{\pi}\delta\left(\mathrm{Im}\ z_1\right)\delta\left(\mathrm{Im}\ z_2\right)\Theta(\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2)\int_0^\infty d\alpha Q\left(\sqrt{\mathrm{Re}^2(z_1+z_2)+4\alpha^2},\mathrm{Re}\ z_1\,\mathrm{Re}\ z_2\right)$$

Again we made some substitutions namely $\lambda_- = \lambda_+\sqrt{\widehat{\alpha}}$ in line (1) and $\alpha = \sqrt{\lambda_+^2-\mathrm{Re}^2(z_1+z_2)/4}$ in line (3). Step (2) is the same as in Eq. (A6). Please notice that this result is almost the same as in the case $s = -1$.

For the complex conjugated pair we have

$$\widehat{Q}_{\mathrm{c}}(z_1,z_2) = \frac{8}{\pi}\delta^{(2)}\left(z_1-z_2^*\right)\int_0^\infty d\lambda_+^2\int_0^{\lambda_+^2} d\lambda_-^2\, Q(2\lambda_+,\lambda_+^2-\lambda_-^2) \tag{A8}$$

$$\times\frac{\Theta\left(\lambda_+-|\mathrm{Re}\ z_1|\right)}{\sqrt{\lambda_+^2-\mathrm{Re}^2\,z_1}}\Theta\left(\lambda_+^2-\lambda_-^2-\mathrm{Re}^2\,z_1\right)\delta\left(\sqrt{\lambda_+^2-\lambda_-^2-\mathrm{Re}^2\,z_1}-\mathrm{Im}\ z_1\right)$$

$$\overset{(1)}{=}\frac{16|\,\mathrm{Im}\ z_1|}{\pi}\delta^{(2)}\left(z_1-z_2^*\right)\int_0^\infty d\lambda_+^2\int_0^1 d\widehat{\alpha}Q(2\lambda_+,\lambda_+^2(1-\widehat{\alpha}))\lambda_+^2\frac{\Theta\left(\lambda_+^2-\mathrm{Re}^2\,z_1\right)}{\sqrt{\lambda_+^2-\mathrm{Re}^2\,z_1}}\delta\left(\lambda_+^2(1-\widehat{\alpha})-|z_1|^2\right)$$

$$\overset{(2)}{=}\frac{16|\,\mathrm{Im}\ z_1|}{\pi}\delta^{(2)}\left(z_1-z_2^*\right)\int_{|z_1|^2}^\infty d\lambda_+^2\frac{Q(2\lambda_+,|z_1|^2)}{\sqrt{\lambda_+^2-\mathrm{Re}^2\,z_1}}$$

$$\overset{(3)}{=}\frac{16|z_1-z_2|}{\pi}\delta^{(2)}\left(z_1-z_2^*\right)\int_{|\,\mathrm{Im}\ z_1|}^\infty d\alpha\, Q(2\sqrt{\mathrm{Re}^2\,z_1+\alpha^2},|z_1|^2)$$

The steps (1) and (3) are the same as in Eq. (A7). In the step (2) we recognize the fact that the Dirac δ-function yields only something non-vanishing if $\lambda_+ \geq |z_1| \geq |\,\mathrm{Re}\ z_1| \geq 0$.

Comparing the results of Eqs. (A6-A8) we notice that the function Q always depends on $\sqrt{(z_1+\mathrm{sign}(z_1z_2)z_2)^2+4\alpha}$ which is the argument for the trace of the original 2×2 matrix Z and on $\mathrm{sign}(z_1z_2)z_1z_2$ which is the determinant of Z. If we want to rewrite the function Q as a function of its singular values $\lambda_{1/2}$ we need the functional dependence of those variables on $z_{1/2}$ and α. This dependence is

$$\lambda_{1/2} = \left|\sqrt{\frac{(z_1+z_2)^2}{4}+\alpha^2}\pm\sqrt{\frac{(z_1-z_2)^2}{4}+\alpha^2}\right|. \tag{A9}$$

This relation readily follows from the system of equations,

$$\lambda_1+\lambda_2 = \sqrt{(z_1+\mathrm{sign}(z_1z_2)z_2)^2+4\alpha}\ \text{and}\ \lambda_1\lambda_2 = \mathrm{sign}(z_1z_2)z_1z_2, \tag{A10}$$

representing the trace and the determinant of Z, respectively. Again we emphasize that z_1 and z_2 are either real or complex conjugate.

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Universal microscopic correlation functions for products of truncated unitary matrices

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Abstract

We investigate the spectral properties of the product of M complex non-Hermitian random matrices that are obtained by removing L rows and columns of larger unitary random matrices uniformly distributed on the group $\mathrm{U}(N+L)$. Such matrices are called truncated unitary matrices or random contractions. We first derive the joint probability distribution for the complex eigenvalues of the product matrix for fixed N, L, and M, given by a standard determinantal point process in the complex plane. The weight however is non-standard and can be expressed in terms of the Meijer G-function. The explicit knowledge of all eigenvalue correlation functions and the corresponding kernel allows us to take various large N (and L) limits at fixed M. At strong non-unitarity, with L/N finite, the eigenvalues condense on a domain inside the unit circle. At the edge and in the bulk we find the same universal microscopic kernel as for a single complex non-Hermitian matrix from the Ginibre ensemble. At the origin we find the same new universality classes labelled by M as for the product of M matrices from the Ginibre ensemble. Keeping a fixed size of truncation, L, when N goes to infinity leads to weak non-unitarity, with most eigenvalues on the unit circle as for unitary matrices. Here we find a new microscopic edge kernel that generalizes the known results for $M = 1$. We briefly comment on the case when each product matrix results from a truncation of different size L_j.

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1 Introduction

The topic of products of random matrices first introduced in [1] has seen a certain renaissance in recent years. Such products have been applied in a variety of disciplines ranging from the problem of entanglement in quantum mechanics [2, 3], quantum chromodynamics with chemical potential [4], combinatorics [5] to finance [6], wireless telecommunications [7] and image processing [8]. See Ref. [9] and references therein for a general and recent overview of random matrix theory (RMT).

In this work we are interested in products of random matrices which are related to the unitary group, namely truncations thereof also called sub-unitary or random contractions. In general, unitary matrices play an important role in time evolution in quantum mechanics or matrix valued diffusion [10]. Another classical problem where (sub-) unitary random matrices have been extensively used is that of chaotic scattering on mesoscopic devices, as reviewed e.g. in [11]. Typically the scattering process is represented by a unitary S-matrix, and the reflection or transmission between the leads of such a device are sub-matrices of the S-matrix. We refer to [12] for a recent review on scattering in chaotic systems using RMT. It is therefore very interesting to study also the spectral properties of such sub-unitary matrices obtained from truncating the full S-matrix, and such an analysis has been already made for a single random matrix in [13]. Subunitary matrices have also been considered in the context of many-body quantum states and a one-component plasma of charges on the pseudo-sphere [14]. Previously an alternative representation of subunitary matrices was constructed in [15]. Here the concept of weak non-unitarity was introduced by adding an imaginary finite rank perturbation to the Gaussian Unitary Ensemble, leading to the same correlations as in [13] for truncated unitary matrices, in the limit where the size of truncation is kept fixed while the matrix size goes to infinity. Truncations of orthogonal matrices have also been considered, see [13, 16], which are used when the system fulfills additional symmetries (e.g. time reversal invariance).

Very recently the joint probability distribution and spectral properties of the complex eigenvalues [17, 18, 19, 20, 21, 22] and the real singular values [23, 24, 25, 26] of products of M random matrices of size $N \times N$ have been computed for finite N and M. The matrices that were multiplied were drawn from the Ginibre ensemble [27] of non-Hermitian matrices with a Gaussian probability distribution. Such exact results open up the possibility of a rigorous asymptotic large N analysis. First steps in that direction have already been taken in these works. Prior to these finite-N results it had already been observed, that when multiplying random matrices from different ensembles a large degree of universality can be seen [28], at least regarding the limiting mean density that was derived in [7, 22, 29, 30].

It is the purpose of this paper to extend this approach to the product of several truncated unitary random matrices. Several questions may serve as a further motivation. Can the aforementioned analytic solution for finite N be extended to such ensembles of matrices, and what are the corresponding universality classes in the large N limit? Second, in [13] the truncation of a unitary matrix was viewed as a mean to introduce decoherence in a quantum system. How does such a system evolve in several steps?

In order to partly answer some of these questions we consider the product of M truncated (or sub-) unitary matrices X_j, with $j = 1, \ldots, M$, which originate each from truncating a unitary matrix of size $N + L_j$ distributed with respect to the Haar measure of the unitary group $\mathrm{U}(N + L_j)$ by removing L_j rows and columns. Obviously the resulting product matrix is non-unitary and its complex eigenvalues lie inside the unit disk. We want to study the spectral properties of the product matrix, first at fixed N, L_j, and M, and then in various large N limits, always keeping M fixed in this work. When writing up this paper a related paper [20] appeared, where among other results the same product of truncated unitary matrices is considered, for fixed N, L_j, and M, only. Whenever we can compare the results from the first part of our paper they agree with their findings. Moreover very recently another

work [22] co-authored by one of the authors was published as a preprint discussing the general case of products of rectangular matrices. Those products also comprise truncated unitary matrices which were given as an example. However the derivation for those matrices was not as thoroughly discussed as it is done here. In [22] only the macroscopic limit with $L_j \propto N \to \infty$ and fixed M was presented. Neither a discussion of local fluctuations nor an analysis of universality were given in [22] or in [20] as it will be considered here. The focus of [22] lied on weak commutation relations of random matrices and general algebraical structures of them which were derived for all three Dyson indices ($\beta = 1, 2, 4$) in a unifying way.

The remainder of this paper is organized as follows. In section 2 we calculate the joint probability density of the product matrix as well as of its complex eigenvalues in subsection 2.1. Because it leads to a standard determinantal point process, with the eigenvalues repelling each other through the modulus square of the Vandermonde determinant, all k-point density correlation functions immediately follow, once the weight function is determined in subsection 2.2. In appendix C we also consider a more general setting, when the individual factors X_j result from truncations of random unitary matrices of different sizes, $N + L_j$, down to the dimension N. Section 3 is devoted to various large N limits starting with the strong non-unitary limit in subsection 3.1. Here L/N remains fixed and positive, and various universal results are recovered for the eigenvalue correlation functions, depending on whether the fluctuations at the edge, in the bulk or at the origin are considered. The weak non-unitarity large N limit when keeping L fixed is performed in subsection 3.2. Here we find a new class of correlation functions for $M > 1$ at the edge of the unit circle. We conclude in section 4. Some further technical details about the measure and the joint probability distribution are collected in two further appendices A and B, respectively.

2 The solution for finite N

We consider the product of M random matrices $X^{(M)} = X_M X_{M-1} \cdots X_1$. Each of these random matrices X_j has the size $N \times N$ and is truncated from a larger $(N + L) \times (N + L)$ unitary matrix U_j. The unitary matrices U_j are identically, independently distributed via the normalized Haar-measure on $\mathrm{U}(N + L)$. We recall the measure of the product matrix $X^{(M)}$ in subsection 2.1 and derive the joint probability density of its complex eigenvalues as well as all k-point density correlation functions. In subsection 2.2 the weight function is determined in terms of the Meijer G-function, giving several examples. In appendix C we generalize this result to different truncations of the individual X_j, resulting from unitary matrices of different sizes $N + L_j$ truncated to N, i.e. $L_j \neq L_i$.

2.1 Probability measure and joint probability density

Consider a square $N \times N$ sub-block X of a single unitary random matrix U distributed according to the Haar-measure on the unitary group $\mathrm{U}(N + L)$ [13]

$$U = \begin{pmatrix} X & W \\ V & Y \end{pmatrix} \in \mathrm{U}(N + L) \ . \tag{2.1}$$

This block is a random matrix which is usually referred to as truncated unitary random matrix, or random contraction, cf. [13, 14, 31]. The probability measure for truncated matrices with $L \geq 0$ is

$$d\mu(X) \propto d[X] \int \det{}^{-L}[\imath H - \mathbb{1}_N] \exp[\mathrm{tr}\,(XX^\dagger - \mathbb{1}_N)(\imath H - \mathbb{1}_N)]\ d[H], \tag{2.2}$$

where we integrate over a $N \times N$ Hermitian matrix H. Let $L \geq 1$ in the following, since $L = 0$ corresponds to the full Haar measure of $\mathrm{U}(N)$ defined by a Dirac delta function, see eq. A.3. In the

case $L \geq N$ this integral can be readily performed,

$$d\mu(X) \propto \det{}^{L-N}(\mathbb{1}_N - XX^\dagger)\Theta(\mathbb{1}_N - X^\dagger X)d[X]. \tag{2.3}$$

This measure is also known as Jacobi measure [32]. The symbol Θ denotes the matrix Heaviside function which is equal to one for positive definite matrices and zero otherwise; $d[X]$ is a shorthand notation for the flat measure $d[X] = \prod_{ij} d\Re e X_{ij} d\Im m X_{ij}$. In appendix A we briefly recall the derivation of eq. (2.2) and of the Jacobi measure (2.3) for truncated matrices.

We are interested in the product $X \equiv X^{(M)} = X_M \cdots X_1$ of M independent truncated matrices X_j, for $j = 1, \ldots, M$. The probability measure for such a product is

$$d\nu(X) = d[X] \int \delta(X - X_M \cdots X_1) \prod_{j=1}^{M} d\mu(X_j) \ , \tag{2.4}$$

with $d\mu(X_j)$ given by eq. (2.2). The Dirac delta function of a complex matrix A is defined as the product of the Dirac delta functions of its real independent variables, i.e. $\delta(A) = \prod_{i,j} \delta^{(2)}(A_{ij})$, where the two-dimensional Dirac delta function for a complex variable z is given as $\delta^{(2)}(z) \equiv \delta(\Re e(z))$ $\delta(\Im m(z))$.

We are going to derive the joint probability density function for the eigenvalues of the product matrix X. To this end we parameterize the measure (2.4) using the generalized Schur decomposition [17, 33] $X_j = U_j^\dagger (Z_j + T_j) U_{j-1}$ for $j = 1, \ldots, M$ with $U_j \in \mathrm{U}(N)$, $U_0 = U_M$, Z_j being complex diagonal $Z_j = \mathrm{diag}(z_{j1}, \ldots, z_{jN})$, and T_j complex strictly upper triangular. In this parametrization the matrix X takes the form $X = U_M^\dagger (Z + T) U_M$, where $Z = Z_M \cdots Z_1$ is diagonal and T is upper triangular, i.e. $Z + T = (Z_M + T_M) \cdots (Z_1 + T_1)$. The diagonal elements of the Z-matrix play the role of the complex eigenvalues of X. In other words the eigenvalues of X can be calculated as products of diagonal elements of the diagonal matrices Z_j as $z_n = \prod_{j=1}^{M} z_{jn}$, for $n = 1, \ldots, N$. Note that although the z_n's are the eigenvalues of X, the z_{jn}'s are not the eigenvalues of X_j. In this parametrization the measure $d\nu(X)$, see eq. (2.4), yields

$$\begin{aligned} d\nu(\{Z_i, T_i, U_i\}) \propto & \left| \Delta\left(\prod_{k=1}^{M} Z_k \right) \right|^2 \\ & \times \prod_{i=1}^{M} \left(\int \det{}^{-L}[\imath H_i - \mathbb{1}_N] \exp[\mathrm{tr}\,(U_i^\dagger (Z_i + T_i)(Z_i + T_i)^\dagger U_i - \mathbb{1}_N)(\imath H_i - \mathbb{1}_N)] \ d[H_i] \right) d[Z_i] d[T_i] d\chi(U_i) \ , \end{aligned} \tag{2.5}$$

where

$$\Delta(Z) = \prod_{1 \leq a < b \leq N} (z_a - z_b) \tag{2.6}$$

is the Vandermonde determinant for $Z = \mathrm{diag}(z_1, \ldots, z_N)$, $d\chi(U_i)$ is the Haar measure for the unitary group $\mathrm{U}(N)$, and $d[Z_i]$, $d[T_i]$ and $d[H_i]$ are flat measures for Z_i, T_i and the $N \times N$ Hermitian matrix $H_i = H_i^\dagger$. The square of the Vandermonde determinant arises as a Jacobian for this parametrization [17].

The joint probability density function for the eigenvalues of the product matrix X can be calculated by integrating out all degrees of freedom but Z in the measure (2.5),

$$P^{(N,L,M)}(z_1, \ldots, z_N) = \int \delta(Z - Z_M \cdots Z_1) d\nu(\{Z_i, T_i, U_i\}). \tag{2.7}$$

The integration over the U_i's completely factorizes in eq. (2.7) since they can be absorbed in the Hermitian matrices H_i's. Then the integration over the T_i's and U_i's (2.7) reads

$$P^{(N,L,M)}(z_1,\ldots,z_N) \propto \int \delta(Z - Z_M \cdots Z_1) \left| \Delta\left(\prod_{k=1}^{M} Z_k\right)\right|^2 \prod_{i=1}^{M} W_1^{(L)}(Z_i) d[Z_i], \tag{2.8}$$

where

$$W_1^{(L)}(Z_i) \propto \int \det{}^{-L}[\imath H_i - \mathbb{1}_N] \exp[\mathrm{tr}\left((Z_i+T_i)(Z_i+T_i)^\dagger - \mathbb{1}_N\right)(\imath H_i - \mathbb{1}_N)]\; d[H_i] d[T_i] \tag{2.9}$$

for $i = 1,\ldots,M$. The integration over each T_i and H_i can be done explicitly [13],

$$W_1^{(L)}(Z_i) \propto \prod_{n=1}^{N} w_1^{(L)}(z_{in}) = \prod_{n=1}^{N} \frac{L}{\pi}(1-|z_{in}|^2)^{L-1}\Theta(1-|z_{in}|). \tag{2.10}$$

Note that this result is true for any $L > 0$ independently of whether or not the condition $L \geq N$ is fulfilled. We briefly recall the calculation of this integral in appendix B. The constant L/π is added for convenience in order to ensure the normalization $\int w_1^{(L)}(z) d^2z = 1$. Inserting this result into (2.8) we obtain

$$P^{(N,L,M)}(z_1,\ldots,z_N) \propto |\Delta_N(Z)|^2 W_M^{(L)}(Z)\;, \tag{2.11}$$

with a weight $W_M^{(L)}(Z)$ which factorizes into a product over one-point weights for the diagonal elements of $Z = \mathrm{diag}(z_1,\ldots,z_n)$,

$$W_M^{(L)}(Z) = \prod_{n=1}^{N} w_M^{(L)}(z_n)\;. \tag{2.12}$$

The individual one-point weights are given by

$$w_M^{(L)}(z_n) = \int_{\mathbb{C}^M} \delta^{(2)}\left(z_n - \prod_{j=1}^{M} z_{jn}\right) \prod_{i=1}^{M} w_1^{(L)}(z_{in}) d^2 z_{in}. \tag{2.13}$$

The weight $w_M^{(L)}$ to be computed in the next subsection 2.2 only depends on the modulus of the argument $w_M^{(L)}(z) = w_M^{(L)}(|z|)$. The one-point weights $w_M^{(L)}$ for the product of M matrices are constructed from the one-point weights for the single matrices, see $w_1^{(L)}$ in eq. (2.13). Their definition is equivalent to the one of the probability density function for a random variable being the product of M independently, identically distributed complex random numbers. The integral equation (2.13) can be transformed into a factorized form via the Mellin transform as we shall see in the next subsection. The joint probability density function (2.11) with the overall normalization factor can be written in the standard form [34]

$$P^{(N,L,M)}(z_1,\ldots,z_N) = \frac{1}{N!\prod_{j=0}^{N-1} h_j} \prod_{n=1}^{N} w_M^{(L)}(z_n) \prod_{a<b}^{N} |z_b - z_a|^2 \tag{2.14}$$

with

$$h_j = \int w_M^{(L)}(z)|z|^{2j} d^2z = \left(\int w_1^{(L)}(z)|z|^{2j} d^2z\right)^M = \binom{L+j}{L}^{-M}\;. \tag{2.15}$$

This normalization can be deduced by applying the orthogonal polynomial method in the complex plane [31, 34] to the weight $w_M^{(L)}(z)$. Since the weight only depends on the modulus of z, the corresponding orthogonal polynomials $p_j(z)$, defined through $\int w_M^{(L)}(z)p_i(z)p_j^*(z)d^2z = h_i\delta_{ij}$, are monic, $p_j(z) = z^j$. The joint probability density function (2.14) can then be written in a determinantal form

$$P^{(N,L,M)}(z_1,\ldots,z_N) = \frac{1}{N!}\det_{1\le a,b\le N}\left[K^{(N,L,M)}(z_a,z_b)\right] \ , \tag{2.16}$$

with the kernel

$$K^{(N,L,M)}(u,v) = \sqrt{w_M^{(L)}(u)w_M^{(L)}(v)}\ T^{(N-1,L,M)}(uv^*) \tag{2.17}$$

which is given in terms of the following truncated series

$$T^{(N-1,L,M)}(x) = \sum_{j=0}^{N-1}\binom{L+j}{j}^M x^j \ . \tag{2.18}$$

The upper limit of the sum corresponds to the value of the first superscript of $T^{(N-1,L,M)}$, in this case $N-1$. The determinantal form (2.16) is particularly helpful when one wants to calculate the k-point correlation function [34]

$$R_k^{(N,L,M)}(z_1,\ldots,z_k) \equiv \frac{N!}{(N-k)!}\int\limits_{\mathbb{C}^{N-k}} P^{(N,L,M)}(z_1,\ldots,z_N)\ d^2z_{k+1}\ldots d^2z_N = \det_{1\le a,b\le k}\left[K^{(N,L,M)}(z_a,z_b)\right] . \tag{2.19}$$

The eigenvalue density can be readily read off from Eq. (2.19) as

$$R_1^{(N,L,M)}(z) = K^{(N,L,M)}(z,z) = w_M^{(L)}(|z|)T^{(N-1,L,M)}(|z|^2). \tag{2.20}$$

In Sec. 3 we study the asymptotic behavior of the kernel (2.17) and of the eigenvalue density (2.20) in various limits. To this end we need to independently calculate the asymptotic behavior of the one-point weight functions $w_M^{(L)}(z)$, eq. (2.13), and of the truncated series $T^{(N-1,L,M)}(z)$ in eq. (2.18). In the ensuing subsections we discuss the one-point weight function and then the truncated series.

For the joint probability density function and the kernel in the more general case of truncations resulting from matrices of different sizes $N + L_j$ we refer the reader to appendix C.

2.2 One-point weight functions

We will now determine the family of one-point weights (2.13) $w_M^{(L)}(z)$ recursively, which only depend on the modulus $|z|$. For $M = 1$ the weight is given by $w_1^{(L)}(z) = (L/\pi)(1-|z|^2)^{L-1}\Theta(1-|z|)$, see eq. (2.10). For $M \ge 2$ we proceed as follows. Using polar coordinates $z_i = r_i e^{i\varphi_i}$ one can represent the integral (2.13) as an M-fold Mellin-convolution

$$w_M^{(L)}(z) \ = \ \frac{(2L)^M}{2\pi}\int\limits_{[0,1]^M}\delta\left(|z| - \prod_{m=1}^M r_m\right)\prod_{i=1}^M\left(1-r_i^2\right)^{L-1}dr_i \ , \tag{2.21}$$

which can be turned into the standard M-fold convolution by a further substitution, $r_i = e^{-\vartheta_i/2}$. This changes the multiplicative constraint in the Dirac delta function to an additive one

$$w_M^{(L)}(z) \ = \ \frac{L^M}{\pi}\int\limits_{\mathbb{R}_+^M}\delta\left(2\ln|z| + \sum_{m=1}^M\vartheta_m\right)\prod_{i=1}^M\left(1-e^{-\vartheta_i}\right)^{L-1}d\vartheta_i \ . \tag{2.22}$$

These expressions will be helpful in the following sections to derive the asymptotic behavior of the weight for large argument. Alternatively one can take advantage of the following recursion relations in order to determine the weight,

$$w_{M+1}^{(L)}(z) = 2\pi \int_0^1 w_1^{(L)}(r) w_M^{(L)}\left(\frac{z}{r}\right)\frac{dr}{r} = 2\pi \int_0^1 w_1^{(L)}\left(\frac{z}{r}\right) w_M^{(L)}(r)\frac{dr}{r} \ . \tag{2.23}$$

They directly follow from eq. (2.13). As we show below these relations are especially helpful when one wants to explicitly determine the form of weights for given L and M in terms of elementary functions. Note that for weights with the support $|z| \leq 1$ the integrand in the last equation is non-zero only for $r \in [|z|, 1]$, so one can, in this case, replace the lower integration limit by $|z|$.

Since the weight function $w_M^{(L)}(z)$ depends on its argument z only through the modulus $|z|$, it is convenient to introduce an auxiliary function $\Omega_M^{(L)}(x)$ with a real positive argument

$$w_M^{(L)}(z) \equiv \frac{1}{\pi}\Omega_M^{(L)}(|z|^2) \ . \tag{2.24}$$

The support of $\Omega_M^{(L)}$ is $[0, 1]$. The factor $1/\pi$ ensures its normalization,

$$\int_0^1 \Omega_M^{(L)}(x)dx = \int_0^1 \Omega_M^{(L)}\left(|z|^2\right) d|z|^2 = \int_{|z|\leq 1} w_M^{(L)}(z)d^2z = 1 \ , \tag{2.25}$$

and keeps the recurrence relation simple,

$$\Omega_{M+1}^{(L)}(x) = \int_0^1 \Omega_1^{(L)}(y)\Omega_M^{(L)}\left(\frac{x}{y}\right)\frac{dy}{y} = \int_0^1 \Omega_1^{(L)}\left(\frac{x}{y}\right)\Omega_M^{(L)}(y)\frac{dy}{y} \ . \tag{2.26}$$

The lower integration limit for this particular combination of weights can be replaced by x since the integrand is zero for $y < x$. The Mellin transform

$$\mathrm{M}_M(s) = \int_0^1 \Omega_M^{(L)}(x)x^{s-1}dx \tag{2.27}$$

is the key to solve the recursion (2.26). The recursion (2.26) assumes the form

$$\mathrm{M}_{M+1}(s) = \mathrm{M}_M(s)\mathrm{M}_1(s) = (\mathrm{M}_1(s))^{M+1}. \tag{2.28}$$

The simplicity of this relation uncovers the importance of the Mellin transform for product matrices. The inverse Mellin transform[1] yields the following one-point weight function

$$w_M^{(L)}(z) = \frac{1}{\pi}\Omega_M^{(L)}\left(|z|^2\right) = \frac{1}{\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\mathrm{M}_1(s))^M|z|^{-2s}ds \ . \tag{2.29}$$

It remains to calculate the Mellin transform of the weight function for $M = 1$, $w_1^{(L)}(z)$. It reads

$$\mathrm{M}_1(s) = \pi\int_0^1 w_1^{(L)}(\sqrt{x})x^{s-1}dx = L\int_0^1 (1-x)^{L-1}x^{s-1}dx = \frac{\Gamma(s)\Gamma(L+1)}{\Gamma(s+L)} = \binom{s+L-1}{s-1}^{-1}. \tag{2.30}$$

[1]The constant in the contour is chosen as $c > k+1$, such that the k-th moment of the weight is bounded.

Employing Eq. (2.29), we see that the one-point weight is

$$\begin{aligned} w_M^{(L)}(z) &= \frac{(L!)^M}{\pi}\int_{\mathcal{C}}\left(\frac{\Gamma(-u)}{\Gamma(L-u)}\right)^M |z|^{2u}\frac{du}{2\pi\imath}\;\Theta(1-|z|) \\ &= \frac{(L!)^M}{\pi}G^{M,\,0}_{M,\,M}\left(\begin{matrix}L,\ldots,L\\0,\ldots,0\end{matrix}\middle|\,|z|^2\right)\Theta(1-|z|) \\ &= \frac{(L!)^M}{\pi}|z|^{2L}G^{M,\,0}_{M,\,M}\left(\begin{matrix}0,\ldots,0\\-L,\ldots,-L\end{matrix}\middle|\,|z|^2\right)\Theta(1-|z|)\;. \end{aligned} \tag{2.31}$$

The expression involves a Meijer G-function, see [35], for which we have used an identity in the last line. The contour $\mathcal{C}$ goes from $-\imath\infty$ to $+\imath\infty$ and leaves the poles of the Gamma functions in the numerator to the right of the contour line. For different truncations $N+L_j\to N$ of the single matrices in the product matrix this weight can be trivially generalized by replacing the prefactor $(L!)^M$ by $L_1!\cdots L_M!$ and the indices L in the upper row of the Meijer G-function by the L_j's, see appendix C.

For small M and L the formulae (2.31) can be expressed in terms of elementary functions. Let us first illustrate this for $L=1$. In this case $\Omega_1^{(1)}(x)=1$ for $x\in[0,1]$ and 0 otherwise. The function $\Omega_M^{(1)}(x)$ vanishes for $x\notin[0,1]$, too. In particular we have

$$w_2^{(1)}(z)=\frac{1}{\pi}\int_0^1\Omega_1^{(1)}(y)\Omega_1^{(1)}\left(\frac{|z|^2}{y}\right)\frac{dy}{y}=\frac{1}{\pi}\int_{|z|^2}^1\frac{dy}{y}\;\Theta(1-|z|)=\frac{\ln\left(|z|^{-2}\right)}{\pi}\;\Theta(1-|z|)\;, \tag{2.32}$$

and generally

$$w_{M+1}^{(1)}(z)=\frac{1}{\pi}\int_{|z|^2}^1\Omega_M^{(1)}\left(\frac{|z|^2}{y}\right)\frac{dy}{y}\Theta(1-|z|)=\frac{\ln^M\left(|z|^{-2}\right)}{\pi M!}\;\Theta(1-|z|)\;, \tag{2.33}$$

as can be easily seen by induction. Similarly one can find the consecutive $\Omega_M^{(2)}$'s for $L=2$

$$\begin{aligned} \Omega_1^{(2)}(x) &= 2(1-x)\;\Theta(1-x)\;, \\ \Omega_2^{(2)}(x) &= \Big(-8(1-x)-4(1+x)\ln(x)\Big)\Theta(1-x)\;, \\ \Omega_3^{(2)}(x) &= \Big(48(1-x)+24(1+x)\ln(x)+4(1-x)\ln^2(x)\Big)\Theta(1-x)\;,\quad \text{etc.} \end{aligned} \tag{2.34}$$

or do the same for larger L. The function $\Omega_M^{(L)}$ always takes the form of a polynomial of order $L-1$ in x and of order $M-1$ in $\ln(x)$ because of the general relation

$$\begin{aligned} \Omega_M^{(L)}(x) &= (L!)^M\int_{\mathcal{C}}\frac{x^u}{\prod_{l=0}^{L-1}(l-u)^M}\frac{du}{2\pi\imath}\Theta(1-x) \\ &= \frac{(L!)^M}{(M-1)!}\sum_{l=0}^{L-1}\frac{\partial^{M-1}}{\partial\mu^{M-1}}\left(\frac{x^{l-\mu}}{\prod_{\substack{1\le j\le L\\ j\neq l}}(j-l+\mu)^M}\right)\Bigg|_{\mu=0}\Theta(1-x)\;, \end{aligned} \tag{2.35}$$

where we have employed the residue theorem. In general the term $\Omega_{M+1}^{(L)}(x)\sim\ln^M(1/x)$ dominates the region around the origin which is exactly the same as for $L=1$. For $L=1$ we obtain eq. (2.33), and for $L=2$ we have

$$\begin{aligned} \Omega_M^{(2)}(x) &= \frac{2^M}{(M-1)!}\,\frac{\partial^{M-1}}{\partial\mu^{M-1}}\left(\frac{x^{-\mu}}{(\mu+1)^M}+\frac{x^{1-\mu}}{(\mu-1)^M}\right)\Bigg|_{\mu=0}\Theta(1-x) \\ &= \frac{2^M}{(M-1)!}\sum_{m=0}^{M-1}(-1)^{M+m}\frac{(2M-2-m)!}{(M-1-m)!m!}\left(x+(-1)^{m+1}\right)\ln^m(x)\;\Theta(1-x)\;, \end{aligned} \tag{2.36}$$

agreeing with the special cases in eqs. (2.34).

2.3 Truncated series

In this subsection we list some useful representations of the truncated series (2.18) which is simply the kernel (2.17) at finite-N without the weight functions,

$$T^{(N,L,M)}(x) = \sum_{j=0}^{N} \binom{L+j}{j}^M x^j \tag{2.37}$$

$$= \frac{1}{(L!)^M} \left(\prod_{j=1}^{M} \frac{\partial^L}{\partial y_j^L} \right) \left(\frac{1-(y_1 \cdots y_M)^{N+L+1}}{1-y_1 \cdots y_M} \right) \Bigg|_{y_j = x^{1/M}} \tag{2.38}$$

$$= \int_0^{2\pi} \frac{d\varphi}{2\pi} \, {}_M F_{M-1} \left(\begin{matrix} L+1,\ldots,L+1 \\ 1,\ldots,1 \end{matrix} \Bigg| \, \mathrm{e}^{-\imath\varphi} \right) \frac{1-(x\mathrm{e}^{\imath\varphi})^{(N+1)}}{1-x\mathrm{e}^{\imath\varphi}} \tag{2.39}$$

$$= N \int_0^1 \binom{L+N\xi}{N\xi}^M x^{N\xi} \sum_{j=0}^{N} \delta(j-N\xi) d\xi. \tag{2.40}$$

Each representation will become useful when deriving specific asymptotic expansions.

Note that apart from the representation (2.38) all representations of the truncated series $T^{(N,L,M)}(x)$ can be easily generalized to a product of matrices with different truncations as it is discussed in appendix C. In the representations (2.37) and (2.40) one has only to replace the exponentiated binomials by a product of binomials with different L_j's and in the representation (2.39) one has to replace the indices $L+1$ in the upper row of the hypergeometric function ${}_M F_{M-1}$ by L_j+1. This generalization is not that simple for the representation (2.38) as can be easily checked by writing the rational function as a geometric sum.

3 Asymptotic behavior for large N

In this section we consider the asymptotic behavior of the model for large $N \to \infty$ and for fixed M. Other important limits are postponed to forthcoming publications. We distinguish two cases: (1) an extensive truncation where the number of truncated columns and rows, L, is of order N, and (2) a weak truncation where L is constant. The former one corresponds to an ensemble of matrices that strongly breaks the unitarity of the matrices drawn from the original ensemble while the latter one corresponds to an ensemble of weakly non-unitary matrices.

3.1 Strong non-unitarity: $N, L \to \infty$ and $\alpha = L/N$ fixed

Let us denote the ratio of the size N of the truncated matrices and the size $N+L$ of the original unitary matrices before truncation by

$$\mu \equiv \frac{N}{N+L} = \frac{1}{1+\alpha} < 1 \quad \text{and} \quad 0 < \alpha \equiv \frac{L}{N}. \tag{3.1}$$

We keep $\mu < 1$ fixed while taking the limit $N \to \infty$.

First we study the macroscopic behavior in subsection 3.1.1 and derive the limiting eigenvalue density. In particular we show that the limiting eigenvalue density covers a disk centered at the origin of the complex plane with radius $\mu^{M/2} < 1$. This is very different from the eigenvalues of a unitary matrix distributed on the unit circle. We call this limit with such an extensive truncation, $L \sim N$, the strong non-unitarity regime.

In subsections 3.1.2 and 3.1.3 we turn to the microscopic limit, showing that the eigenvalue fluctuations in the bulk and at the edge of the support of the macroscopic eigenvalue density have the same local universal behavior as the Ginibre ensemble. Finally in subsection 3.1.4 we discuss the behavior of the kernel at the origin and show that it falls into the same universal classes labeled by M as the product of Ginibre matrices [17].

3.1.1 Macroscopic regime

Here we calculate the mean limiting eigenvalue density for $N \to \infty$. We first derive the asymptotic behavior of the truncated series (2.18) and, then, of the one-point weights (2.31) in the large N limit while keeping the eigenvalues of order unity, $0 < |z| < 1$. These two results constitute the eigenvalue density (2.20).

For the truncated series we employ the representation (2.40). The sum of Dirac delta-functions can be omitted to leading order in the $1/N$-expansion. This amounts to substituting the sum over j by an integral over $\xi = j/N$ running from zero to one and normalized to unity. The binomial symbol containing ξ in the remaining integral is approximated by Stirling's formula, i.e.

$$\binom{L+N\xi}{N\xi} \approx \sqrt{\frac{\alpha+\xi}{2\pi\xi L}}\ \alpha^{-L} \exp\left[N\left((\alpha+\xi)\ln(\alpha+\xi) - \xi\ln\xi\right)\right]\ , \tag{3.2}$$

so we have (2.40)

$$T^{(N,L,M)}(|z|^2) \approx \sqrt{\frac{N^2}{(2\pi L)^M}}\ \alpha^{-ML} \int_0^1 \left(\frac{\alpha+\xi}{\xi}\right)^{M/2} \exp\left[Nf(\xi)\right] d\xi \tag{3.3}$$

with the real function

$$f(\xi) = M\left((\alpha+\xi)\ln(\alpha+\xi) - \xi\ln\xi + \xi\ln|z|^{2/M}\right). \tag{3.4}$$

The large N behavior of this integral can be found by employing Laplace' method (saddle point method for a real function in the exponent). The idea is to replace the exponential function of the integrand by a Gaussian with the width of order $1/\sqrt{N}$ and the maximum located at ξ_0 given by the saddle point equation

$$\ln\left(\frac{\alpha+\xi_0}{\xi_0}\right) + \ln\left(|z|^{2/M}\right) = 0\ . \tag{3.5}$$

This yields for the saddle point

$$\xi_0 = \frac{\alpha}{|z|^{-2/M}-1} = \frac{\mu^{-1}-1}{|z|^{-2/M}-1}\ . \tag{3.6}$$

In case of varying truncations, $L_j \neq L_i$ for $j \neq i$, (see appendix C) the corresponding saddle point equation would be more involved. The saddle point equation (3.5) is generically equivalent to a polynomial equation of order M.

For $|z| < \mu^{M/2}$ the maximum of $f(\xi)$ is located inside the integration range $[0,1]$ of the integral (3.3). When $|z|$ is close to $\mu^{M/2}$, or more precisely when $|z| - \mu^{M/2}$ is of order $1/\sqrt{N}$, a part of the Gaussian lies inside the integration range and a considerable part outside, while when $|z| - \mu^{M/2} \ll 1/\sqrt{N}$ the whole Gaussian lies outside the integration range. This means that in this approximation the integral (3.3) is given by the integral over the Gaussian times a factor corresponding to the fraction of the Gaussian that lies inside the integration range. This factor is approximated by a complementary

error function, erfc $\left(a\sqrt{N}(|z|-\mu^{M/2})\right)$, with some positive constant a. This function changes its value from one to zero for $|z|$ in a narrow interval whose width is of order $1/\sqrt{N}$. The width tends to zero for $N\to\infty$ and the function reduces to the Heaviside step function

$$T^{(N,L,M)}(|z|^2) \;\approx\; \sqrt{\frac{1}{M(2\pi L)^{M-1}}}|z|^{(1-M)/M}\left(1-|z|^{2/M}\right)^{-LM-1}\Theta(\mu^{M/2}-|z|). \tag{3.7}$$

One should remember that for large but finite N the Heaviside step function in the last equation should be rather replaced by the complementary error function. We will come back to this point in subsection 3.1.3 when discussing the fluctuations in the microscopic limit at the edge of the spectrum.

To complete the derivation of the large N limit for the kernel (2.17) we also need to calculate the limiting weight $w_M^{(L)}(z)$, eq. (2.13). We determine it using eq. (2.22). By integrating out ϑ_M we obtain an integral representation

$$w_M^{(L)}(z)=\frac{L^M}{\pi}\Theta(1-|z|)\int\limits_{\mathbb{R}_+^{M-1}}\left(1-|z|^2\exp\left[\sum_{m=1}^{M-1}\vartheta_m\right]\right)^{L-1}\prod_{i=1}^{M-1}\left(1-\mathrm{e}^{-\vartheta_i}\right)^{L-1}d\vartheta_i\ , \tag{3.8}$$

which is well suited for a saddle point analysis for large L and fixed $|z|$. The maximum of the $(M-1)$-dimensional integrand is located at the point $(\vartheta_1^{(0)},\ldots,\vartheta_{M-1}^{(0)})$ given by the following $M-1$ equations:

$$\frac{1}{1-\mathrm{e}^{\vartheta_j^{(0)}}}=\frac{|z|^2}{|z|^2-\exp[-\sum_{m=1}^{M-1}\vartheta_m^{(0)}]},\quad \forall j\in 1,\ldots,M-1\ . \tag{3.9}$$

They have a unique symmetric solution $\vartheta_1^{(0)}=\ldots=\vartheta_{M-1}^{(0)}=-(2\ln|z|)/M$. When expanding around this saddle point we have to calculate the determinant of the $(M-1)\times(M-1)$ Hessian matrix at the saddle point, i.e.

$$\det\left[\frac{|z|^{-2/M}}{(1-|z|^{-2/M})^2}(1+\delta_{ab})\right]_{1\le a,b\le M-1}=\frac{M|z|^{2(M-1)/M}}{(1-|z|^{2/M})^{2(M-1)}}. \tag{3.10}$$

Summarizing the steps we find the asymptotic formula for $N\to\infty$

$$w_M^{(L)}(z) \;\approx\; \sqrt{\frac{(2\pi L)^{M-1}}{M}}\frac{L}{\pi}|z|^{(1-M)/M}\left(1-|z|^{2/M}\right)^{ML-1}\Theta(1-|z|) \tag{3.11}$$

for the one-point weights. One should note that the corresponding expression for the one-point weights would be more complicated for varying truncations L_j since in that case the saddle point equation for the counterpart of the integral (3.8) would have generically a non-symmetric solution. The symmetric solution is particular and does not apply to the general case of different truncations proposed in appendix C.

Inserting the eq. (3.7) and eq. (3.11) in eq. (2.20) we obtain the normalized level density

$$\rho^{(\mu,M)}(z)\equiv\lim_{N,L\to\infty}\frac{1}{N}R_1^{(N,L,M)}(z)=\frac{\alpha}{\pi M}\frac{|z|^{2(1-M)/M}}{(1-|z|^{2/M})^2}\Theta(\mu^{M/2}-|z|). \tag{3.12}$$

It agrees with the result derived in Ref. [13, 36] for $M=1$ and [37] for general M. We see that the support of this function is a disk of radius $\mu^{M/2}$ which is smaller than the radius $\sqrt{\mu}$ of a single matrix

$(M=1)$ in the product since $\mu \leq 1$. Note that by changing variables, $z = |z|e^{\imath\phi} \to v = |z|^{1/M} e^{\imath\phi}$, one obtains

$$d^2 z \rho^{(\mu,M)}(z) = d^2 v \rho^{(\mu,1)}(v) \ , \tag{3.13}$$

which means that the product of M independent truncated matrices has the same limiting density as the M-th power of a single random matrix drawn from the same ensemble for $N \to \infty$. This is due to the self-averaging property of isotropic matrices [37]. This also agrees with what was found for the product of M Ginibre matrices [7, 17, 22, 29, 30]. One should also note that the density can be made globally flat on its support by the following reparameterization $z \to \widehat{z} = \sqrt{(1-\mu)/(1-|z|^{2/M})} e^{\imath\phi}$ that gives

$$\widehat{\rho}^{(\mu,M)}(\widehat{z}) = \begin{cases} \dfrac{1}{\pi\mu} \ , & \sqrt{1-\mu} \leq |\widehat{z}| \leq 1, \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{3.14}$$

This is the well-known limiting ring distribution of the Ginibre ensemble of rectangular matrices of dimensions $N \times (N+L)$ whose ratio $\mu = N/(N+L)$ is kept constant in the large N limit, see ref. [38].

We conclude this subsection by considering truncations much larger than the remaining matrix, $L \gg N$. Here the parameter μ becomes very small, being approximately equal to $\mu \approx N/L \ll 1$. In this case the radius of the support of the eigenvalue density of the product $X^{(M)}$ is of order $(N/L)^{M/2} \ll 1$. This can be fixed by rescaling all matrices in the product by a factor $(N/L)^{1/2}$, $X_j = (N/L)^{1/2} Y_j$, and we obtain a product matrix $Y \equiv Y^{(M)} = Y_M \cdots Y_1$ whose eigenvalue density has a support of radius one. Moreover the probability measure for individual matrices Y_j in the product becomes Gaussian in this limit, as follows from eq. (2.3)

$$\begin{aligned} d\mu\left(\sqrt{\frac{N}{L}} Y\right) &\propto \det{}^{L-N}\left(\mathbb{1}_N - \frac{N}{L} Y^\dagger Y\right) \Theta\left(\mathbb{1}_N - \frac{N}{L} Y^\dagger Y\right) d[Y] \\ &\approx \exp\left[-N \mathrm{Tr} Y^\dagger Y\right] d[Y] \ . \end{aligned} \tag{3.15}$$

In the second step of the calculation we assumed that $N/L \ll 1$ and used the following approximation $\det(\mathbb{1} - \alpha^{-1} A) = \exp(\mathrm{Tr}\ln(\mathbb{1} - \alpha^{-1} A)) \approx \exp(-\alpha^{-1} \mathrm{Tr} A)$ that holds for $|\alpha^{-1}| \ll 1$. We also omitted the Heaviside function because the matrix in its argument becomes automatically positive definite in the limit $\alpha^{-1} = N/L \to 0$. In other words in the regime $L \gg N$ the behavior of the product of independent truncated unitary matrices is equivalent to the product of independent Ginibre matrices.

3.1.2 Bulk fluctuations

In the previous subsection we determined the large N asymptotic behavior of the truncated series (3.7),

$$T^{(N,L,M)}(uv^*) \approx \frac{1}{\sqrt{M(2\pi L)^{M-1}}} (uv^*)^{(1-M)/(2M)} \left(1 - (uv^*)^{1/M}\right)^{-LM-1} \Theta(\mu^{M/2} - |z|) \ , \tag{3.16}$$

and of the one-point weight (3.11),

$$w_M^{(L)}(u) \approx \sqrt{\frac{(2\pi L)^{M-1}}{M}} \frac{L}{\pi} |u|^{(1-M)/M} \left(1 - |u|^{2/M}\right)^{ML-1} \Theta(1-|z|) \ , \tag{3.17}$$

which we repeat here for completeness. Together they build up the kernel (2.17) which is given by

$$K^{(N,L,M)}(u,v) = \sqrt{w_M^{(L)}(u) w_M^{(L)}(v)}\ T^{(N-1,L,M)}(uv^*) \ . \tag{3.18}$$

Note that with complex arguments the truncated series contains a phase, whereas the weights only depending on the modulus do not. These formulae hold inside the disk of radius $\mu^{M/2}$. The kernel vanishes when either u or v lie outside the disk.

In this subsection we determine the microscopic large N limit in the bulk of the spectrum. This can be analyzed by studying the kernel (3.18) for two neighboring points,

$$u = z + \frac{1}{\sqrt{N}}\delta u \quad \text{and} \quad v = z + \frac{1}{\sqrt{N}}\delta v, \tag{3.19}$$

both located in the vicinity of the point z in the bulk, i.e. $0 < |z| < \mu^{M/2}$, with $|z|$, $|\delta u|$ and $|\delta v|$ of order unity. The calculation is carried out as a $1/\sqrt{N}$-expansion up to second order, by inserting (3.19) into the expressions for the limiting truncated series (3.16) and weights (3.17). Technically it is easier when the u- and v-dependent factors are first exponentiated and then Taylor expanded in the two variables in the standard way,

$$\begin{aligned} \exp[g(u,v^*)] \;\approx\; & \exp\left[g(z,z^*) + \frac{1}{\sqrt{N}}(\delta u, \delta v^*)(\partial_u g, \partial_{v^*} g)^T \right. \\ & \left. + \frac{1}{2N}(\delta u, \delta v^*) \begin{pmatrix} \partial_u\partial_u g & \partial_u\partial_{v^*} g \\ \partial_{v^*}\partial_u g & \partial_{v^*}\partial_{v^*} g \end{pmatrix} (\delta u, \delta v^*)^T \right] \end{aligned} \tag{3.20}$$

up to higher order terms, where g is a general action. In the limit of large N and L only the factors with an exponent proportional to L will contribute. We obtain the following result for the truncated series

$$\begin{aligned} T^{(N,L,M)}(uv^*) \;\approx\; & \frac{1}{\sqrt{M(2\pi L)^{M-1}}}\, |z|^{(1-M)/(M)} \left(1-|z|^{2/M}\right)^{-LM-1} \\ & \times \exp\left[\frac{L|z|^{2(1-M)/M}}{\sqrt{N}(1-|z|^{2/M})}(\delta u z^* + \delta v^* z) + \frac{L|z|^{2(1-M)/M}}{NM(1-|z|^{2/M})^2}\delta u \delta v^* \right] \\ & \times \exp\left[-\frac{L|z|^{2(1-2M)/M}}{2N(1-|z|^{2/M})}\left(1 - \frac{1}{M(1-|z|^{2/M})}\right)(\delta u^2 z^{*\,2} + \delta v^{*\,2} z^2) \right], \end{aligned} \tag{3.21}$$

and for the weight at argument u

$$\begin{aligned} w_M^{(L)}(u) \;\approx\; & \sqrt{\frac{(2\pi L)^{M-1}}{M}}\frac{L}{\pi}|z|^{(1-M)/M}\left(1-|z|^{2/M}\right)^{ML-1} \\ & \times \exp\left[-\frac{L|z|^{2(1-M)/M}}{\sqrt{N}(1-|z|^{2/M})}(\delta u z^* + \delta u^* z) - \frac{L|z|^{2(1-M)/M}}{NM(1-|z|^{2/M})^2}\delta u \delta u^* \right] \\ & \times \exp\left[\frac{L|z|^{2(1-2M)/M}}{2N(1-|z|^{2/M})}\left(1 - \frac{1}{M(1-|z|^{2/M})}\right)(\delta u^2 z^{*\,2} + \delta u^{*\,2} z^2) \right]. \end{aligned} \tag{3.22}$$

Inserting these two asymptotic results into eq. (3.18) we obtain the following expression for the limiting microscopic kernel in the bulk of the spectrum,

$$\begin{aligned} K_{\text{bulk}}^{(\mu,M)}(\delta u, \delta v) \;\equiv\; & \lim_{N,L\to\infty} \frac{1}{N} K^{(N,L,M)}(u,v) \\ =\; & \rho^{(\mu,M)}(z) \mathrm{e}^{\imath\Phi(\delta u,\delta v)} \exp\left[-2\pi\rho^{(\mu,M)}(z)\left(\frac{1}{2}|\delta u|^2 + \frac{1}{2}|\delta v|^2 - \delta u \delta v^*\right)\right] \end{aligned} \tag{3.23}$$

where we have introduced the phase

$$\begin{aligned}\Phi(\delta u,\delta v) \;\equiv\; & \frac{L|z|^{2(1-M)/M}}{\sqrt{N}(1-|z|^{2/M})}\Im m((\delta u-\delta v)z^*) \\ & -\frac{L|z|^{2(1-2M)/M}}{2N(1-|z|^{2/M})}\left(1-\frac{1}{M(1-|z|^{2/M})}\right)\Im m\Big((\delta u^2-\delta v^2)z^{*\,2}\Big)\;. \end{aligned} \tag{3.24}$$

It is a real function which is antisymmetric $\Phi(\delta u,\delta v)=-\Phi(\delta v,\delta u)$ under exchanging its arguments. Thus the phase factor cancels out in the expression for the k-point correlation functions given by the determinants of the kernel (2.19). The remaining part of the expression (3.23) contains the macroscopic density (3.12) at the point z where we zoom in. It is equivalent to the Ginibre kernel [27] which is universal [39, 40, 41]. When changing variables to

$$\delta\hat{u}=\sqrt{2\pi\rho^{(\mu,M)}(z)}\;\delta u \quad , \quad \delta\hat{v}=\sqrt{2\pi\rho^{(\mu,M)}(z)}\;\delta v \tag{3.25}$$

we obtain the identical expression for the microscopic bulk kernel as in the Ginibre ensemble [27].

As a last remark, note that in the previous subsection 3.1.1 on the macroscopic behavior we found in the limit $\mu\to 0$ the product of Ginibre matrices. In this limit, the universal microscopic bulk kernel of a product of Ginibre matrices was already found [17]. Additionally universality in the bulk holds for all values of $1>\mu\geq 0$ including the limit $\mu\to 0$.

3.1.3 Edge fluctuations

We are going to derive the asymptotic behavior of the kernel (2.17) in the large N limit for fixed $\alpha=N/L$, for u,v lying close to each other and close to the edge. More precisely, we consider

$$u=\mu^{M/2}\mathrm{e}^{\imath\varphi}+\frac{1}{\sqrt{N}}\delta u \;\text{ and }\; v=\mu^{M/2}\mathrm{e}^{\imath\varphi}+\frac{1}{\sqrt{N}}\delta v \tag{3.26}$$

in the vicinity of a point $z=\mu^{M/2}\mathrm{e}^{\imath\varphi}$ on the edge of the support of the distribution (3.12), and $|\delta u|$ and $|\delta v|$ are both of order unity. Since the level density is rotationally invariant we assume that $\varphi=0$ without loss of generality.

As before we begin with calculating the truncated series (3.3) using the saddle point method, except that now we do this for the sum $T^{(N,L,M)}(uv^*)$ of a complex argument uv^*. We obtain an analogous expression as (3.3) except that $\ln|z|^2$ is replaced by the principal value of $\ln(uv^*)$

$$\ln(uv^*)\approx M\ln(\mu)+\frac{\mu^{-M/2}}{\sqrt{N}}(\delta u+\delta v^*)-\frac{\mu^{-M}}{2N}(\delta u^2+\delta v^{*\,2})\equiv M\ln(\mu)+A+B\;. \tag{3.27}$$

Here we define the terms A and B of order $1/\sqrt{N}$ and $1/N$, respectively, while neglecting higher order terms in the expansion. The saddle point ξ_0 fulfils an equation analogous to (3.5), except that $|z|^{-2/M}$ is replaced by $(uv^*)^{-1/M}$, and is expanded as in eq. (3.27),

$$0=f'(\xi_0)=M\ln\left(\frac{\alpha+\xi_0}{\xi_0}\right)+M\ln(\mu)+A+B. \tag{3.28}$$

Recall the definition of the action f in eq. (3.4). This leads to the following expression for the solution ξ_0 of the saddle point equation, expanded up to leading order:

$$\xi_0\approx 1+\frac{\mu^{-M/2}}{\sqrt{N}\;M(1-\mu)}(\delta u+\delta v^*)\equiv 1+\frac{A}{M(1-\mu)}. \tag{3.29}$$

This expansion can be also directly obtained from eq. (3.6). The saddle point $|\xi_0| \leq 1$ is located in the vicinity of the upper integration interval at $\xi = 1$. Therefore we can no longer replace the integral by a Gaussian integration over the whole real axis when expanding $\xi = \xi_0 - \delta\xi/\sqrt{N}$. Instead we have to set a lower bound of the integral over $\delta\xi$ at $A/(M(1-\mu))$. Integrating over $\delta\xi$ we obtain a complementary error function,

$$T^{(N,L,M)}(u,v^*) \approx \sqrt{\frac{N^2}{(2\pi L)^M}}\,\alpha^{-ML}\exp[Nf(\xi_0)]\sqrt{\frac{\pi}{-2Nf''(\xi_0)}}\,\mathrm{erfc}\left[\frac{A}{M(1-\mu)}\sqrt{-\frac{Nf''(\xi_0)}{2}}\right], \tag{3.30}$$

given in terms of the notation from eqs. (3.28) and (3.29) above, cf. eq. (3.3). While in the last two terms the limiting value $-f''(\xi_0 = 1) = M(1-\mu)$ is legitimate, we still need to expand the first exponential factor $\exp[Nf(\xi_0)]$ up to second order in $1/\sqrt{N}$ using eqs. (3.27) and (3.29). Collecting all terms to that order we finally arrive at

$$\begin{aligned} T^{(N,L,M)}(uv^*) &\approx \frac{(1-\mu)^{-LM-1}\mu^{-(M-1)/2}}{(2\pi L)^{(M-1)/2}\sqrt{M}}\exp\left[\sqrt{N}\mu^{-M/2}(\delta u+\delta v^*)\right] \\ &\times \exp\left[-\frac{\mu^{-M}}{2}(\delta u^2+\delta v^{*\,2})+\frac{\mu^{-M}}{2M(1-\mu)}(\delta u+\delta v^*)^2\right]\mathrm{erfc}\left[\frac{\mu^{-M/2}}{\sqrt{2M(1-\mu)}}(\delta u+\delta v^*)\right]. \end{aligned} \tag{3.31}$$

Let us come to the calculation of the asymptotic behavior of the one-point weight for large N and L. Applying the result (3.11) to u and v in our edge scaling regime, see eq. (3.26), the two leading terms of the $1/\sqrt{N}$ expansion yield

$$\begin{aligned} w_M^{(L)}(u) \approx \sqrt{\frac{(2\pi L)^{M-1}}{M}}\frac{L}{\pi}\mu^{-(M-1)/2}\,(1-\mu)^{ML-1}\exp\left[-\sqrt{N}\mu^{-M/2}(\delta u+\delta u^*)\right] \\ \times\exp\left[\frac{\mu^{-M}}{2}\left(1-\frac{1}{M(1-\mu)}\right)(\delta u^2+\delta u^{*\,2})-\frac{\mu^{-M}}{M(1-\mu)}\delta u\delta u^*\right] \end{aligned} \tag{3.32}$$

and analogously for $w_M^{(L)}(v)$. Summarizing the results (3.32) and (3.31) the microscopic limit of the kernel at the edge is

$$\begin{aligned} K_{\mathrm{edge}}^{(\mu,M)}(\delta u,\delta v) &\equiv \lim_{N,L\to\infty}\frac{1}{N}K^{(N,L,M)}(u,v) \\ &= \frac{\mu^{-M}}{\pi M(1-\mu)}\exp\left[\imath\Phi(\delta u,\delta v)\right]\exp\left[-\frac{\mu^{-M}}{2M(1-\mu)}\left(|\delta u|^2+|\delta v|^2-2\delta u\delta v^*\right)\right] \\ &\times \mathrm{erfc}\left[\frac{\mu^{-M/2}}{\sqrt{2M(1-\mu)}}(\delta u+\delta v^*)\right], \end{aligned} \tag{3.33}$$

where $\Phi(\delta u,\delta v) = -\Phi(\delta v,\delta u)$ is a real phase,

$$\Phi(\delta u,\delta v) = \sqrt{N}\mu^{-M/2}\Im m(\delta u-\delta v)+\frac{\mu^{-M}(1-M+\mu M)}{2M(1-\mu)}\Im m(\delta u^2-\delta v^2)\,. \tag{3.34}$$

The phase factors $\exp[\imath\Phi]$ cancel out in the determinantal structure of the k-point correlation function (2.19) and, thus, are irrelevant in the spectral statistics. What remains is the universal error function kernel that agrees with that of the Ginibre ensemble [34, 42, 43] after changing to new variables

$$\delta\hat{u} \equiv \frac{\mu^{-M/2}}{\sqrt{M(1-\mu)}}\delta u. \tag{3.35}$$

In particular the microscopic density at the edge is then given by

$$\rho_{\rm edge}^{(\mu,M)}(\delta u) \;=\; K_{\rm edge}^{(\mu,M)}(\delta u,\delta u) = \frac{\mu^{-M}}{\pi M(1-\mu)}{\rm erfc}\left[\sqrt{\frac{2\mu^{-M}}{M(1-\mu)}}|\delta u|\right] \;, \tag{3.36}$$

which is the universal result [34, 42, 43].

We expect that the universal results (3.33) and (3.36) also hold in the general setting with different truncations, see appendix C. Since the saddle point equations are highly complicated in this case we have not proven it here.

3.1.4 Fluctuations at the origin

While studying the asymptotic behavior of the kernel in the vicinity of the origin for $N\to\infty$ it is convenient to introduce a rescaled variable δz with $|\delta z|$ of order one,

$$z = L^{-M/2}\delta z \;, \tag{3.37}$$

and express results in δz. One could alternatively use a variable $\delta z'$ in the scaling formula $z = N^{-M/2}\delta z'$ but since $\alpha = L/N$ is kept fixed in the limit $N\to\infty$, δz and $\delta z'$ differ by an inessential constant $\delta z = \alpha^{M/2}\delta z'$ which does not affect the N-dependence of the scaling.

For $N\to\infty$ the truncated series asymptotically behaves like

$$T^{(N,L,M)}(|z|^2) \approx \sum_{j=0}^{\infty}\frac{|\delta z|^{2j}}{[j!]^M} = {}_0F_{M-1}\left(\begin{matrix}-\\ \overline{1,\ldots,1}\end{matrix}\middle|\, |\delta z|^2\right) \tag{3.38}$$

resulting from (2.37). Note that this asymptotic result does not change at all when choosing different truncations of the matrices in the product matrix $X^{(M)}$, see appendix C.

The asymptotic behavior of the one-point weights (2.13) can be derived from the first line of the integral representation (2.31) using the following asymptotic behavior of the Gamma function:

$$\lim_{L\to\infty}\frac{\Gamma(L-u)}{\Gamma(L)}\;{\rm e}^{u\ln(L)} = 1 \;. \tag{3.39}$$

This directly leads to

$$w_M^{(L)}(z) \approx \frac{L^M}{\pi}G_{0,\,M}^{M,\,0}\left(\begin{matrix}-\\ \overline{0,\ldots,0}\end{matrix}\middle|\, |\delta z|^2\right) \;. \tag{3.40}$$

Also this result can be readily generalized to the situation of different truncations $N+L_j\to N$ of the matrices in the product matrix, see appendix C, by replacing the prefactor L^M by $L_1\cdots L_M$.

Now we have all constituents needed for the microscopic limit of the kernel (2.17) at the origin. Combining eqs. (3.38) and (3.40) we obtain

$$\begin{aligned} K_{\rm origin}^{(\mu,M)}(\delta u,\delta v) \;&\equiv\; \lim_{N,L\to\infty} L^{-M}K^{(N,L,M)}(u+L^{-M/2}\delta u, v+L^{-M/2}\delta v) \\ &=\; \frac{1}{\pi}\sqrt{G_{0,\,M}^{M,\,0}\left(\begin{matrix}-\\ \overline{0,\ldots,0}\end{matrix}\middle|\, |\delta u|^2\right)G_{0,\,M}^{M,\,0}\left(\begin{matrix}-\\ \overline{0,\ldots,0}\end{matrix}\middle|\, |\delta v|^2\right)}\;{}_0F_{M-1}\left(\begin{matrix}-\\ \overline{1,\ldots,1}\end{matrix}\middle|\, \delta u\delta v^*\right). \end{aligned} \tag{3.41}$$

The kernel agrees with the microscopic kernel at the origin for the complex eigenvalues of products of Ginibre matrices [17] hinting to a universal property of product matrices. The same result follows for the model with different truncations described in appendix C, whenever all $L_j\to\infty$. Only the rescaling has to be changed from $L^{-M/2}$ to $\prod_{j=1}^M L_j^{-1/2}$.

As a check, one can regain the Ginibre kernel [27] from eq. (3.41) for $M = 1$,

$$K^{(\mu,M=1)}(u,v) = \frac{1}{\pi}\exp\left[-\frac{|\delta u|^2}{2} - \frac{|\delta v|^2}{2} + \delta u \delta v^*\right]. \tag{3.42}$$

Indeed it is identical with the kernel (3.23) in the bulk of the spectrum since the origin is not a distinguished point for $M = 1$. For $M = 2$ the Meijer G-function and hypergeometric function reduce to a K-Bessel and I-Bessel function respectively, which agree with the kernel found previously in a two-matrix model describing the low-energy limit of QCD [4], with large chemical potential, see e.g. [44] as well as [45].

Using the expression (3.41) the microscopic level density at the origin is given by

$$\rho^{(\mu,M)}_{\rm origin}(\delta z) = K^{(\mu,M)}_{\rm origin}(\delta z, \delta z) = \frac{1}{\pi} G^{M,0}_{0,M}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\Big| |\delta z|^2\right) {}_0F_{M-1}\left(\begin{matrix}-\\ 1,\ldots,1\end{matrix}\Big| |\delta z|^2\right) . \tag{3.43}$$

Note that this rescaled density has a logarithmic singularity

$$\lim_{|\delta z|\to 0} \rho^{(\mu,M)}_{\rm origin}(\delta z) \approx \ln^{M-1}|\delta z| \tag{3.44}$$

at the origin for $M > 1$. Indeed the following two limits hold:

$$\begin{aligned} &\lim_{|\delta z|\to 0} {}_0F_{M-1}\left(\begin{matrix}-\\ 1,\ldots,1\end{matrix}\Big| |\delta z|^2\right) \to 1 , \\ &\lim_{|\delta z|\to 0} G^{M,0}_{0,M}\left(\begin{matrix}-\\ 0,\ldots,0\end{matrix}\Big| |\delta z|^2\right) \approx (\ln^{M-1}|\delta z|^2)/(M-1)! . \end{aligned} \tag{3.45}$$

For $M = 1$ however this singularity is absent, as can be seen from eq. (3.42). Thus the level density takes a constant value at the origin, $\rho^{(\mu,M=1)}(\delta z) = 1/\pi$, as in [13] and the Ginibre ensemble [27].

We emphasize and briefly comment on the case of more general truncations L_j from appendix C. Due to the lack of the corresponding large N expressions for the truncated series (3.7) and one-point weight (3.8) an expansion is non-trivial. However, from universality arguments we expect that both bulk and edge fluctuations should also match with the universal Ginibre results in this case as it is the case for the fluctuations at the origin.

3.2 Weak non-unitarity: $N \to \infty$ and L fixed

In this section we consider a limit called weak non-unitarity limit introduced and studied for $M = 1$ in [13], where N is taken to infinity while L is kept fixed. For fixed L many things simplify since the matrices X_j and their product $X^{(M)}$ are almost unitary in the limit $N \to \infty$. Indeed we shall see that almost the whole eigenvalue spectrum of $X^{(M)}$ is concentrated in the vicinity of the unit circle in the complex plane. The macroscopic density becomes a delta function of the modulus as shown in the next subsection, whereas nontrivial universal correlations will be found when studying the inner edge of the unit circle in subsection 3.2.2.

3.2.1 Macroscopic regime

The macroscopic level density can be readily derived via its moments

$$\langle z^a z^{*\,b}\rangle = \frac{1}{N}\int\limits_{\mathbb{C}} z^a z^{*\,b} R_1^{(N,L,M)}(z) d^2z = \frac{\delta_{ab}}{N}\sum_{j=0}^{N-1}\binom{L+j}{j}^M \binom{L+j+a}{j+a}^{-M}, \tag{3.46}$$

see Eqs. (2.18), (2.31), (2.20), and (2.15). The large N asymptotics (L is fixed) of these moments is given by

$$\langle z^a z^{*\,b}\rangle \approx \delta_{ab}\int_0^1 \left[\frac{\Gamma(L+Ny+1)\Gamma(Ny+a+1)}{\Gamma(Ny+1)\Gamma(L+Ny+a+1)}\right]^M dy \approx \delta_{ab}. \tag{3.47}$$

These moments correspond to a level density uniformly distributed on the complex unit circle,

$$\rho^{(L,M)}(z) = \frac{1}{\pi}\delta(1-|z|^2). \tag{3.48}$$

This result is indeed also true for the general case of a product of random matrices originating from different truncations, see appendix C.

Let us visualize what the meaning of the quite straightforward result (3.48) is. For $|z|$ of order unity and for large N, L the density can be well approximated by the expression

$$\rho^{(N,L,M)}(z) = \frac{1}{N}K^{(N,L,M)}(z,z) \approx \frac{L}{\pi NM}\frac{|z|^{2/M-2}}{(1-|z|^{2/M})^2}\Theta_N\left(\left(\frac{N}{N+L}\right)^{M/2} - |z|\right), \tag{3.49}$$

which can be obtained from (3.12) by replacing α by L/N and μ by $N/(N+L)$. Here $\Theta_N(x)$ is a sigmoidal function that changes between zero and one in a narrow crossover region of width $\sim 1/\sqrt{N}$. In the limit $N\to\infty$ it approaches the step function $\Theta(x)$. The exact form of this function depends on N, L, M but it is not essential for the argument that we are going to give below. In the first order approximation one can think of $\Theta_N(x)$ as of the step function $\Theta(x)$. For fixed L, M the limit $N\to\infty$ is non-uniform and has to be taken carefully. If one takes the limit point-wise one obtains $\lim_{N\to\infty}\rho^{(N,L,M)}(z) = 0$ which is of course wrong since the integral of the eigenvalue density has to be normalized. In fact the formula (3.49) gives the correct normalization

$$\int \rho^{(N,L,M)}(z)d^2z \approx \frac{L}{\pi NM}\int_0^{\left(\frac{N}{N+L}\right)^{M/2}} \frac{r^{2/M-2}}{(1-r^{2/M})^2}2\pi r dr = 1\ . \tag{3.50}$$

To resolve this discrepancy it is instructive to calculate the integral of the density $\rho^{(N,L,M)}(z)$ over a disk with radius slightly smaller than one, for example $r = (1-cN^{-1/2})^{M/2}$ where c is a positive constant. When N is large enough this radius is smaller than the cut-off $R = (N/(N+L))^{M/2}$ in the Heaviside-distribution in eq. (3.49) so the disk lies entirely inside the support of the eigenvalue density. While choosing the constant c one should also take into account that the function Θ_N in (3.49) does not have a sharp threshold at R, but rather a sigmoidal one that extends on an interval $[R-\sigma N^{-1/2}, R+\sigma N^{-1/2}]$, which represents a smeared cut-off. One should choose $c>\sigma$ to keep the disk radius smaller than the lower value of the smeared cut-off to avoid interference with the finite N boundary effects. For all points inside such a disk Θ_N in equation (3.49) can be replaced by one and the fraction of eigenvalues inside the disk is given for large N by the following integral

$$p \approx \frac{L}{\pi NM}\int_{|z|\le\left(1-cN^{-1/2}\right)^{M/2}} \rho^{(N,L,M)}(z)d^2z = \frac{L}{\pi NM}\int_0^{\left(1-cN^{-1/2}\right)^{M/2}} \frac{r^{2/M-2}}{(1-r^{2/M})^2}2\pi r dr\ . \tag{3.51}$$

Changing the integration variable to $x = r^{2/M}$ we find that for $N\to\infty$

$$p \approx \frac{L}{N}\int_0^{1-cN^{-1/2}} \frac{dx}{(1-x)^2} = \frac{L}{N}\left(c^{-1}N^{1/2}-1\right) \longrightarrow 0\ . \tag{3.52}$$

The fraction of eigenvalues inside the disk of radius $r = (1 - cN^{-1/2})^{M/2} \approx 1 - (cM/2)/N^{1/2}$ tends to zero as $1/N$ for $N \to \infty$ and the disk radius becomes one. This means that the whole interior of the disk contains almost no eigenvalues and all of them are squeezed in a narrow strip around the unit circle. In the limit $N \to \infty$ the disk becomes an open disk. It contains no eigenvalues as shown above (3.52) while the integral over the whole disk of radius one contains all eigenvalues (3.50). This means that all eigenvalues condense on the unit circle for $N \to \infty$ and hence

$$\lim_{N\to\infty} \frac{1}{N} K^{(N,L,M)}(z,z) = \rho^{(L,M)}(z) = \frac{1}{\pi}\delta(1-|z|^2), \tag{3.53}$$

cf. the result (3.48). For this reason we will only zoom into the vicinity of the unit circle in the next subsection. In the microscopic limit it can be shown that the correlations in the bulk and at the origin of the complex unit disk are highly suppressed, too.

3.2.2 Edge fluctuations

Consider a point on a unit circle $z = \mathrm{e}^{\imath\phi}$. Due to the rotational symmetry we can choose $z = 1$ ($\phi = 0$) without loss of generality. We are interested in eigenvalue correlations measured at two points in the vicinity of $z = 1$

$$u = 1 - \frac{\delta u}{N}, \quad \text{and} \quad v = 1 - \frac{\delta v}{N}, \tag{3.54}$$

where $|\delta u|$ and $|\delta v|$ are of order unity. As usual, the first step of the calculation is to determine the large N behavior of the truncated series (2.18) with argument

$$uv^* = 1 - \frac{1}{N}\left(\delta u + \delta v^*\right) + \frac{1}{N^2}\delta u \delta v^* \ . \tag{3.55}$$

We employ the representation (2.38). Changing variables in eq. (2.38) $y_i = 1 - t_i/N$, where the t_i's are of order one, leads to the following asymptotic expression:

$$T^{(N,L,M)}(uv^*) \approx \frac{(-N)^{ML} N}{(L!)^M} \left(\prod_{j=1}^{M} \frac{\partial}{\partial t_j}\right)^L \left(\frac{1-\exp\left[-(t_1+\cdots+t_M)\right]}{t_1+\cdots+t_M}\right)\Bigg|_{t_j=(\delta u+\delta v^*)/M} . \tag{3.56}$$

Hereby we used the following approximation formulae for large N : $y_1 \cdots y_M \approx 1 - (t_1 + \ldots + t_N)/N$, $y_i^{N+L+1} = (1 - t_i/N)^{N+L+1} \approx \mathrm{e}^{-t_i}$, and $(uv^*)^{1/M} \approx 1 - (\delta u + \delta v^*)/(NM)$ neglecting $O(N^{-2})$ terms. The differential operator in eq. (3.56) acts on the function which is effectively a function of $t = t_1 + \ldots + t_M$. Changing the derivatives in this operator to $\partial/\partial t_i = (\partial t/\partial t_i)\partial/\partial t = \partial/\partial t$ we obtain

$$T^{(N,L,M)}(uv^*) \approx \frac{N^{ML+1}}{(L!)^M} \left(-\frac{\partial}{\partial t}\right)^{ML} \left(\frac{1-\mathrm{e}^{-t}}{t}\right)\Bigg|_{t=(\delta u+\delta v^*)/M} . \tag{3.57}$$

This formula cannot be easily extended to a general truncation of the matrices in the product, see appendix C, because of the reason discussed in the paragraph after eq. (2.40). Nevertheless we expect that the simple replacement $ML \to L_1 + \ldots + L_M$ and $(L!)^M \to L_1! \cdots L_M!$ should do the job.

Let us again switch to the calculation of the one-point weight. Starting from eq. (2.22) the weight reads

$$w_M^{(L)}(u) \ \approx \ \frac{L^M}{\pi} \int\limits_{\mathbb{R}_+^M} \delta\left(\sum_{m=1}^{M} \vartheta_m - \frac{1}{N}\delta_R u\right) \prod_{i=1}^{M} \left(1-\mathrm{e}^{-\vartheta_i}\right)^{L-1} d\vartheta_i \tag{3.58}$$

in the weak non-unitarity regime, where we introduced an abbreviation $\delta_R u = \delta u + \delta u^* = 2\Re e \delta u$. We change the integration variables to θ_i given by $\vartheta_i = \frac{1}{N}\delta_R u\, \theta_i$. For large N the expansion $\mathrm{e}^{-\vartheta_i} = 1 - \delta_R u\, \theta_i/N$ is sufficient up to $O(1/N^2)$ corrections. We obtain

$$\begin{aligned} w_M^{(L)}(u) &\approx \frac{L^M}{\pi}\Theta(\delta_R u)\left(\frac{1}{N}\delta_R u\right)^{ML-1} \int\limits_{[0,1]^M} \delta\left(\sum_{m=1}^M \theta_m - 1\right)\prod_{i=1}^M \theta_i^{L-1} d\theta_i \\ &= \frac{L^M N^{1-ML}}{\pi}\Theta(\delta_R u)\,(\delta_R u)^{ML-1}\prod_{j=1}^{M-1}\left(\int_0^1 \theta^{L-1}(1-\theta)^{jL-1}d\theta\right), \end{aligned} \tag{3.59}$$

and finally

$$w_M^{(L)}(u) \approx \frac{(L!)^M N^{1-ML}}{(ML-1)!\pi}\,(\delta_R u)^{ML-1}\,\Theta(\delta_R u)\ . \tag{3.60}$$

This result can indeed be readily extended to a product of random matrices originating from different truncations $N + L_j \to N$, see appendix C, by replacing $ML \to L_1 + \ldots + L_M$ and $(L!)^M \to L_1! \cdots L_M!$. This is the reason why we expect the same generalization of the asymptotic result (3.57) for the truncated series $T^{(N,L,M)}(uv^*)$.

Collecting both asymptotic formulae (3.57) and (3.60) we find

$$\begin{aligned} K_{\mathrm{weak}}^{(L,M)}(\delta u, \delta v) &\equiv \lim_{N\to\infty}\frac{1}{N^2}K^{(N,L,M)}(u,v) \\ &= \frac{\Theta(\Re e\delta u)\Theta(\Re e\delta v)}{\pi(ML-1)!}\big(4(\Re e\delta u)(\Re e\delta v)\big)^{(ML-1)/2}\left(-\frac{\partial}{\partial t}\right)^{ML}\left(\frac{1-\mathrm{e}^{-t}}{t}\right)\Bigg|_{t=(\delta u+\delta v^*)/M}. \end{aligned} \tag{3.61}$$

The factor $1/N^2$ results from the change of variables in the microscopic limit (3.54) which introduces N^{-2} to the Jacobian $d^2\delta u d^2\delta v = N^{-4} d^2u d^2v$. Note that u and v are complex variables. The limiting eigenvalue density (2.20) is

$$\rho_{\mathrm{weak}}^{(L,M)}(\delta u) = K_{\mathrm{weak}}^{(L,M)}(\delta u, \delta u) = \frac{\Theta(\Re e\delta u)\,(2\Re e\delta u)^{ML-1}}{\pi(ML-1)!}\left(-\frac{\partial}{\partial t}\right)^{ML}\left(\frac{1-\mathrm{e}^{-t}}{t}\right)\Bigg|_{t=2(\Re e\delta u)/M}. \tag{3.62}$$

This result agrees with the one found in Ref. [13] for $M = 1$. Note that the correlation functions that follow from eq. (3.61) also agree for $M = 1$ with the correlation functions found in [15] for non-Hermitian finite rank perturbations of rank L of GUE matrices.

As a check one can show that the bulk fluctuations in the strong non-unitarity limit can be recovered from those of the weak non-unitarity regime by applying the following rescaling

$$\delta u \to L r_0 + \sqrt{L}\delta\hat{u} \quad \text{and} \quad \delta v \to L r_0 + \sqrt{L}\delta\hat{v}\ , \quad r_0 \in \mathbb{R} \tag{3.63}$$

to eq. (3.61) and taking the limit of large L. In more detail, the exponential factor can be dropped in the t-derivatives of eq. (3.57) since it is highly suppressed. We obtain

$$\left(-\frac{\partial}{\partial t}\right)^{ML}\left(\frac{1-\exp[-t]}{t}\right)\Bigg|_{t=(\delta u+\delta v^*)/M} \approx (ML)!\,\frac{1}{t^{ML+1}}\Bigg|_{t=(\delta u+\delta v^*)/M}. \tag{3.64}$$

The expansion of the ML-th power in the weight (3.60) is straightforward. Retaining terms up to second order in $1/\sqrt{L}$ we arrive at

$$\begin{aligned}\lim_{L\to\infty} K_{\rm weak}^{(L,M)}(\delta u,\delta v) &\approx \frac{M}{4\pi L r_0^2}\exp\left[-\frac{M}{8r_0^2}(|\delta\hat{u}|^2+|\delta\hat{v}|^2-2\delta\hat{u}\delta\hat{v}^*)\right] \\ &\quad\times\exp\left[-\imath\frac{M\sqrt{L}}{2r_0}\Im m(\delta\hat{u}-\delta\hat{v})+\imath\frac{M}{8r_0^2}\Im m((\delta\hat{u})^2-(\delta\hat{v})^2)\right]. \qquad (3.65)\end{aligned}$$

The phase factor in the last line cancels in the correlation functions. What remains is the Ginibre kernel with some scale factor. This factor is proportional to the macroscopic level density

$$\rho^{(\mu,M)}(z=1-r_0)\approx\frac{M}{4\pi r_0^2}\ , \qquad (3.66)$$

that is obtained from eq. (3.12) by expanding it in r_0 for $\alpha=1$. The reason for $\alpha=1$ lies in the fact that the ratio L/N, cf. its definition (3.1), does not make any sense in the scaling limit $N\gg L\gg 1$. Thus one has to omit this term.

4 Conclusions

We have studied the complex eigenvalues of products of truncated unitary random matrices. These are obtained by removing rows and columns from larger unitary matrices of the same size or of different sizes. Our investigations generalize previous works on spectral properties of a single truncated unitary matrix [13] as well as of products of random matrices taken from the Ginibre ensemble [17]. We mainly concentrated on the question, whether in the large matrix size limit the local spectral fluctuations also called microscopic limit lead to the same universal correlations know from non-Hermitian random matrix theory, namely the Ginibre ensemble, or if they give rise to new universality classes. This study adds an important part to the full picture of the spectral properties of product matrices consisting of a finite number of matrices recently discussed in several works [17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

Our strategy was to first derive exact results for products of M matrices of size N, resulting from truncations of unitary matrices of size(s) $N+L$ (or $N+L_j$) by L (or L_j, with $j=1,\ldots,M$), for all parameters finite. Those results are in agreement with the very recently published works [20, 22]. In the next step two large N limits were identified, namely when both N and L become large with L/N finite, named strong non-unitarity, and when L remains finite at large N, named weak non-unitarity. We emphasize that the large N limits were not done before and are the main focus of our investigations.

In the strong non-unitarity limit, the complex eigenvalues become concentrated in a supporting disk inside the unit circle (the support for unitary matrices). Three regimes were separately analyzed. At the edge and in the bulk of the support we found the same universal spectral correlations as for the Ginibre ensemble (equivalent to a single truncated unitary matrix at $M=1$) [39, 40, 41, 34, 42, 43]. The same feature was previously found for products of M matrices of the Ginibre ensemble [17]. At the origin the same microscopic correlation functions as for M products of Ginibre matrices [17] were derived. Therefore these classes labelled by M are also universal. Note that for a single Ginibre matrix with $M=1$ the origin is not special and displays the same local spectral statistics as in the bulk.

At weak non-unitarity the majority of complex eigenvalues condense on the unit circle, just as for unitary matrices. Nevertheless, microscopically they still spread out inside the complex unit disk, with a width of the order $1/\sqrt{N}$. Zooming into these edge fluctuations a new class of correlations labelled again by M were found, generalizing previous results for a single truncated unitary matrix [13].

It remains to be shown how much of this picture changes when considering different truncations L_j in the large N limit. Only at strong non-unitarity at the origin and in the weak non-unitarity limit we could show that the same class of universal correlations persists compared to using the same truncation L for all matrices. Such investigations are currently under way. It would also be very interesting to try to use our findings in various applications mentioned in the introduction.

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A The measure for truncated unitary matrices

We briefly derive the measure for the ensemble of $N \times N$ matrices obtained by truncation of a single unitary matrix U distributed with respect to the Haar measure $d\chi(U)$ of $\mathrm{U}(N+L)$. The truncation selects the upper left block X which we choose for simplicity to be quadratic[2] of size $N \times N$ from a unitary matrix of size $(N+L) \times (N+L)$

$$U = \begin{pmatrix} X & W \\ V & Y \end{pmatrix} . \tag{A.1}$$

The probability measure for X is obtained by integrating over the remaining blocks: V of size $L \times N$, W of size $N \times L$ and Y of size $L \times L$. It is convenient to represent the Haar measure for U as a flat measure with a matrix Dirac delta function which selects unitary matrices $d\chi(U) = \delta(\mathbb{1}_{N+L} - UU^\dagger) d[U]$. We have

$$UU^\dagger = \begin{pmatrix} XX^\dagger + WW^\dagger & XV^\dagger + WY^\dagger \\ YX^\dagger + YW^\dagger & VV^\dagger + YY^\dagger \end{pmatrix} \tag{A.2}$$

and $d[U] = d[X]d[Y]d[W]d[V]$, to be inserted in

$$d\chi(X) = \int \delta\left(\mathbb{1}_{N+L} - UU^\dagger\right) d[U] . \tag{A.3}$$

This result is equivalent with lemma 7 in [20] proven by a QR-decomposition.

The integration over Y, W and V is performed in two steps. First we integrate out V and Y, by introducing an integral representation of the Dirac delta function for $(N+L) \times (N+L)$ Hermitian matrices

$$\delta(A) = \frac{1}{\pi^{N^2} 2^N} \int \exp\left[\imath \mathrm{tr}\, AH\right] d[H] = \frac{1}{\pi^{N^2} 2^N} \int \exp\left[\mathrm{tr}\, A(\imath H - \mathbb{1}_{N+L})\right] d[H] , \tag{A.4}$$

where $H = H^\dagger$ and $A = A^\dagger$. The shift introduced in the second expression is useful for ensuring the absolute convergence over the integrals in V and Y, see the calculation below. After subdividing H

[2]The more general result readily follows and is quoted at the end of this section.

in appropriate subblocks H_j, $j = 1, 2, 3$, where H_1, H_3 are Hermitian, we have

$$\begin{aligned} d\mu(X) &\propto d[X] \int \exp\left[\mathrm{tr}\left(UU^\dagger - \mathbb{1}_{N+L}\right)\left(\imath \begin{pmatrix} H_1 & H_2 \\ H_2^\dagger & H_3 \end{pmatrix} - \mathbb{1}_{N+L}\right)\right] d[H_1]d[H_2]d[H_3]d[V]d[W]d[Y] \\ &\propto d[X] \int \det{}^{-L-N}[\imath H_3 - \mathbb{1}_L] \exp\left[\mathrm{tr}\,(XX^\dagger + WW^\dagger - \mathbb{1}_N)(\imath H_1 - \mathbb{1}_N)\right] \\ &\quad \times \exp\left[\mathrm{tr}\, H_2^\dagger (XX^\dagger + WW^\dagger) H_2 (\imath H_3 - \mathbb{1}_L)^{-1} - \mathrm{tr}\,(\imath H_3 - \mathbb{1}_L)\right] \; d[H_1]d[H_2]d[H_3]d[W] \\ &\propto d[X] \int \delta(XX^\dagger + WW^\dagger - \mathbb{1}_N)\, \det{}^{-L-N}[\imath H_3 - \mathbb{1}_L] \\ &\quad \times \exp\left[\mathrm{tr}\, H_2^\dagger H_2 (\imath H_3 - \mathbb{1}_L)^{-1} - \mathrm{tr}\,(\imath H_3 - \mathbb{1}_L)\right] \; d[H_2]d[H_3]d[W] \\ &\propto d[X] \int \delta(XX^\dagger + WW^\dagger - \mathbb{1}_N)\; d[W]\;. \end{aligned} \tag{A.5}$$

In the first step we have integrated out V and Y which are Gaussian, leading to determinants with powers $-N$ and $-L$, respectively. In the second step we have rewritten the integral over H_1 as a Dirac delta function of the invariant $XX^\dagger + WW^\dagger = \mathbb{1}_N$. The Gaussian integral over H_2 results in an additional power of $+L$ of the same determinant and finally the integral over H_3 decouples and merely contributes to the global normalization constant. The only nontrivial contribution to the measure of X comes from the remaining constraint of the upper left corner in eq. (A.2).

The remaining integral can be computed by writing the Dirac delta function as an integral employing eq. (A.4) and finally integrating over W, i.e.

$$\begin{aligned} d\mu(X) &\propto d[X] \int \exp[\mathrm{tr}\,(XX^\dagger + WW^\dagger - \mathbb{1}_N)(\imath H_1 - \mathbb{1}_N)]\; d[H_1]d[W] \\ &\propto d[X] \int \det{}^{-L}[\imath H_1 - \mathbb{1}_N] \exp[\mathrm{tr}\,(XX^\dagger - \mathbb{1}_N)(\imath H_1 - \mathbb{1}_N)]\; d[H_1] \\ &= d[X] \int \det{}^{-L}[\imath H_1 - \mathbb{1}_N] \exp[\mathrm{tr}\,(X^\dagger X - \mathbb{1}_N)(\imath H_1 - \mathbb{1}_N)]\; d[H_1]. \end{aligned} \tag{A.6}$$

The last two integrals over H_1 are versions of the Ingham-Siegel integral as discussed in [46]. However we have to distinguish two cases. For the case $L \geq N$ we find

$$d\mu(X) \;\propto\; d[X] \det{}^{L-N}[\mathbb{1}_N - XX^\dagger]\Theta(\mathbb{1}_N - XX^\dagger) \tag{A.7}$$

yielding the result given at the beginning of the paper (2.3). Note that one can equivalently derive the result (A.7) from eq. (A.5) by the following change of variables $W \to \sqrt{\mathbb{1}_N - XX^\dagger}W$. Exactly in this kind of derivation the condition $L \geq N$ becomes immediate. The matrix $WW^\dagger$ has $N - L$ zero modes in the case $N > L$ such that the remaining Dirac delta function $\delta(WW^\dagger - \mathbb{1}_N)$ can never be satisfied. Thus for the case $N > L$ we have to remain with the integral representation (A.6).

For completeness we mention that repeating all the steps given above one obtains exactly the same result for a rectangular truncation, that is when the block X is an $N_1 \times N_2$ rectangular matrix. Denoting the complementary dimensions of truncated blocks by L_1 and L_2 fulfilling the condition $L_1 - N_2 = L_2 - N_1 \geq 0$ where $N_1 + L_1 = N_2 + L_2$ is the dimension of the original Haar distributed unitary matrix, the result reads

$$\begin{aligned} d\mu(X) &\propto d[X]\, \det{}^{L_1-N_2}(\mathbb{1}_{N_2} - X^\dagger X)\; \Theta(\mathbb{1}_{N_2} - X^\dagger X) \\ &\propto d[X]\, \det{}^{L_2-N_1}(\mathbb{1}_{N_1} - XX^\dagger)\; \Theta(\mathbb{1}_{N_1} - XX^\dagger). \end{aligned} \tag{A.8}$$

B Joint probability density of the eigenvalues

In this appendix we briefly recall how to integrate out the strictly upper triangular matrix T_i in the integral (2.9). We are looking for an explicit expression only depending on the elements of the diagonal matrix Z_i as given in eq. (2.10). Such an expression can be derived using a Schur decomposition (we pursue this idea, too) in [13, 20, 22], see also [14] for the case $L \geq N$. Here we repeat the derivation for the sake of self-consistency and completeness.

For simplicity we will drop the index i in this appendix. Let us only recall that T is strictly upper triangular, that is $T_{ab} = 0$ for $N \geq a \geq b \geq 1$ on the diagonal and below. Furthermore we introduce a matrix $S = Z + T$ which is (not-strictly) upper triangular and becomes quite convenient in the ensuing discussion. Our goal is to calculate the following integral

$$I_N = \int \det{}^{-L}[\imath H - \mathbb{1}_N] \exp[\mathrm{tr}\,(S^\dagger S - \mathbb{1}_N)(\imath H - \mathbb{1}_N)]\; d[H]d[T] \tag{B.1}$$

which is equivalent to eq. (2.9) up to a constant. We keep an explicit index N on the left hand side since we are going to do this integral recursively in the matrix size N. In the calculations below we will also use reduced matrices of size $N-1$. They will be denoted by $\mathbb{1}_{N-1}$, H', T', and S', respectively. They are obtained from $\mathbb{1}_N$, H, T, and S by removing the N-th column and the N-th row. In particular we split the matrices S and H in four blocks

$$S = \begin{pmatrix} S' & \vec{v} \\ \vec{0}^T & z_N \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} H' & \vec{w} \\ \vec{w}^\dagger & h_N \end{pmatrix} \tag{B.2}$$

built of $(N-1)\times(N-1)$ blocks S' and H' and 1×1 blocks consisting of single elements $z_N \in \mathbb{C}$ and $h_N \in \mathbb{R}$. Additionally the off-diagonal blocks correspond to vertical and horizontal vectors consisting of $N-1$ elements. The elements of the vertical vectors $\vec{v}$ and $\vec{w}$ are equal to $v_j = S_{jN}$ and $w_j = H_{jN}$, $j = 1, \ldots, N-1$, respectively. All elements of the horizontal vector $\vec{0}^T$ are equal to zero. Using these conventions we can reduce the matrix dimension from N to $N-1$ in the integral (B.1)

$$\begin{aligned} I_N \;=\; & \int \det{}^{-L}\left[\imath H' - \mathbb{1}_{N-1} + \frac{\vec{w}\vec{w}^\dagger}{\imath h_N - 1}\right](\imath h_N - 1)^{-L} \\ & \times \exp[\mathrm{tr}\,(S'^\dagger S' - \mathbb{1}_{N-1})(\imath H' - \mathbb{1}_{N-1}) + (\vec{v}^\dagger\vec{v} + |z_N|^2 - 1)(\imath h_N - 1)] \\ & \times \exp[\imath\vec{w}^\dagger S'^\dagger \vec{v} + \imath\vec{v}^\dagger S'\vec{w}] d[H']d[T']d[w]d[v]d\,h_N \end{aligned} \tag{B.3}$$

Here we applied the following identity

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D)\;\det\left(A - BD^{-1}C\right), \tag{B.4}$$

where in our case the first factor is the determinant of a 1×1 matrix. In the first step we integrate over $\vec{v}$ such that we have

$$\begin{aligned} I_N \;\propto\; & \int \det{}^{-L}\left[\imath H' - \mathbb{1}_{N-1} + \frac{\vec{w}\vec{w}^\dagger}{\imath h_N - 1}\right](\imath h_N - 1)^{-L-N+1} \exp[(|z_N|^2 - 1)(\imath h_N - 1)] \\ & \times \exp\left[\mathrm{tr}\,(S'^\dagger S' - \mathbb{1}_{N-1})(\imath H' - \mathbb{1}_{N-1}) - \frac{\vec{w}^\dagger S'^\dagger S'\vec{w}}{1 - \imath h_N}\right] d[H']d[T']d[w]d\,h_N. \end{aligned} \tag{B.5}$$

In the next step one can shift $H' \to H' - \imath\vec{w}\vec{w}^\dagger/(1-\imath h_N)$ which is legitimate since we do not cross any pole of the determinant. Then we end up with a Gaussian integral in $\vec{w}$ which can be done,

$$\begin{aligned} I_N \;\propto\; & \int \det{}^{-L}\left[\imath H' - \mathbb{1}_{N-1}\right](\imath h_N - 1)^{-L} \exp[(|z_N|^2 - 1)(\imath h_N - 1)] \\ & \times \exp\left[\mathrm{tr}\,(S'^\dagger S' - \mathbb{1}_{N-1})(\imath H' - \mathbb{1}_{N-1})\right] d[H']d[T']d\,h_N. \end{aligned} \tag{B.6}$$

The integral over h_N can be performed via the residue theorem and we find the recursion

$$I_N \propto w_1^{(L)}(z_N) I_{N-1} \tag{B.7}$$

with

$$w_1^{(L)}(z_N) \propto \left(1-|z_N|^2\right)^{L-1} \Theta(1-|z_N|^2) \ . \tag{B.8}$$

Note that the function on the right hand side is independent of N. The only part that depends on N is the coefficient which was skipped. Inserting this result into eq. (B.7) we eventually obtain

$$I_N \propto \prod_{n=1}^{N} w_1^{(L)}(z_n) \propto \prod_{n=1}^{N} \left(1-|z_n|^2\right)^{L-1} \Theta(1-|z_n|^2) \ , \tag{B.9}$$

in agreement with eq. (2.10) and [13, 14, 20, 22].

C The model with different truncations

Let us consider a neat generalization where the matrices X_j have still the same size $N \times N$ but they were truncated from unitary matrices of different size, $U_j \in \mathrm{U}(N+L_j)$, with $L_j > 0$ for all $j = 1, \ldots, M$. Then the measure of the product matrix X reads

$$\begin{aligned} d\nu(X) &= d[X] \int \det{}^{-L_j}[\imath H_1 - \mathbb{1}_N] \exp[\mathrm{tr}\,(X^\dagger X - \mathbb{1}_N)(\imath H_1 - \mathbb{1}_N)]\ d[H_1] \\ &= \prod_{j=1}^{M} \det{}^{L_j-N}(\mathbb{1}_N - X_j^\dagger X_j)\Theta(\mathbb{1}_N - X_j^\dagger X_j) d[X_j], \end{aligned} \tag{C.1}$$

where the second line only applies in the case $L_j \geq N$. The question is what the one-point weight function will look like. To reach this goal we go along the same lines as in the previous subsections.

Notice that the Vandermonde determinant in the joint probability density (2.14) remains unchanged since the generalized Schur decomposition still applies, such that we have

$$P^{(N,L_1,\ldots,L_M,M)}(Z) = \frac{1}{N!} \prod_{j=0}^{M} \prod_{l=0}^{N-1} \binom{L_j+l}{l}^{-1} \prod_{a<b}^{N} |z_a - z_b|^2 \prod_{n=1}^{N} w_M^{(L_1,\ldots,L_M)}(z_n) \ , \tag{C.2}$$

with the normalized one point weight function

$$w_M^{(L_1,\ldots,L_M)}(z_n) = \int_{\mathbb{C}^M} \delta^{(2)}\left(z_n - \prod_{j=1}^{M} z_{jn}\right) \prod_{i=1}^{M} \frac{L_j}{\pi} (1-|z_{in}|^2)^{L_j-1} \Theta(1-|z_{in}|^2) d^2 z_{in} \ . \tag{C.3}$$

It is obviously symmetric in the indices L_j. For $L_1 = \ldots = L_M = L$ we have $w_M^{(L,\ldots,L)} = w_M^{(L)}$ in comparison to the former sections.

Again the one-point weight satisfies a recurrence relation

$$w_M^{(L_1,\ldots,L_M)}(z) = 2\pi \int_0^1 w_1^{(L_M)}(r') w_{M-1}^{(L_1,\ldots,L_{M-1})}\left(\frac{z}{r'}\right) \frac{dr'}{r'} \ , \tag{C.4}$$

as well as with other permutations of the indices L_j on the right hand side which is very similar to the one in Eq. (2.23). The recurrence relation in terms of the Mellin transform

$$M_M^{(L_1,\ldots,L_M)}(s) = \int_0^1 \Omega_M^{(L_1,\ldots,L_M)}(s) x^{s-1} dx, \text{ with } \Omega_M^{(L_1,\ldots,L_M)}(|z|^2) = \frac{1}{\pi} w_M^{(L_1,\ldots,L_M)}(z) \tag{C.5}$$

reads

$$M_M^{(L_1,\ldots,L_M)}(s) = M_{M-1}^{(L_1,\ldots,L_{M-1})}(s) M_1^{(L_M)}(s) = \prod_{m=1}^{M} M_1^{(L_m)}(s) \ . \tag{C.6}$$

The inverse Mellin transform yields the one-point weight $w_M^{(L_1,\ldots,L_M)}$. Hereby we can benefit from the known result of the Mellin transform $M_1^{(L_j)}$, cf. eq. (2.30). Thus the one-point weight is

$$\begin{aligned} w_M^{(L_1,\ldots,L_M)}(z) &= \frac{1}{\pi} \int_{\mathcal{C}} \prod_{l=1}^{M} L_l! \, \frac{\Gamma(-u)}{\Gamma(L_l - u)} |z|^{2u} \frac{du}{2\pi\imath} \, \Theta(1-|z|) \\ &= \frac{1}{\pi} \prod_{l=1}^{M} L_l! \, G^{M,\,0}_{M,\,M} \left(\begin{matrix} L_1,\ldots,L_M \\ 0,\ldots,0 \end{matrix} \middle| \, |z|^2 \right) \, \Theta(1-|z|) \ . \end{aligned} \tag{C.7}$$

Indeed our former result (2.31) can be easily retained by setting $L_1 = \ldots = L_M = L$. Also the moments of this weight can be easily computed and we find

$$\int_{\mathbb{C}} w_M^{(L_1,\ldots,L_M)}(z) |z|^k d^2 z = \prod_{m=1}^{M} \binom{L_m + k/2}{k/2}^{-1}, \tag{C.8}$$

which is a generalization of Eq. (2.15). The corresponding kernel from which the k-point correlation functions are built up reads

$$K^{(N,L_1,\ldots,L_M,M)}(u,v) = \sqrt{w_M^{(L_1,\ldots,L_M)}(u) w_M^{(L_1,\ldots,L_M)}(v)} \sum_{j=0}^{N-1} \prod_{m=1}^{M} \binom{L_m + j}{j} (uv^*)^j, \tag{C.9}$$

cf. eq. (2.19). The joint probability density, weight function and kernel agrees with the results from [20, 22].

Note that most of the discussion of the asymptotics and the universality of the kernel (2.17) carries over or can be trivially generalized to the kernel (C.9). Only the large N-asymptotics in the strong non-unitarity regime where one has to zoom into the local scale of the bulk and of the soft edge is quite involved due to the non-trivial saddle point equations arising from the choice $L_1 \neq L_2 \neq \ldots \neq L_M$.

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Supersymmetry Method for Chiral Random Matrix Theory with Arbitrary Rotation Invariant Weights

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Abstract. In the past few years, the supersymmetry method was generalized to real-symmetric, Hermitean, and Hermitean self-dual random matrices drawn from ensembles invariant under the orthogonal, unitary, and unitary symplectic group, respectively. We extend this supersymmetry approach to chiral random matrix theory invariant under the three chiral unitary groups in a unifying way. Thereby we generalize a projection formula providing a direct link and, hence, a 'short cut' between the probability density in ordinary space and the one in superspace. We emphasize that this point was one of the main problems and critiques of the supersymmetry method since only implicit dualities between ordinary and superspace were known before. As examples we apply this approach to the calculation of the supersymmetric analogue of a Lorentzian (Cauchy) ensemble and an ensemble with a quartic potential. Moreover we consider the partially quenched partition function of the three chiral Gaussian ensembles corresponding to four-dimensional continuum QCD. We identify a natural splitting of the chiral Lagrangian in its lowest order into a part of the physical mesons and a part associated to source terms generating the observables, e.g. the level density of the Dirac operator.

Random Matrix Theory, Supersymmetry, Multivariate Statistics, Correlated Wishart Matrices, Universality, chiral Lagrangian, Multicritical Ensembles, generalized Hubbard-Stratonovich transformation, superbosonization formula

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1. Introduction

Chiral random matrix theory is the oldest of all random matrix ensembles. It was introduced by Wishart [1] in the 1920's to model generic properties of correlation matrices. Since then chiral random matrix theory was applied to many other fields of physics and beyond because of its versatility. One important application is the study of correlation matrices in time series analysis [2, 3, 4, 10, 11, 12, 13, 14]. Chiral random matrix theory serves as a benchmark model for empirical correlation matrices and is used to extract the system specific correlations from the generic statistical fluctuations. Another famous development is the introduction of chiral random matrix theory to QCD by Shuryak and Verbaarschot [15, 16, 17]. They showed the equivalence of the microscopic limit of the QCD-Dirac operator with chiral random matrix theory. In particular chiral random matrix theory explained the statistical fluctuations of the smallest eigenvalues of the Dirac operator and predicted relations between low energy constants and observables which are confirmed by lattice QCD data [18, 19]. Recent applications of chiral random matrix theory can be also found in condensed matter theory [20], telecommunication [21, 22, 23], and quantum information theory [24] but its range is by far not restricted to those examples.

For the sake of simplicity, a Gaussian function is often used within the context of random matrix theory. Due to universality [25, 26, 27, 28], this choice is quite often legitimized as long as the interest lies in correlations on the local scale of the mean level spacing. To prove universality as well as to modify random matrix theory to describe particular systems many technical tool were developed. For example, the supersymmetry method, originally introduced for Gaussian weights [29, 30, 31, 32], is established as a versatile tool in the field of random matrix theory because of its broad applicability to non-Gaussian ensembles. For the history of the supersymmetry method and its variants, we refer the reader to Ref. [32]. Moreover, one is not always interested in the local scale, e.g. see the analysis of universality on macroscopic scales as it is discussed with free probability [33, 34]. Insofar a generalization to arbitrary statistical weights is of particular interest. Other important techniques are the orthogonal polynomial method [6], Toda lattice structures [7], free probability theory [35] and maps to Hamiltonian systems [8, 9]. For a comprehensive overview see [5, 64, 51] and references therein.

Here, we focus on the supersymmetry method, not on aspects related to other methods such as orthogonal polynomials. We start from a close connection between matrix invariants in ordinary and superspace which was first observed in Ref. [36]. In particular for chiral random matrix models we investigate how probability densities which only depend on matrix invariants (but are otherwise arbitrary) are uniquely mapped from ordinary to superspace. This is the issue at stake.

An exact map from ordinary space to superspace for arbitrary isotropic ensembles for real symmetric, Hermitean and Hermitean self-dual matrices was provided in two different but related approaches, a few years ago. Isotropy is the invariance under the orthogonal, unitary or unitary symplectic group, respectively, see Ref. [35]. One approach pursues the idea to generalize the original Hubbard-Stratonovich transformation in superspace for Gaussian weights [29, 30, 31, 32] to arbitrary weights [36, 37]. In another approach one tries to find a direct, exact identity between integrals over dyadic supermatrices and integrals over cosets. This second approach is known as the superbosonization formula [38, 39]. Both approaches are completely equivalent [40] and both have their advantages as well as disadvantages. One crucial

disadvantage they both share is that they do not directly relate the probability density in ordinary space with the one in superspace. They only become explicit when the characteristic function (Fourier transform of the probability density) is known in a closed form. Hence one has to calculate the statistical weight for each random matrix ensemble, separately. This is exactly the problem we want to address.

The extension of the generalized Hubbard-Stratonovich transformation as well as the superbosonization formula to the other seven classes in the tenfold classification via the Cartan scheme [41, 42] is still unsolved. We address three of these seven classes in a unifying way, namely chiral random matrices generated by non-Gaussian probability densities. In particular we derive a projection formula explicitly relating the probability density in ordinary space with the one in superspace. Thus we present a solution to the disadvantage of the generalized Hubbard-Stratonovich transformation and the superbosonization formula where one has to study each ensemble separately. Such a projection formula was already accomplished for real symmetric, Hermitean, and Hermitean self-dual matrices, see Ref. [43]. In Sec. 2, we briefly summarize the idea behind such a projection formula for ensembles in the original classification by Dyson [41] and put it into contrast with the well established generalized Hubbard-Stratonovich transformation and the superbosonization formula. In Sec. 3, we generalize this approach to the three chiral random matrix theories of real, complex and quaternion rectangular matrices in a unifying way.

To underline that the projection formula is a powerful tool we apply it to a selection of ensembles encountered in different fields of random matrix theory, in Sec. 4. Some of these ensembles, as the Lorentz (Cauchy)-like ensembles and the ensemble with a quartic potential, are not at all trivial and it is not immediately clear what their supersymmetric counterpart will look like. The other examples are the norm-dependent ensembles without and with empirical correlations and the unquenched chiral Gaussian random matrix ensembles modelling QCD with quarks. In particular for the partially quenched partition function we derive a representation whose microscopic limit agrees with QCD and shows a natural splitting into physical mesons and those corresponding to the source term generating the observables like the level density or higher order correlations. The explicit calculation of this result is presented in Appendix A. The article is concluded with a summary in Sec. 5.

2. Main idea of a projection formula

The supersymmetry method is essentially a general relation between partition functions in ordinary space,

$$Z(\kappa) = \int d[H]P(H)\frac{\prod_{j=1}^{k_2}\det(H-\kappa_j^{(2)}\mathbb{1}_N)}{\prod_{j=1}^{k_1}\det(H-\kappa_j^{(1)}\mathbb{1}_N)}, \tag{2.1}$$

and partition functions in superspace, which we expect to be of the form

$$Z(\kappa) = \int d[\sigma]Q(\sigma)\mathrm{sdet}^{\,\mu(N)}(\sigma-\kappa). \tag{2.2}$$

The $N\times N$ matrix H is distributed by P and drawn from one of the Hermitean ensembles classified in the ten-fold way via the Cartan classification scheme [41, 42]. The exponent μ is some affine linear function in the former ordinary dimension N.

The supermatrix σ has a dimension related to the number of determinants in Eq. (2.1). It fulfills certain symmetries depending on the ones of the ordinary matrix H, and is drawn from a probability density Q in superspace. The source variables

$$\kappa := \begin{cases} \mathrm{diag}(\kappa_1, \kappa_2), & \beta = 2, \\ \mathrm{diag}(\kappa_1, \kappa_1, \kappa_2, \kappa_2), & \beta = 1, 4, \end{cases} \tag{2.3}$$

with

$$\kappa_1 := \mathrm{diag}(\kappa_1^{(1)}, \ldots, \kappa_{k_1}^{(1)}), \qquad \kappa_2 := \mathrm{diag}(\kappa_1^{(2)}, \ldots, \kappa_{k_2}^{(2)}), \tag{2.4}$$

are distinguished by the Dyson index $\beta = 1, 2, 4$. We notice that κ is always a supermatrix. In the context of QCD, it comprises masses of the physical fermions as well as masses of the valence fermions usually denoted by m_j [17]. The masses of the valence fermions consist of source variables for differentiation to generate the matrix Green functions often denoted by J_j and markers for the eigenvalues of H which are usually denoted by x_j [32]. Additionally we have to assume that $\kappa_j^{(1)}$ has a non-zero imaginary part, since the spectrum of H lies on the real axis.

The main task is to derive two things. First of all, the corresponding supermatrix space, $\sigma \in \mathcal{M}_{\mathrm{SUSY}}$, has to be identified which is independent of the probability density P. This identification was already done in Ref. [42]. Second, one has to calculate the probability distribution Q which crucially depends on the ordinary matrix space, $H \in \mathcal{M}_{\mathrm{ord}}$, and on the probability density P. Exactly the second task is the hardest one and is up to now only known in a closed form when H is real symmetric, Hermitean, or Hermitean self-dual [43].

After recalling the standard supersymmetry method in subsection 2.1, we briefly rederive a projection formula for ensembles of real symmetric, Hermitean, and Hermitean self-dual matrices in subsection 2.2 to point out the main idea of such a projection formula.

2.1. Standard supersymmetry approach

Let us introduce three abbreviations,

$$\mathrm{U}^{(\beta)}(n) := \begin{cases} \mathrm{O}(n), & \beta = 1, \\ \mathrm{U}(n), & \beta = 2, \\ \mathrm{USp}(2n), & \beta = 4, \end{cases} \tag{2.5}$$

$$\mathrm{Herm}^{(\beta)}(n) := \begin{cases} \mathrm{Gl}(n, \mathbb{R})/\mathrm{O}(n) \cong \mathrm{U}(n)/\mathrm{O}(n), & \beta = 1, \\ \mathrm{Gl}(n, \mathbb{C})/\mathrm{U}(n), & \beta = 2, \\ \mathrm{Gl}(n, \mathbb{H})/\mathrm{USp}(2n) \cong \mathrm{U}(2n)/\mathrm{USp}(2n), & \beta = 4, \end{cases} \tag{2.6}$$

and

$$\gamma := \begin{cases} 1, & \beta = 1, 2, \\ 2, & \beta = 4, \end{cases} \quad \text{and } \widetilde{\gamma} := \begin{cases} 2, & \beta = 1, \\ 1, & \beta = 2, 4, \end{cases} \tag{2.7}$$

such that we can deal with all three Dyson indices $\beta = 1, 2, 4$ in a unifying way. Equation (2.6) is an abbreviation for the set of real symmetric, Hermitean, and Hermitean self-dual matrices, respectively. Here, $\mathbb{H}$ is the quaternion number field which we represent via the Pauli matrices and the two-dimensional unit matrix $\mathbb{1}_2$ throughout the work.

The aim is to identify a partition function in superspace starting from a partition function in ordinary space,

$$Z(\kappa) := \int d[H]P(H)\frac{\prod_{j=1}^{k_2}\det(H-\kappa_j^{(2)}\mathbb{1}_{\gamma n})}{\prod_{j=1}^{k_1}\det(H-\kappa_j^{(1)}\mathbb{1}_{\gamma n})} \tag{2.8}$$
$$= \int d[H]P(H)\mathrm{sdet}^{-1/(\gamma\widetilde{\gamma})}(H\otimes\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}-\mathbb{1}_{\gamma n}\otimes\kappa),$$

with $H\in\mathrm{Herm}^{(\beta)}(n)$ and P fulfilling the rotation invariance (also known as isotropy [35])

$$P(H)=P(UHU^{-1}),\quad \forall\, U\in\mathrm{U}^{(\beta)}(n). \tag{2.9}$$

Let for simplicity $\mathrm{Im}\,\kappa_1>0$ in this subsection. We will weaken this condition later on.

In the original supersymmetry method one introduces a rectangular complex supermatrix V [32] of dimension $(\gamma n)\times(\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2)$ and uses the crucial identity

$$\mathrm{sdet}^{-1/(\gamma\widetilde{\gamma})}(H\otimes\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}-\mathbb{1}_{\gamma n}\otimes\kappa)=\frac{\int d[V]\exp[\imath\mathrm{str}\,V^\dagger V\kappa-\imath\mathrm{str}\,V^\dagger HV]}{\imath^{\gamma n(k_2-k_1)}\int d[V]\exp[-\mathrm{str}\,V^\dagger V]}. \tag{2.10}$$

Recall the definition of κ in Eq. (2.3) and of γ and $\widetilde{\gamma}$ in Eq. (2.7). The rescaling by the imaginary unit $\imath$ is needed to ensure the convergence of the integral over V. The supermatrix V consists of independent complex random variables as well as complex Grassmann variables (anti-commuting variables) and fulfills some symmetries under complex conjugation if the Dyson index is $\beta=1,4$, i.e. the complex conjugate of V is

$$V^*=\begin{cases} V\,\mathrm{diag}\,(\mathbb{1}_{2k_1},\imath\tau_2\otimes\mathbb{1}_{k_2}), & \beta=1,\\ (-\imath\tau_2\otimes\mathbb{1}_n)V\,\mathrm{diag}\,(\imath\tau_2\otimes\mathbb{1}_{k_1},\mathbb{1}_{2k_2}), & \beta=4,\end{cases} \tag{2.11}$$

where τ_2 is the second Pauli matrix. The case $\beta=1$ is some kind of reality condition and for $\beta=4$ it is some kind of generalization of quaternions.

When plugging Eq. (2.10) into the partition function (2.8) the integration over H reduces to a Fourier transform of the probability density P. We assume that the Fourier transform,

$$\Phi(A):=\int d[H]P(H)\exp[-\imath\,\mathrm{tr}\,HA], \tag{2.12}$$

exists for any $(\gamma n)\times(\gamma n)$ matrix A sharing the same symmetries as H apart from relations involving complex conjugations. The invariance property (2.9) of P carries over to one of Φ, i.e.

$$\Phi(A)=\Phi(UAU^{-1})\quad\forall\, U\in\mathrm{U}^{(\beta)}(n) \tag{2.13}$$

meaning that the function Φ can be written as a function of the traces of A. Identifying the matrix $A=VV^\dagger$, one can show that there is a superfunction $\widetilde{\Phi}$, which is by far not unique (see Ref. [37]), such that another essential identity of the supersymmetry method holds [32],

$$\Phi(VV^\dagger)=\widetilde{\Phi}(V^\dagger V). \tag{2.14}$$

Note that the tilde emphasizes that $\widetilde{\Phi}$ is not the same as but related to the function Φ. The partition function reads

$$Z(\kappa)=\frac{\int d[V]\exp[\imath\mathrm{str}\,V^\dagger V\kappa]\widetilde{\Phi}(V^\dagger V)}{\imath^{\gamma n(k_2-k_1)}\int d[V]\exp[-\mathrm{str}\,V^\dagger V]} \tag{2.15}$$

which is already a representation in superspace.

Two different ways can be pursued from this point. One approach is the superbosonization formula [38, 39]. With help of the superbosonization formula the integral over $V^\dagger V$ is replaced by an integral over a $(\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2) \times (\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2)$ supermatrix U fulfilling some symmetries under the transposition if $\beta = 1, 4$, i.e.

$$U^T = \begin{cases} \mathrm{diag}\,(\mathbb{1}_{2k_1}, -\imath\tau_2 \otimes \mathbb{1}_{k_2}) U \mathrm{diag}\,(\mathbb{1}_{2k_1}, \imath\tau_2 \otimes \mathbb{1}_{k_2}), & \beta = 1, \\ \mathrm{diag}\,(-\imath\tau_2 \otimes \mathbb{1}_{k_1}, \mathbb{1}_{2k_2}) U \mathrm{diag}\,(\imath\tau_2 \otimes \mathbb{1}_{k_1}, \mathbb{1}_{2k_2}), & \beta = 4, \end{cases} \tag{2.16}$$

which means that U_{BB} is symmetric (self-dual) and U_{FF} is self-dual (symmetric) for $\beta = 1$ ($\beta = 4$). Additionally, the matrix U consists of four blocks,

$$U = \begin{bmatrix} U_{\mathrm{BB}} & \eta^\dagger \\ \eta & U_{\mathrm{FF}} \end{bmatrix}, \tag{2.17}$$

whose off-diagonal blocks η and $\eta^\dagger$ contain independent Grassmann variables apart from the condition (2.16), the boson-boson block is positive definite, $U_{\mathrm{BB}} > 0$, and Hermitean, $U_{\mathrm{BB}}^\dagger = U_{\mathrm{BB}}$, and the fermion-fermion block is unitary, $U_{\mathrm{FF}}^\dagger = U_{\mathrm{FF}}^{-1}$. Hence the supermatrix U is in one of the three cosets [38, 39, 40]

$$\mathrm{Herm}_{\odot}^{(\beta)}(\widetilde{\gamma}k_1|\gamma k_2) := \begin{cases} \mathrm{U}\,(2k_1|2k_2)/\mathrm{UOSp}^{\,(+)}(2k_1|2k_2), & \beta = 1, \\ \mathrm{Gl}\,(k_1|k_2)/\mathrm{U}\,(k_1|k_2), & \beta = 2, \\ \mathrm{U}\,(2k_1|2k_2)/\mathrm{UOSp}^{\,(-)}(2k_1|2k_2), & \beta = 4, \end{cases} \tag{2.18}$$

where $\mathrm{U}\,(p|q)$ is the unitary supergroup and $\mathrm{Gl}\,(p|q)$ is the general linear, complex supergroup. The two supergroups $\mathrm{UOSp}^{\,(\pm)}(2k_1|2k_2)$ for $\beta = 1, 4$ are the two independent matrix-representations of the unitary ortho-symplectic supergroup $\mathrm{UOSp}\,(2k_1|2k_2)$. Matrices in this group are real in the boson-boson block and quaternion in the fermion-fermion block for $\beta = 1$ denoted by the superscript "(+)" and vice versa for $\beta = 4$ denoted by the superscript "(−)", see Ref. [37]. The subscript "$\odot$" refers to the kind of embedding of the coset which is a contour-integral around the origin for the fermion-fermion block U_{FF} in the case of the superbosonization formula.

The superbosonization formula can be summarized to the following simple equation,

$$Z(\kappa) = \frac{\int d\mu(U) \mathrm{sdet}^{\,n/\widetilde{\gamma}} U \exp[\imath \mathrm{str}\, U\kappa] \widetilde{\Phi}(U)}{\imath^{\gamma n(k_2-k_1)} \int d\mu(U) \mathrm{sdet}^{\,n/\widetilde{\gamma}} U \exp[-\mathrm{str}\, U]}, \tag{2.19}$$

see Refs. [38, 39]. The measure $d\mu(U)$ is the Haar measure of the corresponding coset.

The second supersymmetric approach is the generalized Hubbard-Stratonovich transformation [36, 37]. Instead of replacing $V^\dagger V$ by a supermatrix one assumes that the superfunction $\widetilde{\Phi}$ is a Fourier transform of another superfunction Q as well, i.e.

$$\widetilde{\Phi}(B) = \int d[\sigma] Q(B) \exp[-\imath \mathrm{str}\, \sigma B], \tag{2.20}$$

for some supermatrix B. The integration domain of σ is very important. First of all it fulfills the same symmetries under transposition as U in the superbosonization formula, see Eq. (2.16), i.e.

$$\sigma^T = \begin{cases} \mathrm{diag}\,(\mathbb{1}_{2k_1}, -\imath\tau_2 \otimes \mathbb{1}_{k_2}) \sigma \mathrm{diag}\,(\mathbb{1}_{2k_1}, \imath\tau_2 \otimes \mathbb{1}_{k_2}), & \beta = 1, \\ \mathrm{diag}\,(-\imath\tau_2 \otimes \mathbb{1}_{k_1}, \mathbb{1}_{2k_2}) \sigma \mathrm{diag}\,(\imath\tau_2 \otimes \mathbb{1}_{k_1}, \mathbb{1}_{2k_2}), & \beta = 4, \end{cases} \tag{2.21}$$

which is again equivalent that σ_{BB} is symmetric (self-dual) and σ_{FF} is self-dual (symmetric) for $\beta = 1$ ($\beta = 4$). However the blocks of

$$\sigma = \begin{bmatrix} \sigma_{\mathrm{BB}} & \eta^\dagger \\ \eta & \sigma_{\mathrm{FF}} \end{bmatrix} \tag{2.22}$$

are drawn from different supports as for U. The off-diagonal blocks η and $\eta^\dagger$ are again independent Grassmann variables apart from the condition (2.21) while the boson-boson block is now only Hermitean, $\sigma_{\mathrm{BB}}^\dagger = \sigma_{\mathrm{BB}}$. The fermion-fermion block can be diagonalized by $\widehat{U} \in \mathrm{U}^{\,(4/\beta)}(\gamma k_2)$, i.e. $\sigma_{\mathrm{FF}} = \widehat{U} s_{\mathrm{FF}} \widehat{U}^\dagger$. The eigenvalues $\{s_{\mathrm{FF}}\}_j$ live on contours such that the integral over them converges. For a Gaussian ensemble the standard Wick-rotation, i.e. $\{s_{\mathrm{FF}}\}_j \in \imath\mathbb{R}$, does the job. For other polynomial potentials one has to choose other Wick-rotations, e.g. for $P(H) \propto \exp[-\operatorname{tr} H^{2m}]$ it is $\{s_{\mathrm{FF}}\}_j \in e^{\imath\pi/(2m)}\mathbb{R}$. Therefore the supermatrix σ lies also in an embedding of the cosets (2.18) but the set will be now denoted by $\mathrm{Herm}^{\,(\beta)}_{\mathrm{Wick}}(\widetilde{\gamma}k_1|\gamma k_2)$ where the subscript "Wick" reflects the nature of the integration domain.

Reading off $B = V^\dagger V$ and integrating over V one obtains the final result for the generalized Hubbard-Stratonovich transformation,

$$Z(\kappa) = \int d[\sigma] Q(\sigma) \mathrm{sdet}^{\,-n/\widetilde{\gamma}}(\sigma - \kappa), \tag{2.23}$$

see Refs. [36, 37]. The measure $d[\sigma]$ is the flat one, i.e. the product of the differential of all independent matrix elements.

Both approaches, the superbosonization formula as well as the generalized Hubbard-Stratonovich transformation, have a crucial weakness. Without an explicit knowledge of the Fourier transform Φ no direct functional relation between the probability density P, the superfunction $\widetilde{\Phi}$, and the superfunction Q is known. The reason is the duality relation (2.13) between ordinary and superspace. Particularly for the generalized Hubbard-Stratonovich transformation, the dyadic matrices $VV^\dagger$ and $V^\dagger V$ are in different matrix spaces. Hence, one cannot expect that the Fourier transforms (2.12) and (2.20) yield the same functional dependence of P and Q. The projection formula [43] briefly rederived in subsection 2.2 circumvents this problem.

2.2. Projection formula for Dyson's threefold way

The key idea to find a direct relation between P and Q is to extend the original matrix set $H \in \mathrm{Herm}^{(\beta)}(n)$ to a larger matrix set also comprising the target set $\sigma \in \mathrm{Herm}^{\,(\beta)}_{\mathrm{Wick}}(\widetilde{\gamma}k_1|\gamma k_2)$. Let $\widetilde{\gamma}k_1 \leq \widetilde{\gamma}k_2 + n$ to keep the calculation as simple as possible otherwise we have to do a case discussion. This condition is usually the case when applying supersymmetry to random matrix theory. Nevertheless we underline that this condition is not at all a restriction since the other case can be taken care by slightly modifying the ensuing discussion, see Ref. [43].

The idea of our approach is based on a Cauchy-like integration formula for supermatrices in the coset $\mathrm{Herm}^{\,(\beta)}_{\mathrm{Wick}}(p|q)$ with $p, q \in \mathbb{N}_0$ which was first derived by Wegner [44], see also Refs. [45, 46, 47] for slightly modified versions. Let $l \in \mathbb{N}$ be a positive integer and f be an integrable and smooth superfunction on the set of supermatrices $\mathrm{Herm}^{\,(\beta)}_{\mathrm{Wick}}(p + \widetilde{\gamma}l|q + \gamma l)$ and invariant under

$$f(\widetilde{U}\Sigma\widetilde{U}^{-1}) = f(\Sigma) \tag{2.24}$$

for all $\Sigma \in \mathrm{Herm}^{\,(\beta)}_{\mathrm{Wick}}(p + \widetilde{\gamma}l|q + \gamma l)$ and

$$\widetilde{U} \in \mathrm{U}^{\,(\beta)}(p + \widetilde{\gamma}l|q + \gamma l) := \begin{cases} \mathrm{UOSp}^{\,(+)}(p + 2l|2q + 2l), & \beta = 1, \\ \mathrm{U}\,(p + l|q + l), & \beta = 2, \\ \mathrm{UOSp}^{\,(-)}(2p + 2l|q + 2l), & \beta = 4. \end{cases} \tag{2.25}$$

Employing the following splitting of

$$\Sigma = \begin{bmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} + \widehat{\Sigma} \quad \text{with} \quad \widehat{\Sigma} = \begin{bmatrix} 0 & \widehat{V} \\ \widehat{V}^\dagger & \sigma \end{bmatrix} \tag{2.26}$$

such that $\widetilde{\Sigma} \in \mathrm{Herm}^{(\beta)}_{\mathrm{Wick}}(p|q)$ and $\sigma \in \mathrm{Herm}^{(\beta)}_{\mathrm{Wick}}(\widetilde{\gamma}l|\gamma l)$, the Cauchy-like integral identity [44, 45, 46, 47] reads

$$\frac{\int d[\widehat{\Sigma}] f(\Sigma)}{\int d[\widehat{\Sigma}] \exp[-\mathrm{str}\, \widehat{\Sigma}^2]} = f\left(\begin{bmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \right) \tag{2.27}$$

reducing a large supermatrix, Σ, to a smaller one, $\widetilde{\Sigma}$, independent of the concrete form of the superfunction f. The notation Σ, $\widetilde{\Sigma}$ and $\widehat{\Sigma}$ has no deeper meaning. It only underlines that all three matrices are essentially of the same form apart from their different dimensions.

Equation (2.27) is at the heart of our approach. Let us consider the partition function (2.8) in ordinary space. From now on, we lift the condition $\mathrm{Im}\,\kappa_1 > 0$ to emphasize that our idea works in general and define $L = \mathrm{sign}\,\mathrm{Im}\,\kappa$. We assume that the probability density P is rotation invariant, see Eq. (2.9). Moreover we assume that a contour like the Wick-rotation and an extension of P, denoted by $\widetilde{P}$, from the ordinary matrix set $\mathrm{Herm}^{(\beta)}(n)$ to the supermatrix set $\mathrm{Herm}^{(\beta)}_{\mathrm{Wick}}(n+\widetilde{\gamma}k_2|\gamma k_2)$ exists such that the superfunction $\widetilde{P}$ is integrable and smooth on $\mathrm{Herm}^{(\beta)}_{\mathrm{Wick}}(n+\widetilde{\gamma}k_2|\gamma k_2)$. Then we can extend the integral (2.8) to an integral in superspace, i.e.

$$Z(\kappa) = \frac{\int d[\Sigma] \widetilde{P}(\Sigma) \mathrm{sdet}^{\,-1/(\gamma\widetilde{\gamma})}(H \otimes \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \mathbb{1}_{\gamma n} \otimes \kappa)}{\int d[\widehat{\Sigma}] \exp[-\mathrm{str}\, \widehat{\Sigma}^2]}, \tag{2.28}$$

where we employ a splitting similar to Eq. (2.26), i.e.

$$\Sigma = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} + \widehat{\Sigma} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix} + \Sigma' \text{ with } \widehat{\Sigma} = \begin{bmatrix} 0 & \widehat{V} \\ \widehat{V}^\dagger & \widehat{\sigma} \end{bmatrix} \text{ and } \Sigma' = \begin{bmatrix} H' & V' \\ V'^\dagger & 0 \end{bmatrix} \tag{2.29}$$

with $H \in \mathrm{Herm}^{(\beta)}_{\mathrm{Wick}}(n|0) = \mathrm{Herm}^{(\beta)}(n)$, $H' \in \mathrm{Herm}^{(\beta)}_{\mathrm{Wick}}(n+\widetilde{\gamma}(k_2-k_1)|0) = \mathrm{Herm}^{(\beta)}(n+\widetilde{\gamma}(k_2-k_1))$, $\widehat{\sigma} \in \mathrm{Herm}^{(\beta)}_{\mathrm{Wick}}(\widetilde{\gamma}k_2|\gamma k_2)$, and $\sigma \in \mathrm{Herm}^{(\beta)}_{\mathrm{Wick}}(\widetilde{\gamma}k_1|\gamma k_2)$. The second splitting becomes more important later on. Notice that we extended $H \to \Sigma$ in the probability density $\widetilde{P}$, only.

From now on we pursue the ideas of the standard supersymmetry method, see subsection 2.1. We introduce the same rectangular supermatrix V as in Eq. (2.10), i.e.

$$\mathrm{sdet}^{\,-1/(\gamma\widetilde{\gamma})}(H \otimes \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \mathbb{1}_{\gamma n} \otimes \kappa) = \frac{\int d[V] \exp[\imath \mathrm{str}\, V^\dagger V L \kappa - \imath \mathrm{str}\, V^\dagger H V L]}{\imath^{\gamma n(k_2-k_1)} \mathrm{sdet}^{\,-n/\widetilde{\gamma}} L \int d[V] \exp[-\mathrm{str}\, V^\dagger V]}. \tag{2.30}$$

In terms of Σ the partition function reads

$$Z(\kappa) = \frac{\int d[\Sigma] \widetilde{P}(\Sigma) \int d[V] \exp[\imath \mathrm{str}\, V^\dagger V L \kappa - \imath \mathrm{str}\, \Sigma \widehat{A}]}{\imath^{\gamma n(k_2-k_1)} \mathrm{sdet}^{\,-n/\widetilde{\gamma}} L \int d[V] \exp[-\mathrm{str}\, V^\dagger V] \int d[\widehat{\Sigma}] \exp[-\mathrm{str}\, \widehat{\Sigma}^2]} \tag{2.31}$$

with

$$\widehat{A} = \begin{bmatrix} VLV^\dagger & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & V\sqrt{L} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{L}V^\dagger & 0 \end{bmatrix} \tag{2.32}$$

and $\sqrt{L}$ the positive root of the diagonal elements of L. The block structure of $\widehat{A}$ corresponds to the first splitting of Σ in Eq. (2.29). The Fourier-Laplace transform

$$\widehat{\Phi}(\widehat{A}) = \int d[\Sigma]\widetilde{P}(\Sigma)\exp[-\imath\mathrm{str}\,\Sigma\widehat{A}] \tag{2.33}$$

is assumed to exist such that we can interchange the integrals over Σ and V. Employing the same symmetry arguments as in Eq. (2.13) we have

$$\widehat{\Phi}(\widehat{A}) = \widehat{\Phi}(\widehat{B}) \quad \text{with} \quad \widehat{B} = \begin{bmatrix} 0 & 0 \\ \sqrt{L}V^\dagger & 0 \end{bmatrix}\begin{bmatrix} 0 & V\sqrt{L} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{L}V^\dagger V\sqrt{L} \end{bmatrix}. \tag{2.34}$$

The block structure of $\widehat{B}$ is the one of the second splitting of Σ in Eq. (2.29). The advantage of Eq. (2.34) in contrast to Eq. (2.14) is that the superfunction $\widehat{\Phi}$ is still the same since $\widehat{A}$ and $\widehat{B}$ are in the same supermatrix set. Hence the inverse Fourier transform is still $\widetilde{P}$ and *not* some new superfunction.

The only technical difficulty grows from a non-trivial L because we cannot simply exchange the integrations over Σ and V again. To overcome this problem we introduce an auxiliary supermatrix $\sigma_{\mathrm{aux}} \in \mathrm{Herm}_\imath^{(\widetilde{\beta})}(\widetilde{\gamma}k_1|\gamma k_2)$ drawn from a Gaussian distribution where the subscript "$\imath$" denotes the standard Wick-rotation [29, 30, 31, 32] by the imaginary unit. This Gaussian models some kind of Dirac δ-function, i.e. we can "simplify"

$$\exp[-\imath\mathrm{str}\,\sqrt{L}\sigma\sqrt{L}V^\dagger V] = \lim_{t\to 0}\frac{\int d[\sigma_{\mathrm{aux}}]\exp[-\mathrm{str}\,(\sigma_{\mathrm{aux}}-\sqrt{L}\sigma\sqrt{L})^2/t - \imath\mathrm{str}\,\sigma_{\mathrm{aux}}V^\dagger V]}{\int d[\sigma_{\mathrm{aux}}]\exp[-\mathrm{str}\,\sigma_{\mathrm{aux}}^2/t]}, \tag{2.35}$$

where $t/2$ is the variance of the Gaussian distribution. Assuming that the integral of $\widetilde{P}$ multiplied with $\exp[|\mathrm{str}\,\sigma^2|]$ exists, we are allowed to interchange the integrations over Σ, V, and σ_{aux}. We underline that the integrability of $\widetilde{P}$ with $\exp[|\mathrm{str}\,\sigma^2|]$ is a weak restriction which can be lifted at the end of the day; for example a modification of $P(H)$ to $P(H)\exp[-\delta\,\mathrm{tr}\,H^4]$ ($\delta > 0$) does the job and we can take $\delta \to 0$ in the end.

After introducing σ_{aux} we interchange the integrals and integrate over V first. Shifting σ_{aux} by $\sqrt{L}\sigma\sqrt{L}$ we can take the limit $t \to 0$. Finally the partition function takes the simple form

$$Z(\kappa) = \frac{\int d[\Sigma]\widetilde{P}(\Sigma)\mathrm{sdet}^{\,-n/\widetilde{\gamma}}(\sigma-\kappa)}{\int d[\widehat{\Sigma}]\exp[-\mathrm{str}\,\widehat{\Sigma}^2]}. \tag{2.36}$$

Notice that the superdeterminant only depends on σ and not anymore on the ordinary matrix H.

In the last step we identify the superfunction Q by comparing the result (2.36) with the result of the generalized Hubbard-Stratonovich transformation (2.23) yielding the final result of this section which is the projection formula

$$Q(\sigma) = \frac{\int d[\Sigma']\widetilde{P}\left(\begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix} + \Sigma'\right)}{\int d[\widehat{\Sigma}]\exp[-\mathrm{str}\,\widehat{\Sigma}^2]}. \tag{2.37}$$

We integrate over different splittings of Σ in the numerator and the denominator. Recall the definition (2.29) of the matrices $\widehat{\Sigma}$ and Σ'. The superfunction $\widetilde{\Phi}$ in the superbosonization formula (2.19) can be obtained by the Fourier transformation (2.20) of Q.

We underline that the projection formula also holds if the source κ is chosen non-diagonal as it sometime happens in QCD [48] or if we add an external operator H_0 to the original random matrix H often consider in transition ensembles [49, 50]. In both cases the integral (2.36) is slightly modified but the fundamental functional relation (2.37) still remains the same.

The projection formula (2.37) has one big advantage which the results of the superbosonization formula (2.19) and of the generalized Hubbard-Stratonovich transformation (2.23) are lacking. With the aid of the projection formula one can study deformations of the probability weight in a quite elegant way. Exactly such an advantage we want to achieve for the chiral ensembles, too.

Finally, we emphasize that the projection formula (2.37), after extending P to $\widetilde{P}$, yields one of infinitely many probability weights in superspace corresponding to the same partition function in ordinary space (2.8). This ambiguity of the weight in superspace is well known [37]. Moreover other extensions of P to superspace certainly result into other superfunctions Q. Thus an interesting mathematical question is: When varying over all possible extensions $\widetilde{P}$ of P, do we get all possible probability weights Q in superspace obtained by the generalized Hubbard-Stratonovich transformation, agreeing with exactly the same partition functions in ordinary space?

3. Projection formula for chiral ensembles

The aim is to generalize the projection formula (2.37) to chiral ensembles. We introduce the chiral matrix

$$H_\chi = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix}, \tag{3.1}$$

where the matrix entries of W are either real, complex, or quaternion independent random variables for $\beta = 1, 2, 4$, respectively. The chiral matrix H_χ is related to the anti-Hermitean, chiral random matrix

$$\mathcal{D} \longrightarrow D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} = \gamma_5 H_\chi, \quad \text{with } \gamma_5 = (\mathbb{1}_n, -\mathbb{1}_{n+\nu}) \tag{3.2}$$

modelling the Euclidean Dirac operator $\mathcal{D}$ in four dimensions [15, 16, 17]. The modulus of the index $\nu \in \{-n, 1-n, 2-n, \ldots\}$ is equal to the number of generic zeros of H_χ which can be identified with the topological charge in continuum theory. The random matrix W is drawn from the coset

$$\mathrm{Gl}^{(\beta)}(n; n+\nu) := \mathrm{U}^{(\beta)}(2n+\nu)/[\mathrm{U}^{(\beta)}(n) \times \mathrm{U}^{(\beta)}(n+\nu)] \tag{3.3}$$

distributed by P_χ such that $H_\chi \in \mathrm{Herm}^{(\beta)}(2n+\nu)$. The probability density is assumed to be invariant under

$$P_\chi(W) = P_\chi(UW), \quad \forall\, U \in \mathrm{U}^{(\beta)}(n). \tag{3.4}$$

Notice that we do not assume invariance under right transformations as well which is usually the case [51, 17]. The reason is that we also want to study correlated random matrix ensembles as they naturally appear in the analysis of one-sided correlated Wishart ensembles where the invariance is broken by an empirical correlation matrix, see Refs. [2, 3, 4, 11, 12, 13, 14].

Due to the invariance (3.4) we can reduce the functional dependence of P_χ on W to one of $WW^\dagger$. Thus there is a function P such that

$$P_\chi(W) = P(W^\dagger W). \tag{3.5}$$

Moreover we assume that the chiral partition function,

$$Z_\chi(\kappa) := \int d[W] P_\chi(H_\chi) \frac{\prod_{j=1}^{k_2} \det(H_\chi - \kappa_j^{(2)} \mathbb{1}_{\gamma(2n+\nu)})}{\prod_{j=1}^{k_1} \det(H_\chi - \kappa_j^{(1)} \mathbb{1}_{\gamma(2n+\nu)})} \tag{3.6}$$

$$= \int d[W] P_\chi(H_\chi) \mathrm{sdet}^{-1/(\gamma\widetilde{\gamma})}(H_\chi \otimes \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \mathbb{1}_{\gamma(2n+\nu)} \otimes \kappa),$$

can be reduced to one for $WW^\dagger$ or/and $W^\dagger W$,

$$\begin{aligned} Z_\chi(\kappa) &= (-1)^{\gamma(n+\nu)(k_2-k_1)} \mathrm{sdet}^{-\nu/\widetilde{\gamma}} \kappa \\ &\quad \times \int d[W] P(W^\dagger W) \mathrm{sdet}^{-1/(\gamma\widetilde{\gamma})}(WW^\dagger \otimes \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \mathbb{1}_{\gamma n} \otimes \kappa^2), \\ &= (-1)^{\gamma(n+\nu)(k_2-k_1)} \mathrm{sdet}^{\nu/\widetilde{\gamma}} \kappa \\ &\quad \times \int d[W] P(W^\dagger W) \mathrm{sdet}^{-1/(\gamma\widetilde{\gamma})}(W^\dagger W \otimes \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \mathbb{1}_{\gamma(n+\nu)} \otimes \kappa^2). \end{aligned} \tag{3.7}$$

One has to understand that those partition functions do not cover all interesting spectral correlation functions. For example QCD with finite chemical potential or/and finite temperature cannot be modelled with this restriction, cf. Refs. [52, 53, 54, 17]. For those partition functions the approach of a projection formula can be modified. Unluckily this modified approach only works for the case $\beta = 2$. We will elaborate more on this problem in a forthcoming publication [55].

To make contact with the projection formula (2.37) for the original ensembles in Dyson's threefold way, we notice that the second representation of the partition function in Eq. (3.7) can be expressed in terms of an integral over $H \in \mathrm{Herm}^{(\beta)}(n)$ if $\nu \leq 0$,

$$\begin{aligned} Z_\chi(\kappa) \propto \mathrm{sdet}^{\nu/\widetilde{\gamma}} \kappa \int d[H] \Theta(H) \det{}^{|\nu|/\widetilde{\gamma}+(\gamma-\widetilde{\gamma})/2} H P(H) \\ \times \mathrm{sdet}^{-1/(\gamma\widetilde{\gamma})}(H \otimes \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \mathbb{1}_{\gamma(n+\nu)} \otimes \kappa^2), \end{aligned} \tag{3.8}$$

with the matrix version of the Heaviside Θ function. It is unity if H is positive definite and otherwise vanishes. Apart from the similarity of Eq. (3.8) with Eq. (2.8) by identifying $\Theta(H)\det{}^{|\nu|/\widetilde{\gamma}+(\gamma-\widetilde{\gamma})/2} H P(H)$ as the new probability density, the crucial differences are the non-isotropy of P, i.e. Eq. (2.9) does not necessarily apply, and the Heaviside Θ function which is by far not smooth. Thus the original projection formula (2.37) is not applicable anymore.

In subsection 3.1, we pursue a similar idea as presented in subsection 2.2 to find a projection formula for partition functions of the form (3.7). This formula is simplified via a combination with the superbosonization formula in subsection 3.2.

3.1. Projection formula

The key idea to derive a projection formula is again to apply one of the Cauchy-like integration theorems for supermatrices first derived by Wegner [44], see also Refs. [45, 46, 47]. This time we need a Cauchy-like integration theorem for extending the set of rectangular matrices $\mathrm{Gl}^{(\beta)}(n; n+\nu)$ to a space of rectangular supermatrices which is the coset

$$\mathrm{Gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l; n+\nu) := \mathrm{U}^{(\beta)}(2n+\nu+\widetilde{\gamma}l|\gamma l)/[\mathrm{U}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l) \times \mathrm{U}^{(\beta)}(n+\nu)] \tag{3.9}$$

with $l \in \mathbb{N}$.

Let $p_1, p_2, q, l \in \mathbb{N}_0$. We split a rectangular $(p_1 + \widetilde{\gamma} l | q + \gamma l) \times p_2$ supermatrix Ω in the following way

$$\Omega = \left[\begin{array}{c} \widetilde{\Omega} \\ \widehat{\Omega} \end{array} \right] \in \mathrm{Gl}^{(\beta)}(p_1 + \widetilde{\gamma} l | q + \gamma l; p_2) \tag{3.10}$$

with $\widetilde{\Omega} \in \mathrm{Gl}^{(\beta)}(p_1 | q; p_2)$ and $\widehat{\Omega} \in \mathrm{Gl}^{(\beta)}(\widetilde{\gamma} l | \gamma l; p_2)$. Assuming a smooth superfunction f integrable on the set $\mathrm{Gl}^{(\beta)}(p_1 + \widetilde{\gamma} l | q + \gamma l; p_2)$ and invariant under

$$f(\Omega) = f(\widetilde{U}\Omega), \quad \forall\ \widetilde{U} \in \mathrm{U}^{(\beta)}(p_1 + \widetilde{\gamma} l | q + \gamma l) \text{ and } \Omega \in \mathrm{Gl}^{(\beta)}(p_1 + \widetilde{\gamma} l | q + \gamma l; p_2), \tag{3.11}$$

the Cauchy-like integration theorem for rectangular supermatrices [44, 45, 46, 47] reads

$$\frac{\int d[\widehat{\Omega}] f(\Omega)}{\int d[\widehat{\Omega}] \exp[-\operatorname{tr} \widehat{\Omega}^{\dagger} \widehat{\Omega}]} = f\left(\left[\begin{array}{c} \widetilde{\Omega} \\ 0 \end{array} \right] \right). \tag{3.12}$$

We notice that no Wick-rotation is needed for this theorem in contrast to Eq. (2.27), simplifying the derivation by getting rid of one technical detail.

We apply the identity (3.12) to the partition function

$$\begin{aligned} Z_{\chi}(\kappa) = (-1)^{\gamma(n+\nu)(k_2-k_1)} & \operatorname{sdet}^{-\nu/\widetilde{\gamma}} \kappa \\ & \times \int d[W] P(W^{\dagger} W) \operatorname{sdet}^{-1/(\gamma\widetilde{\gamma})} (W W^{\dagger} \otimes \mathbb{1}_{\gamma\widetilde{\gamma} k_1 | \gamma\widetilde{\gamma} k_2} - \mathbb{1}_{\gamma n} \otimes \kappa^2). \end{aligned} \tag{3.13}$$

We have chosen the first version of Eq. (3.7), the reason for this choice becomes clearer later on. The product $W^{\dagger} W$ and, hence, the function $P(W^{\dagger} W)$ are obviously invariant under left multiplication of W with unitary matrices and can, thus, generally be extended to $\Omega^{\dagger}\Omega$ and $P(\Omega^{\dagger}\Omega)$ by the integration theorem (3.12), respectively. The only thing we assume is that $P(\Omega^{\dagger}\Omega)$ has to be smooth and integrable on $\mathrm{Gl}^{(\beta)}(n + \widetilde{\gamma} k_2 | \gamma k_2; n + \nu)$ where we again restrict ourself to the case $\widetilde{\gamma} k_1 \leq \widetilde{\gamma} k_2 + n$. The other, usually less interesting case $\widetilde{\gamma} k_1 \geq \widetilde{\gamma} k_2 + n$ can be derived in a slightly modified discussion.

In the first step we apply the Cauchy-like integration theorem to the partition function to extend the integral over the ordinary space $\mathrm{Gl}^{(\beta)}(n; n + \nu)$ to an integral over the superspace $\mathrm{Gl}^{(\beta)}(n + \widetilde{\gamma} k_2 | \gamma k_2; n + \nu)$, i.e.

$$\begin{aligned} Z_{\chi}(\kappa) = (-1)^{\gamma(n+\nu)(k_2-k_1)} & \operatorname{sdet}^{-\nu/\widetilde{\gamma}} \kappa \\ & \times \frac{\int d[\Omega] P(\Omega^{\dagger}\Omega) \operatorname{sdet}^{-1/(\gamma\widetilde{\gamma})} (W W^{\dagger} \otimes \mathbb{1}_{\gamma\widetilde{\gamma} k_1 | \gamma\widetilde{\gamma} k_2} - \mathbb{1}_{\gamma n} \otimes \kappa^2)}{\int d[\widehat{\Omega}] \exp[-\operatorname{tr} \widehat{\Omega}^{\dagger} \widehat{\Omega}]}, \end{aligned} \tag{3.14}$$

where we employ the following splitting of the rectangular supermatrix,

$$\Omega = \left[\begin{array}{c} W \\ \widehat{\Omega} \end{array} \right] = \left[\begin{array}{c} W' \\ \Omega' \end{array} \right] \tag{3.15}$$

with $W \in \mathrm{Gl}^{(\beta)}(n|0; n + \nu) = \mathrm{Gl}^{(\beta)}(n; n + \nu)$, $W' \in \mathrm{Gl}^{(\beta)}(n + \widetilde{\gamma}(k_2 - k_1)|0; n + \nu) = \mathrm{Gl}^{(\beta)}(n + \widetilde{\gamma}(k_2 - k_1); n + \nu)$, $\widehat{\Omega} \in \mathrm{Gl}^{(\beta)}(\widetilde{\gamma} k_2 | \gamma k_2; n + \nu)$, and $\Omega' \in \mathrm{Gl}^{(\beta)}(\widetilde{\gamma} k_1 | \gamma k_2; n + \nu)$. The second splitting corresponds to the embedding of the superspace we aim at.

Let $\widetilde{L} = \operatorname{sign} \operatorname{Im} \kappa^2$ be the sign of the squared source variables arrayed on a diagonal matrix. In the next step of our approach we introduce Gaussian integrals

over exactly the same rectangular supermatrix V as in Eq. (2.10) yielding

$$Z_\chi(\kappa) = (-1)^{\gamma(n+\nu)(k_2-k_1)}\mathrm{sdet}^{\,-\nu/\widetilde{\gamma}}\kappa \tag{3.16}$$
$$\times \frac{\int d[\Omega]P(\Omega^\dagger\Omega)\int d[V]\exp[\imath\mathrm{str}\,V^\dagger V\widetilde{L}\kappa^2 - \imath\mathrm{str}\,\Omega\Omega^\dagger\widetilde{A}]}{\imath^{\gamma n(k_2-k_1)}\mathrm{sdet}^{\,-n/\widetilde{\gamma}}\widetilde{L}\int d[V]\exp[-\mathrm{str}\,V^\dagger V]\int d[\widehat{\Omega}]\exp[-\mathrm{tr}\,\widehat{\Omega}^\dagger\widehat{\Omega}]}$$

with

$$\widetilde{A} = \begin{bmatrix} V\widetilde{L}V^\dagger & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & V\sqrt{\widetilde{L}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \sqrt{\widetilde{L}}V^\dagger & 0 \end{bmatrix} \tag{3.17}$$

cf. Eqs. (2.31) and (2.32). The dyadic matrix $\widetilde{A}$ has again a dual matrix

$$\widetilde{B} = \begin{bmatrix} 0 & 0 \\ \sqrt{\widetilde{L}}V^\dagger & 0 \end{bmatrix}\begin{bmatrix} 0 & V\sqrt{\widetilde{L}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\widetilde{L}}V^\dagger V\sqrt{\widetilde{L}} \end{bmatrix}, \tag{3.18}$$

cf. Eq. (2.34). Interchanging the integrals over Ω and V in Eq. (3.16) we arrive at the following integral transform of P,

$$\Psi(\widetilde{A}) = \int d[\Omega]P(\Omega^\dagger\Omega)\exp[-\imath\mathrm{str}\,\Omega\Omega^\dagger\widetilde{A}], \tag{3.19}$$

which plays the role of the Fourier-Laplace transform (2.33) in the case of Dyson's threefold way. Now the invariance of Ω under multiplication from the left with unitary supermatrices enters, implying

$$\Psi(\widetilde{U}\widetilde{A}\widetilde{U}^{-1}) = \Psi(\widetilde{A}), \quad \forall\ \widetilde{U} \in \mathrm{U}^{\,(\beta)}(n+\widetilde{\gamma}k_2|\gamma k_2). \tag{3.20}$$

Hence, the following identity is true

$$\Psi(\widetilde{A}) = \Psi(\widetilde{B}), \tag{3.21}$$

connecting the ordinary matrix space with the superspace. This identity is remarkable, as it relates both spaces with one and the same superfunction Ψ. We notice that the supermatrices $\widetilde{A}$ and $\widetilde{B}$ are of the same size corresponding to the first and second splitting of Eq. (3.15), respectively, while their non-zero blocks are not.

The duality relation (3.21) can be plugged into the partition function which reads

$$Z_\chi(\kappa) = (-1)^{\gamma(n+\nu)(k_2-k_1)}\mathrm{sdet}^{\,-\nu/\widetilde{\gamma}}\kappa \tag{3.22}$$
$$\times \frac{\int d[V]\int d[\Omega]P(\Omega^\dagger\Omega)\exp[\imath\mathrm{str}\,V^\dagger V\widetilde{L}\kappa^2 - \imath\mathrm{str}\,\Omega\Omega^\dagger\widetilde{B}]}{\imath^{\gamma n(k_2-k_1)}\mathrm{sdet}^{\,-n/\widetilde{\gamma}}\widetilde{L}\int d[V]\exp[-\mathrm{str}\,V^\dagger V]\int d[\widehat{\Omega}]\exp[-\mathrm{tr}\,\widehat{\Omega}^\dagger\widehat{\Omega}]}.$$

Due to convergence of the integrals we can again not easily switch the integration of Ω and V unless the boson-boson block of $\widetilde{L}$ is proportional to the identity. However this problem can be circumvented as it was discussed in subsection 2.2 by introducing an auxiliary Hermitean supermatrix. We skip this here because it is exactly the same procedure explained in subsection 2.2. Hence we end up with the partition function

$$Z_\chi(\kappa) = (-1)^{\gamma(n+\nu)(k_2-k_1)}\mathrm{sdet}^{\,-\nu/\widetilde{\gamma}}\kappa\frac{\int d[\Omega]P(\Omega^\dagger\Omega)\mathrm{sdet}^{\,-n/\widetilde{\gamma}}(\Omega'\Omega'^{\,\dagger}-\kappa^2)}{\int d[\widehat{\Omega}]\exp[-\mathrm{tr}\,\widehat{\Omega}^\dagger\widehat{\Omega}]}, \tag{3.23}$$

which is one of the main results of this section. We emphasize a few things about this formula. The supermatrices Ω' in the numerator and $\widehat{\Omega}$ in the denominator have different sizes, see the splittings (3.15). Moreover the index ν can take negative values as well since we have not at all used an assumption like $WW^\dagger$ is smaller than $W^\dagger W$. Equation (3.23) can be slightly modified such that the supermatrix κ can be easily

assumed to be non-diagonal, e.g. in QCD you need a non-diagonal κ to generate mixed pion condensates [48], or we can think of a symmetry breaking term in the determinant of Eq. (3.14) which may happen by circumventing the problem of a two-sided correlated Wishart ensemble as it appears for modelling spatial-time correlation matrices [56, 57, 58], see subsection 4.1.

The superdeterminant in Eq. (3.23) only depends on the the product $\Omega'\Omega'^{\dagger}$. Therefore the integral over W' defines a new probability distribution $\widehat{Q}$ on the superspace $\mathrm{Gl}^{(\beta)}(\widetilde{\gamma}k_1|\gamma k_2; n+\nu)$, i.e.

$$\widehat{Q}(\Omega'^{\dagger}\Omega') = \frac{\int d[W']P(\Omega^{\dagger}\Omega)}{\int d[\widehat{\Omega}]\exp[-\operatorname{tr}\widehat{\Omega}^{\dagger}\widehat{\Omega}]} = \frac{\int d[W']P(W'^{\dagger}W' + \Omega'^{\dagger}\Omega')}{\int d[\widehat{\Omega}]\exp[-\operatorname{tr}\widehat{\Omega}^{\dagger}\widehat{\Omega}]}. \tag{3.24}$$

Notice that there is one crucial disadvantage of this projection formula to the one of Dyson's threefold way, cf. Eq. (2.37). The superfunction $\widehat{Q}$ is still a function depending on a matrix $\Omega'^{\dagger}\Omega'$ with ordinary dimensions. It is easy to get rid of this flaw if the original probability density P is also invariant under right multiplication of W. Such a restriction becomes a problem for two-sided correlated Wishart matrices. For one-sided correlated Wishart matrix ensemble we can circumvent this problem, see subsection 4.1.

3.2. Rotation invariant probability densities

In this subsection we further simplify the projection formula by assuming that the probability density P is rotation invariant, i.e.

$$P(W^{\dagger}W) = P(\widetilde{U}W^{\dagger}W\widetilde{U}^{-1}), \quad \forall\ \widetilde{U} \in \mathrm{U}^{(\beta)}(n+\nu) \text{ and } W \in \mathrm{Gl}^{(\beta)}(n; n+\nu). \tag{3.25}$$

Then this invariance is obviously true by replacing $W \to \Omega$, too. Therefore there is certainly a supersymmtric extension of P denoted by $\widetilde{P}$ with

$$P(W'^{\dagger}W' + \Omega'^{\dagger}\Omega') = \widetilde{P}\left(\begin{bmatrix} W'W'^{\dagger} & W'\Omega'^{\dagger} \\ \Omega'W'^{\dagger} & \Omega'\Omega'^{\dagger} \end{bmatrix}\right). \tag{3.26}$$

The reason is that we can write P in terms of matrix invariants like traces which is also a source of ambiguity when extending P to superspace [37].

For further calculations we assume $\nu \geq 0$ which becomes important for convergence of some integrals. Because of the invariance under independent left and right multiplication of W with unitary matrices this is not a restriction at all. One can simply choose W such that it has the smaller dimension n on its left side.

Since the integral (3.24) is invariant under the transformation $\Omega'^{\dagger}\Omega' \to \widetilde{U}\Omega'^{\dagger}\Omega'\widetilde{U}^{-1}$ for all $\widetilde{U} \in \mathrm{U}^{(\beta)}(n)$, too, we can define a probability density on superspace

$$Q(\Omega'\Omega'^{\dagger}) = \frac{\int d[W']\widetilde{P}\left(\begin{bmatrix} W'W'^{\dagger} & W'\Omega'^{\dagger} \\ \Omega'W'^{\dagger} & \Omega'\Omega'^{\dagger} \end{bmatrix}\right)}{\int d[\widehat{\Omega}]\exp[-\operatorname{tr}\widehat{\Omega}^{\dagger}\widehat{\Omega}]}. \tag{3.27}$$

The crucial difference of Eqs. (3.24) and (3.27) is that Q in contrast to $\widehat{Q}$ depends on a $(\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2) \times (\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2)$ supermatrix. Thus, there is a chance to get rid of a number of integration variables which scales with n. This is quite important when taking the limit of large matrices as it is the case when deriving the universal behavior of the spectrum of H_χ.

The aim is to express the integral (3.27) in terms of the combination $\Omega'\Omega'^{\dagger}$ and some integration variables. For this purpose we introduce Dirac δ-functions for the

blocks depending on W',

$$\begin{aligned} Q(\Omega'\Omega'^{\dagger}) \propto & \int d[W'] \int d[H_1] \int d[H_2] \int d[W_1] \int d[W_2] \widetilde{P}\left(\begin{bmatrix} H_1 & W_1 \\ W_1^{\dagger} & \Omega'\Omega'^{\dagger} \end{bmatrix}\right) \\ & \times \exp\left[\mathrm{tr}(H_1 - W'W'^{\dagger})(\imath H_2 + \mathbb{1}_{\gamma(n+\widetilde{\gamma}(k_2-k_1))})\right] \\ & \times \exp\left[\imath\,\mathrm{tr}(W_1 - W'\Omega'^{\dagger})W_2^{\dagger} + \imath\,\mathrm{tr}\, W_2(W_1^{\dagger} - \Omega'W'^{\dagger})\right]. \end{aligned} \tag{3.28}$$

We drop the normalization constant right now and introduce it later on by fixing it with the Gaussian case. The matrices are drawn from $H_1, H_2 \in \mathrm{Herm}^{(\beta)}(n + \widetilde{\gamma}(k_2 - k_1))$ and $W_1^{\dagger}, W_2^{\dagger} \in \mathrm{Gl}^{(\beta)}(\widetilde{\gamma}k_1|\gamma k_2; n + \widetilde{\gamma}(k_2 - k_1))$. Recall the definition of the cosets and the splitting of Ω in Eqs. (2.6), (3.9) and (3.15), respectively. The shift in H_2 guarantees the convergence of the integral over W' which is the first one we perform yielding

$$\begin{aligned} Q(\Omega'\Omega'^{\dagger}) \propto & \lim_{\delta\to 0} \int d[H_1] \int d[H_2] \int d[W_1] \int d[W_2] \widetilde{P}\left(\begin{bmatrix} H_1 & W_1 \\ W_1^{\dagger} & \Omega'\Omega'^{\dagger} \end{bmatrix}\right) \\ & \times \exp\left[\mathrm{tr}\, H_1(\imath H_2 + \mathbb{1}_{\gamma(n+\widetilde{\gamma}(k_2-k_1))}) + \imath\,\mathrm{tr}\, W_1 W_2^{\dagger} + \imath\,\mathrm{tr}\, W_2 W_1^{\dagger}\right] \\ & \times \exp\left[-\mathrm{tr}(\imath H_2 + \mathbb{1}_{\gamma(n+\widetilde{\gamma}(k_2-k_1))})^{-1} W_2(\Omega'\Omega'^{\dagger} + \delta\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}) W_2^{\dagger}\right] \\ & \times \det{}^{-(n+\nu)/\widetilde{\gamma}}(\imath H_2 + \mathbb{1}_{\gamma(n+\widetilde{\gamma}(k_2-k_1))}). \end{aligned} \tag{3.29}$$

The variable δ is a regularization guaranteeing us the convergence of the integrals since $\Omega'\Omega'^{\dagger}$ is not invertible if it contains a fermion-fermion block, i.e. $k_2 \neq 0$. We rescale $W_1 \to W_1\sqrt{\Omega'\Omega'^{\dagger} + \delta\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}}$ and $W_2 \to W_2/\sqrt{\Omega'\Omega'^{\dagger} + \delta\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}}$. The Jacobian of the transformation W_1 and W_2 cancel out and the limit of the regulator $\delta \to 0$ can be made exact. The next integral we perform is over W_2 and we find

$$\begin{aligned} Q(\Omega'\Omega'^{\dagger}) \propto & \int d[H_1] \int d[H_2] \int d[W_1] \widetilde{P}\left(\begin{bmatrix} H_1 & W_1\sqrt{\Omega'\Omega'^{\dagger}} \\ \sqrt{\Omega'\Omega'^{\dagger}} W_1^{\dagger} & \Omega'\Omega'^{\dagger} \end{bmatrix}\right) \\ & \times \exp\left[\mathrm{tr}(H_1 - W_1W_1^{\dagger})(\imath H_2 + \mathbb{1}_{\gamma(n+\widetilde{\gamma}(k_2-k_1))})\right] \\ & \times \det{}^{-(n+\nu)/\widetilde{\gamma}+(k_1-k_2)}(\imath H_2 + \mathbb{1}_{\gamma(n+\widetilde{\gamma}(k_2-k_1))}). \end{aligned} \tag{3.30}$$

We notice that $\widetilde{P}$ depends on invariants only. Hence in an explicit representation of $\widetilde{P}$ we do not encounter the ill-defined matrix $\sqrt{\Omega'\Omega'^{\dagger}}$ but only the supermatrix $\Omega'\Omega'^{\dagger}$. The remaining integral over H_2 is an ordinary Ingham-Siegel integral [59, 60]. Shifting $H_1 \to H_1 + W_1W_1^{\dagger}$ the Ingham-Siegel integral tells us that H_1 has to be positive definite and yields a determinant of H_1 to the power $\nu/\widetilde{\gamma} + (\gamma - \widetilde{\gamma})/2$ (exactly here we need $\nu \geq 0$). The positivity constraint of H_1 is quite often hard to handle such that we replace H_1 by a rectangular matrix $\widehat{W}_1 \in \mathrm{Gl}^{(\beta)}(n + \widetilde{\gamma}(k_2 - k_1); n + \nu + \widetilde{\gamma}(k_2 - k_1))$. Finally, we arrive at the main result of this section and the projection formula for rotation invariant chiral ensembles,

$$Q(\Omega'\Omega'^{\dagger}) = C \int d[\widehat{W}_1] \int d[W_1] \widetilde{P}\left(\begin{bmatrix} \widehat{W}_1\widehat{W}_1^{\dagger} + W_1W_1^{\dagger} & W_1\sqrt{\Omega'\Omega'^{\dagger}} \\ \sqrt{\Omega'\Omega'^{\dagger}} W_1^{\dagger} & \Omega'\Omega'^{\dagger} \end{bmatrix}\right) \tag{3.31}$$

with the normalization constant

$$C = \frac{\int d[W'] \exp[-\mathrm{tr}\, W'W'^{\dagger}]}{\int d[\widehat{W}_1] \exp[-\mathrm{tr}\, \widehat{W}_1\widehat{W}_1^{\dagger}] \int d[W_1] \exp[-\mathrm{tr}\, W_1W_1^{\dagger}] \int d[\widehat{\Omega}] \exp[-\mathrm{tr}\, \widehat{\Omega}^{\dagger}\widehat{\Omega}]}. \tag{3.32}$$

The reason for fixing the normalization with Gaussian weights lies in the universality of the projection formula (3.31). The projection formula is true for almost all ensembles depending on invariants of the rectangular matrix W. Due to this broad applicability Eq. (3.31) is a powerful tool. In Sec. 4, we will present some examples, often encountered in different fields of random matrix theory.

Additionally one can apply the superbosonization formula to the partition function

$$Z_\chi(\kappa) = (-1)^{\gamma(n+\nu)(k_2-k_1)} \mathrm{sdet}^{\,-\nu/\widetilde{\gamma}} \kappa \int d[\Omega'] Q(\Omega'\Omega'^{\,\dagger}) \mathrm{sdet}^{\,-n/\widetilde{\gamma}} (\Omega'\Omega'^{\,\dagger} - \kappa^2), \quad (3.33)$$

which is justified since the whole integral depends on the dyadic supermatrix $\Omega'\Omega'^{\,\dagger}$. Thus we replace $\Omega'\Omega'^{\,\dagger}$ by the supermatrix $\widehat{U} \in \mathrm{Herm}_{\odot}^{(\beta)}(\widetilde{\gamma}k_1|\gamma k_2)$ which has the same structure as the supermatrix U in the original approach of the superbosonization formula (2.19). The partition function reads

$$\begin{aligned} Z_\chi(\kappa) = {} & \frac{(-1)^{\gamma(n+\nu)(k_2-k_1)} \int d[\Omega'] \exp[-\mathrm{str}\, \Omega'\Omega'^{\,\dagger}]}{\int d\mu(\widehat{U}) \exp[-\mathrm{str}\, \widehat{U}] \mathrm{sdet}^{\,(n+\nu)/\widetilde{\gamma}} \widehat{U}} \mathrm{sdet}^{\,-\nu/\widetilde{\gamma}} \kappa \\ & \times \int d\mu(\widehat{U}) Q(\widehat{U}) \mathrm{sdet}^{\,-n/\widetilde{\gamma}} (\widehat{U} - \kappa^2) \mathrm{sdet}^{\,(n+\nu)/\widetilde{\gamma}} \widehat{U} \end{aligned} \quad (3.34)$$

with the superfunction

$$Q(\widehat{U}) = C \int d[\widehat{W}_1] \int d[W_1] \widetilde{P} \left(\begin{bmatrix} \widehat{W}_1 \widehat{W}_1^\dagger + W_1 W_1^\dagger & W_1 \sqrt{\widehat{U}} \\ \sqrt{\widehat{U}} W_1^\dagger & \widehat{U} \end{bmatrix} \right). \quad (3.35)$$

Importantly, one should not confuse the superfunction $\widetilde{\Phi}$ of Eq. (2.19) with the superfunction Q, we mention the different terms in the integrands. The prefactor in Eq. (3.34) is the global normalization constant resulting from the superbosonization formula and strongly depends on the normalization of the Haar-measure $d\mu(\widehat{U})$ of the supersymmetric coset $\mathrm{Herm}_{\odot}^{(\beta)}(\widetilde{\gamma}k_1|\gamma k_2)$.

4. Some examples

We apply the projection formula (3.35) to four non-trivial examples to illustrate how our approach works. Especially it becomes clear what the advantages of the projection formula (3.35) are in comparison to the standard approaches with the generalized Hubbard-Stratonovich transformation [36, 37] and the superbosonization formula [38, 39].

In particular we discuss norm-dependent ensembles and correlated Wishart ensembles in subsection 4.1, Lorentz-like (Cauchy) ensembles in subsection 4.2, the three unquenched chiral Gaussian ensembles in subsection 4.3, and a probability density with a quartic potential in subsection 4.4. The norm-dependent ensembles serve as a check since they can readily be calculated with the previous variants of the supersymmetry method. With help of the correlated Wishart ensembles we show that the projection formula can easily be extended to include a symmetry breaking constant term in the determinants, cf. Eq. (3.7). The Lorentz-like (Cauchy) weight is another standard probability density as the Gaussian weight. It has a particular property namely it exhibits heavy tails and thus not all moments exist. For the unquenched chiral Gaussian ensemble we derive an alternative representation of the chiral Lagrangian, see Refs! . [15, 16, 17] for the common representation. In this

representation the physical mesons are split off from the artificial ones which result from introducing source terms to generate the desired observables. With help of the quartic potential we want to show that one can also study non-trivial potentials via the projection formula (3.35).

4.1. Norm-dependent ensembles and correlated Wishart ensembles

The first class of ensembles we want to look at are the norm-dependent chiral ensembles [51, 61], i.e.

$$P(W^\dagger W) = p(\operatorname{tr} W^\dagger W) \tag{4.1}$$

with an integrable function p. A particular choice is a fixed trace ensemble, namely $p(\operatorname{tr} W^\dagger W) \propto \delta(\operatorname{tr} W^\dagger W - cn)$ with a constant $c > 0$. Such an ensemble naturally appears when modelling lattice QCD [62]. The lattice QCD Dirac operator is build up of unitary matrices and fulfills a fixed-trace condition. However one can readily show that this condition has only a minor effect on the microscopic regime of the Dirac spectrum and is completely suppressed in the exact limit [62]. The choice $p \propto \delta(\operatorname{tr} W^\dagger W - cn)$ only enhances the $1/n$ correction. Also in quantum information it plays an important role [63] since the density operator is normalized.

The corresponding superfunction of the probability density P for an arbitrary p can be simply read off from the projection formula (3.35) and is up to a constant

$$Q(\widehat{U}) \propto \int_0^\infty dr p(r^2 + \operatorname{str} \widehat{U}) r^{\beta(n+\widetilde{\gamma}(k_2-k_1))(n+\nu)}. \tag{4.2}$$

The exponent of the integration variable r is the difference of the number of commuting real variables and anti-commuting Grassmann variables in the rectangular matrices W_1 and $\widehat{W}_1$. Those matrices are of dimension $(\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2) \times (\gamma n + \gamma\widetilde{\gamma}(k_2 - k_1))$ and $(\gamma n + \gamma\widetilde{\gamma}(k_2 - k_1)) \times (\gamma(n+\nu) + \gamma\widetilde{\gamma}(k_2 - k_1))$, respectively, and fulfil certain symmetries similar to Eq. (2.11).

A natural representative of a norm-dependent ensemble is the Gaussian one, i.e. $p(\operatorname{tr} W^\dagger W) \propto \exp[-n \operatorname{tr} W^\dagger W]$. Then the integral over r factorizes in Eq. (4.2). This apparently yields again a Gaussian

$$Q(\Omega'\Omega'^\dagger) \propto \exp\left(-\frac{n}{\widetilde{\gamma}} \operatorname{str} \Omega'\Omega'^\dagger\right) \tag{4.3}$$

in terms of the dyadic supermatrix $\Omega'\Omega'^\dagger$ and reads in terms of the supermatrix $\widehat{U} \in \operatorname{Herm}_{\odot}^{(\beta)}(\widetilde{\gamma}k_1|\gamma k_2)$

$$Q(\widehat{U}) \propto \exp\left(-\frac{n}{\widetilde{\gamma}} \operatorname{str} \widehat{U}\right). \tag{4.4}$$

For a Gaussian weight this result is not surprising but it serves as a simple check for the projection formula (3.35). When plugging Eq. (4.4) into the partition function (3.34), we arrive at

$$Z_\chi(\kappa) \propto \operatorname{sdet}^{-\nu/\widetilde{\gamma}} \kappa \int d\mu(\widehat{U}) \exp\left(-n \operatorname{str} \widehat{U}\right) \operatorname{sdet}^{-n/\widetilde{\gamma}} (\widehat{U} - \kappa^2) \operatorname{sdet}^{(n+\nu)/\widetilde{\gamma}} \widehat{U}. \tag{4.5}$$

The microscopic limit ($n \to \infty$ while ν and $n\kappa$ fixed) connects chiral random matrix theory with QCD [17] and is obtained from our expression by rescaling

$\widehat{U} \to -\imath\kappa\widehat{U}$. After taking the limit $n \to \infty$ we find the well-known chiral Lagrangian [17]

$$Z_\chi(\kappa) \overset{n\gg 1}{\propto} \int d\mu(\widehat{U}) \exp\left(\imath\frac{n}{\widetilde{\gamma}}\mathrm{str}\,\kappa(\widehat{U}+U^{-1})\right)\mathrm{sdet}^{\,\nu/\widetilde{\gamma}}\widehat{U}. \tag{4.6}$$

Surprisingly, we had not to take any saddlepoint approximation with our approach which is usually the case in the other approaches of the supersymmetry method [15, 16]. The reason is that the projection formula already mapped the ordinary space to the correct coset describing the mesons of the chiral Lagrangian in QCD.

Another application of norm-dependent ensembles are correlated Wishart matrices with a non-Gaussian weight. In Sec. 3 we claimed that we can also study one-sided correlated Wishart ensembles with arbitrary weight. Those ensembles appear in many situations where one encounters time series analysis like in finance [11, 12], telecommunication [21], etc. Thus we consider the following partition function

$$\begin{aligned} Z_\chi(\kappa) &= (-1)^{\gamma(n+\nu)(k_2-k_1)}\mathrm{sdet}^{\,-\nu/\widetilde{\gamma}}\kappa \\ &\quad\times\int d[W]p(\mathrm{tr}\,W^\dagger C^{-1}W)\mathrm{sdet}^{\,-1/(\gamma\widetilde{\gamma})}(WW^\dagger\otimes\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}-\mathbb{1}_{\gamma n}\otimes\kappa^2), \end{aligned} \tag{4.7}$$

where the function p is as before arbitrary and C is an empirical correlation matrix and thus positive definite. In the first step we rescale $W \to \sqrt{C}W$ and have

$$\begin{aligned} Z_\chi(\kappa) &= (-1)^{\gamma(n+\nu)(k_2-k_1)}\mathrm{sdet}^{\,-\nu/\widetilde{\gamma}}\kappa\mathrm{det}^{(n+\nu)/\widetilde{\gamma}+(k_2-k_1)}C \\ &\quad\times\int d[W]p(\mathrm{tr}\,W^\dagger W)\mathrm{sdet}^{\,-1/(\gamma\widetilde{\gamma})}(WW^\dagger\otimes\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}-C^{-1}\otimes\kappa^2). \end{aligned} \tag{4.8}$$

In the second step we apply the projection formula (3.35) in combination with a slightly modified version of Eq. (3.34) and find

$$\begin{aligned} Z_\chi(\kappa) &\propto \mathrm{sdet}^{\,-\nu/\widetilde{\gamma}}\kappa\mathrm{det}^{(n+\nu)/\widetilde{\gamma}+(k_2-k_1)}C \\ &\quad\times\int d\mu(\widehat{U})Q(\widehat{U})\mathrm{sdet}^{\,-1/(\gamma\widetilde{\gamma})}(\mathbb{1}_{\gamma n}\otimes\widehat{U}-C^{-1}\otimes\kappa^2)\mathrm{sdet}^{\,(n+\nu)/\widetilde{\gamma}}\widehat{U}. \end{aligned} \tag{4.9}$$

The superfunction Q is the one from the onefold integral (4.2). In the case of a Gaussian weight the one-point correlation function was already studied with help of supersymmetry for $\beta = 1, 2$, see Refs. [13, 14]. Equation (4.9) is an alternative compact representation of this partition function.

4.2. Lorentz (Cauchy)-like ensembles

Another kind of probability density serving as a 'standard candle' in statistical physics is the Lorentz weight. In contrast to the Gaussian weight, almost all moments of the matrix W do not exist for the Lorentzian. In random matrix theory one introduces this weight with a constant $\Gamma \in \mathbb{R}_+$ determining the width of the distribution and an exponent $\mu \in \mathbb{N}$ indicating how rapid the tails fall off, i.e. the Lorentzian ensemble is given by

$$P(W^\dagger W) \propto \det{}^{-\mu}\left(\Gamma^2\mathbb{1}_{n+\nu}+W^\dagger W\right). \tag{4.10}$$

The exponent μ has to be large enough to guarantee the normalizability of the probability density. This ensemble is also known as Cauchy ensemble [64, 65]. Of particular interest is its heavy-tailed behavior which has not been studied in such detail as the exponential cut-off from ensembles with polynomial potentials. Importantly,

one can expect that the universal results may break down. Recent works on heavy tails of random matrices are Refs. [66, 67, 68] and references therein.

Again we are interested in the supersymmtric analogue of P which is given via the projection formula (3.35),

$$\begin{aligned} Q(\widehat{U}) \propto & \int d[\widehat{W}_1] \int d[W_1] \mathrm{sdet}^{\,-\mu} \left(\Gamma^2 \mathbb{1}_{\gamma n+\gamma\widetilde{\gamma}k_2|\gamma\widetilde{\gamma}k_2} + \begin{bmatrix} \widehat{W}_1\widehat{W}_1^\dagger + W_1 W_1^\dagger & W_1\sqrt{\widehat{U}} \\ \sqrt{\widehat{U}}W_1^\dagger & \widehat{U} \end{bmatrix} \right) \\ = & \mathrm{sdet}^{\,-\mu} \left(\Gamma^2 \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} + \widehat{U} \right) \int d[\widehat{W}_1] \int d[W_1] \\ & \times \mathrm{sdet}^{\,-\mu} \left(\Gamma^2 \mathbb{1}_{\gamma n+\gamma\widetilde{\gamma}(k_2-k_1)} + \widehat{W}_1\widehat{W}_1^\dagger + \Gamma^2 W_1 (\Gamma^2 \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} + \widehat{U})^{-1} W_1^\dagger \right). \end{aligned} \tag{4.11}$$

In the second line we pulled out the lower right block of the superdeterminant. Here, we once more observe that one can often calculate with the superdeterminant as it would be a determinant, see Refs. [69]. After rescaling $W_1 \to W_1(\Gamma^2 \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} + \widehat{U})^{1/2}$ the integrals over $\widehat{W}_1$ and W_1 factorize and yield a constant. The projection formula leads to the superfunction (up to a normalization constant)

$$Q(\widehat{U}) \propto \mathrm{sdet}^{\,n/\widetilde{\gamma}+(k_2-k_1)-\mu} \left(\Gamma^2 \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} + \widehat{U} \right). \tag{4.12}$$

Thus the counterpart of the Lorentzian weight (4.10) is also Lorentzian in superspace. Only the exponent changes. Notice that the fermion-fermion block of $\widehat{U}$ is a compact integral such that we do not have any problems of convergence if $n/\widetilde{\gamma}+(k_2-k_1)-\mu \leq 0$. The exponent μ has only to be large enough such that the corresponding partition function,

$$\begin{aligned} Z_\chi(\kappa) \propto & \, \mathrm{sdet}^{\,-\nu/\widetilde{\gamma}} \kappa \int d\mu(\widehat{U}) \mathrm{sdet}^{\,n/\widetilde{\gamma}+(k_2-k_1)-\mu} \left(\Gamma^2 \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} + \widehat{U} \right) \\ & \times \mathrm{sdet}^{\,-n/\widetilde{\gamma}} (\widehat{U} - \kappa^2) \mathrm{sdet}^{\,(n+\nu)/\widetilde{\gamma}} \widehat{U}, \end{aligned} \tag{4.13}$$

exists, namely it has to be larger than $\mu > (n+\nu)/\widetilde{\gamma}$ for this integral. To guarantee the integral of the partition function in ordinary space the exponent has to fulfill $\mu > (n+\nu)/\widetilde{\gamma}+k_2-k_1$. Therefore one has only to take $\mu > (n+\nu)/\widetilde{\gamma}+\max\{0, k_2-k_1\}$ to guarantee the convergence of both integrals.

Interestingly, from Eq. (4.13) immediately follows that in the microscopic limit $n \to \infty$ (ν, $n\Gamma^2$ and $n\kappa$ fixed) for $\mu = n/\widetilde{\gamma} + \widetilde{\mu}$ with $\widetilde{\mu}$ fixed we do not find the universal result (4.6). We already expected that something may change, i.e. the partition function becomes

$$Z_\chi(\kappa) \propto \int d\mu(\widehat{U}) \mathrm{sdet}^{\,(k_2-k_1)-\widetilde{\mu}} \left(n\Gamma^2 \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} + n\kappa\widehat{U} \right) \mathrm{sdet}^{\,\nu/\widetilde{\gamma}} \widehat{U} \exp\left[\frac{n}{\widetilde{\gamma}} \mathrm{str}\, \kappa \widehat{U}^{-1} \right]. \tag{4.14}$$

However one can find the universal result at the hard edge of the spectrum, as the microscopic limit is also known, if $\widetilde{\mu}/n$ and Γ^2 is fixed instead.

4.3. Unquenched chiral Gaussian ensemble

The unquenched partition function is in QCD a statistical weight where additionally to the gauge action we have an interaction with fermionic quarks [17]. They are

equivalent with additional characteristic polynomials in the numerator in the partition function. Hence, the random matrix model is

$$P(WW^\dagger) = \frac{\exp\left(-n\,\mathrm{tr}\,W^\dagger W/\widetilde{\gamma}\right)\prod_{j=1}^{N_\mathrm{f}}\det(W^\dagger W + m_j^2\mathbb{1}_{\gamma(n+\nu)})}{\int d[W]\exp\left(-n\,\mathrm{tr}\,W^\dagger W/\widetilde{\gamma}\right)\prod_{j=1}^{N_\mathrm{f}}\det(W^\dagger W + m_j^2\mathbb{1}_{\gamma(n+\nu)})} \tag{4.15}$$

with the quark masses $m = \mathrm{diag}\,(m_1\mathbb{1}_{\gamma\widetilde{\gamma}},\ldots,m_{N_\mathrm{f}}\mathbb{1}_{\gamma\widetilde{\gamma}})$ of the N_f flavors. This time we explicitly wrote the normalization constant, since it is mass dependent and is, thus, quite essential.

The partition function (3.7) with the probability density (4.15), i.e. the partially quenched partition function

$$\begin{aligned} Z_\chi(\kappa,m) = {} & \frac{(-1)^{\gamma(n+\nu)(k_2-k_1)}\mathrm{sdet}^{\,-\nu/\widetilde{\gamma}}\kappa}{\int d[W]\exp\left(-n\,\mathrm{tr}\,W^\dagger W/\widetilde{\gamma}\right)\prod_{j=1}^{N_\mathrm{f}}\det(W^\dagger W + m_j^2\mathbb{1}_{\gamma(n+\nu)})} \\ & \times\int d[W]\exp\left(-\frac{n}{\widetilde{\gamma}}\,\mathrm{tr}\,W^\dagger W\right)\prod_{j=1}^{N_\mathrm{f}}\det(W^\dagger W + m_j^2\mathbb{1}_{\gamma(n+\nu)}) \\ & \times\mathrm{sdet}^{\,-1/(\gamma\widetilde{\gamma})}(WW^\dagger\otimes\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \mathbb{1}_{\gamma n}\otimes\kappa^2), \end{aligned} \tag{4.16}$$

can be dealt with in two different ways. Either the additional determinants and the determinants generating the correlation functions are computed on equal footing or one can consider the additional determinants as part of the probability density P. We decide for the latter choice since we aim at a separation of the physical quarks from the artificial ones which are also known as valence quarks.

In Appendix A we calculate the partially quenched partition function at finite n. It is a double integral over an ordinary matrix $U_\pi \in \mathrm{Herm}^{(\beta)}_{\odot}(0|\gamma N_\mathrm{f}) = \mathrm{U}^{\,(4/\beta)}(\gamma N_\mathrm{f})$ and the supermatrix $\widehat{U} \in \mathrm{Herm}^{(\beta)}_{\odot}(\widetilde{\gamma}k_1|\gamma k_2)$,

$$\begin{aligned} Z_\chi(\kappa,m) \propto {} & \mathrm{sdet}^{\,-\nu/\widetilde{\gamma}}\kappa\Bigg[\int d\mu(\widehat{U})\int d\mu(U_\pi)\exp\left(-\frac{n}{\widetilde{\gamma}}(\mathrm{str}\,\widehat{U} - \mathrm{tr}\,U_\pi)\right) \\ & \times\mathrm{sdet}^{\,\nu/\widetilde{\gamma}}\widehat{U}\mathrm{sdet}^{\,-n/\widetilde{\gamma}}(\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \kappa^2\widehat{U}^{-1}) \\ & \times\det{}^{\nu/\widetilde{\gamma}}U_\pi\det{}^{(n+\nu)/\widetilde{\gamma}+k_2-k_1}(\mathbb{1}_{N_\mathrm{f}} + m^2U_\pi^{-1}) \\ & \times\mathrm{sdet}^{\,1/(\gamma\widetilde{\gamma})}\left(\widehat{U}\otimes\mathbb{1}_{\gamma\widetilde{\gamma}N_\mathrm{f}} + \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}\otimes(U_\pi + m^2)\right)\Bigg]\Bigg/ \\ & \left[\int d\mu(U_\pi)\exp\left(\frac{n}{\widetilde{\gamma}}\,\mathrm{tr}\,U_\pi\right)\det{}^{(n+\nu)/\widetilde{\gamma}}(U_\pi + m^2)\det{}^{-n/\widetilde{\gamma}}U_\pi\right]. \end{aligned} \tag{4.17}$$

We take the microscopic limit $n\to\infty$ with $n\kappa$ and nm fixed. The partially quenched partition function becomes

$$\begin{aligned} Z_\chi(\kappa,m) \propto {} & \Bigg[\int d\mu(\widehat{U})\int d\mu(U_\pi)\mathrm{sdet}^{\,\nu/\widetilde{\gamma}}\widehat{U}\det{}^{\nu/\widetilde{\gamma}}U_\pi \\ & \times\exp\left(\frac{n}{\widetilde{\gamma}}\,\mathrm{tr}\,m(U_\pi + U_\pi^{-1}) - \frac{n}{\widetilde{\gamma}}\mathrm{str}\,\kappa(\widehat{U} - \widehat{U}^{-1})\right) \\ & \times\mathrm{sdet}^{\,1/(\gamma\widetilde{\gamma})}\left(n\kappa\widehat{U}\otimes\mathbb{1}_{\gamma\widetilde{\gamma}N_\mathrm{f}} + \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}\otimes nmU_\pi\right)\Bigg]\Bigg/ \end{aligned} \tag{4.18}$$

$$\left[\int d\mu(U_\pi)\exp\left(\frac{n}{\widetilde{\gamma}}\operatorname{tr} m(U_\pi+U_\pi^{-1})\right)\det{}^{\nu/\widetilde{\gamma}}U_\pi\right].$$

This partition function has to agree with the well-known results for the three chiral ensembles, see Refs. [15, 16, 17]. It is equal to Eq. (4.6) when the variables κ also comprise the quark masses m. For $\beta = 2$ this can be readily checked due to the knowledge of the Harish-Chandra-Itzykson-Zuber integral [70, 71]. In the real and quaternion case this is not as easy since the corresponding group integrals are not known.

What is the benefit of the representation (4.18) of the partially quenched partition function? The physical quarks are completely separated from the auxiliary particles, i.e. the chiral Lagrangian for the physical mesons can be read off

$$\begin{aligned}\mathcal{L}(U_\pi,\kappa,m) = \frac{n}{\widetilde{\gamma}}\operatorname{tr} m(U_\pi+U_\pi^{-1}) + \ln\Bigg[\int d\mu(\widehat{U})\exp\left(-\frac{n}{\widetilde{\gamma}}\operatorname{str}\kappa(\widehat{U}-\widehat{U}^{-1})\right)\operatorname{sdet}{}^{\nu/\widetilde{\gamma}}\widehat{U}\\ \times\operatorname{sdet}{}^{1/(\gamma\widetilde{\gamma})}\left(n\kappa\widehat{U}\otimes\mathbb{1}_{\gamma\widetilde{\gamma}N_{\mathrm{f}}}+\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}\otimes nmU_\pi\right)\Bigg].\end{aligned}\tag{4.19}$$

The first part of the Lagrangian is the leading order of the unquenched partition function with N_{f} flavors [17]. The second term is the operator corresponding to the generating function for some observables like the level density. Therefore we could split the observable from the physical system, U_π, in the chiral Lagrangian with help of the projection formula. Since random matrix theory only describes the Goldstone bosons with zero momentum a good question is if one can achieve such a splitting (4.19) for the kinetic modes, too.

4.4. Probability density with quartic potential

In the last example we want to consider the probability density with quartic potential

$$P(WW^\dagger)\propto\exp[-\alpha\operatorname{tr}(WW^\dagger)^2-\widehat{\alpha}\operatorname{tr}WW^\dagger],\tag{4.20}$$

$\alpha > 0$ and $\widehat{\alpha}\in\mathbb{R}$. This probability density is the standard one for the analysis of multicritical behavior [72, 73, 74, 75]. Depending on the relation of the two constants α and $\widehat{\alpha}$ the macroscopic level density of $WW^\dagger$ can exhibit a one-cut or two-cut solution which also influences the universality on the local scale of the mean level density where the two cuts are merging to one. We are aiming at a supersymmetric representation of the partition function with the probability density (4.20).

The superfunction Q corresponding to the probability density (4.20) is via the projection formula (3.35)

$$\begin{aligned}Q(\widehat{U})\propto\int d[\widehat{W}_1]\int d[W_1]\exp\left[-\alpha\left(\operatorname{tr}(\widehat{W}_1\widehat{W}_1^\dagger+W_1W_1^\dagger)^2+\operatorname{tr}W_1\widehat{U}W_1^\dagger+\operatorname{str}\widehat{U}^2\right)\right]\\ \times\exp\left[-\widehat{\alpha}\left(\operatorname{tr}\widehat{W}_1\widehat{W}_1^\dagger+\operatorname{tr}W_1W_1^\dagger+\operatorname{str}\widehat{U}\right)\right].\end{aligned}\tag{4.21}$$

The quartic term $\operatorname{tr}(\widehat{W}_1\widehat{W}_1^\dagger+W_1W_1^\dagger)^2$ can be traced back to a quadratic structure by introducing a Gaussian over an auxiliary matrix $H\in\operatorname{Herm}^{(\beta)}(n+\widetilde{\gamma}(k_2-k_1))$. Then the integrals over $\widehat{W}_1$ and W_1 are purely Gaussian and can be performed without any problem, leading to

$$\begin{aligned}Q(\widehat{U})\propto\exp\left[-\alpha\operatorname{str}\widehat{U}^2-\widehat{\alpha}\operatorname{str}\widehat{U}\right]\int d[H]\exp\left[-\frac{1}{4\alpha}\operatorname{tr}(H-\imath(\widehat{\alpha}-1)\mathbb{1}_{\gamma n+\gamma\widetilde{\gamma}(k_2-k_1)})^2\right]\\ \times\det{}^{-(n+\nu)/\gamma+k_1-k_2}(\imath H-\mathbb{1}_{\gamma n+\gamma\widetilde{\gamma}(k_2-k_1)})\end{aligned}$$

$$\times \operatorname{sdet}{}^{-1/(\gamma\widetilde{\gamma})}(\imath H \otimes \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2} - \mathbb{1}_{\gamma n+\gamma\widetilde{\gamma}(k_2-k_1)} \otimes (\alpha\widehat{U} + \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2})). \tag{4.22}$$

The determinant results from the integral over $\widehat{W}_1 \in \mathrm{Gl}^{(\beta)}(n + \widetilde{\gamma}(k_2 - k_1); n + \nu + \widetilde{\gamma}(k_2 - k_1))$ while the superdeterminant results from the integral over $W_1^\dagger \in \mathrm{Gl}^{(\beta)}(\widetilde{\gamma}k_1|\gamma k_2; n + \widetilde{\gamma}(k_2 - k_1))$. We recall the definition of the cosets in Eqs. (2.6), (3.3), and (3.9). The shift in the Gaussian of the auxiliary ordinary matrix H also guarantees the convergence of the integrals over $\widehat{W}_1$ and W_1 for negative $\widehat{\alpha}$.

For $\beta = 2$ the integral (4.22) can be further simplified via various techniques in random matrix theory [51, 76, 77, 78]. In one of these techniques [51, 78] one constructs the orthogonal polynomials of the weight $g(E) = \exp[-(E - \imath(\widehat{\alpha} - 1))^2/(4\alpha)]/(\imath E - 1)^{n+\nu+k_1-k_2}$. Then one obtains a quotient of two determinants of $\max\{k_1, k_2\} \times \max\{k_1, k_2\}$ matrices where the determinant in the numerator depends on the orthogonal polynomials and their Cauchy transform with respect to the weight $g(E)$ whose arguments are the eigenvalues of the supermatrix $(\alpha\widehat{U} + \mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2})$. The determinant in the denominator is the square root of the Berezinian (Jacobian in superanalysis) resulting from a diagonalization of the supermatrix $\widehat{U}$ [77]. See Refs. [51, 78] and references therein for an intro! duction in the application of orthogonal polynomials.

For $\beta = 1, 4$ the situation is not as simple. Though the ordinary matrix H is decoupled from the supermatrix $\widehat{U}$ and no unknown group integrals make the calculation insurmountable, the square root of the superdeterminant hinders the application of orthogonal polynomial theory. The obvious way out of this dilemma is the expansion of the integral (4.22) in the matrix $\widehat{U}$. Then one can calculate each of the expansion coefficients. Since H and $\widehat{U}$ are decoupled such an expansion is trivial. The non-trivial task is to perform the integral over H to find the coefficients. We emphasize that such an expansion is finite if $k_1 = 0$ because the superdeterminant becomes a determinant in the numerator and, thus, a polynomial in $\widehat{U}$.

What is the benefit of Q, see Eq. (4.22), in particular when there is no explicit, simple expression? The advantage of the result (4.22) with the corresponding partition function in superspace is revealed when considering the correlated situation, meaning that we destroy the invariance of W under the multiplication from the right (or left) with unitary matrices by an external correlation matrix C. In contrast to the partition function in ordinary space with the probability weight (4.20) we do not encounter large group integrals (if k_1 and k_2 are small) when diagonalizing H. The resulting partition function is Eq. (4.9) where we replace the norm-dependent superfunction by the superfunction (4.22). Particularly the calculation of the level density is capable in this way for all three Dyson indices, see Refs. [13, 14] for the Gaussian ensemble.

5. Summary and Conclusions

We presented a new variant of the supersymmetry method which directly relates the probability density in ordinary space with the one in superspace via a projection formula. Thereby we briefly rederived this formula, see Eq. (2.37), for the ensembles originally included in Dyson's threefold way [41], namely real symmetric, Hermitean, and Hermitean self-dual matrices, which was first done in Ref. [43]. In a second step we extended the idea behind such a projection formula to the three chiral ensembles. Hereby we found a formula for ensembles whose invariance of the rectangular matrices

under multiplication from the right (or left) is broken, see Eq. (3.24). This formula is quite convenient for those situations when introducing empirical correlation matrices on both sides of the rectangular random matrix as it is the case in spacial-temporal correlations [56, 57, 58].

The result (3.24) is not as compact as the further simplified formula (3.35) which is only possible if we ensure the invariance of the probability density under left and right multiplication of the rectangular random matrix with unitary matrices. The supersymmetric integral in the partition function is over one of the three coset integrals depending on the Dyson index β which already play a crucial role in the standard approach with the superbosonization formula [38, 39]. Nevertheless one should not confuse our approach with the one in Refs. [38, 39].

The projection formula (2.37) for the three non-chiral ensembles agrees with the result of the generalized Hubbard-Stratonovich transformation [36, 37] in the integration domain as well as in the form of the integrand. This is not the case for the chiral ensembles where the projection formula shares the integration domain with the original superbosonization formula [38, 39] while the integrand is of a completely different form and resembles more the one of the generalized Hubbard-Stratonovich transformation [36, 37]. Therefore the projection formulas (3.24) and (3.35) for chiral ensembles represent an alternative approach to the standard supersymmetry methods in random matrix theory.

We applied the projection formula (3.35) to the relatively simple example of norm-dependent ensembles and found a quite compact and explicit dependence of the probability density in superspace on the one in ordinary space which reduces to a onefold integral (4.2). For the Gaussian case we recovered the well known chiral Lagrangian of QCD [15, 16, 17] in the microscopic limit, see Eq. (4.6). Furthermore we showed how to generalize the projection formula in the case of one-sided correlated random matrices, see Eq. (4.9). This underlines that the projection formula (3.35) is not at all restricted to rotation invariant ('isotropic' [35]) ensembles but can also cover a simple, but also the most popular kind of symmetry breaking.

Another ensemble to which we applied the projection formula (3.35) is of Lorentz (Cauchy) type. Surprisingly not only the Gaussian weight is form-invariant under mapping the probability density in ordinary space to one in superspace but also the Lorentz weight. Only the exponent of the determinant changes and has to be taken care of. With help of the representation in superspace we showed that depending on the exponent of the determinant the Lorentzian shows universal behavior in the microscopic limit or not. Hence the projection formula (3.35) provides a new tool to investigate universality issues in chiral random matrix theory, as well.

Moreover, we considered the standard application of chiral random matrix theory to QCD. With help of the projection formula (3.35) we split the chiral Lagrangian of the partially quenched theory in QCD into two parts, see Eq. (4.19). One part consists of the lowest order of the unquenched theory in the physical mesons (the pions for two flavors) which is the well-known linear term in the quark masses [15, 16, 17]. We refer to the expansion scheme of the microscopic limit (the limit of large space-time volume, $V \to \infty$, with fixed rescaled quark masses, $Vm = \text{const.}$), see Refs. [17], which is one kind of a low energy expansion. The other part represents the interaction with the source terms which are artificially introduced to generate the observables. This is some kind of a natural splitting into the physical system and the measurement. It would be quite interesting if such a splitting is also applicable to the kinetic modes of the mesons which are not included in the lowest order description by random matrix

theory. Maybe chiral perturbation theory can shed light to this.

In a fourth example we considered a probability density with a quartic potential emphasizing that the projection formula (3.35) can also deal with more complicated situations. We derived a representation of the probability density in superspace which is still an integral over a Hermitian matrix H, see Eq. (4.22). However the coupling of the ordinary matrix H with the supermatrix $\widehat{U}$ is in an invariant way, meaning that H and $\widehat{U}$ are independently invariant under unitary transformations. In the case of the Dyson index $\beta = 2$ this allows to apply the machinery of orthogonal polynomials [51, 78] and other techniques [76, 77] (whereby Ref. [76] is not limited to $\beta = 2$) to calculate an explicit expression of the probability density Q, see Eq. (4.22). For an elaborate presentation of the calculation methods we refer to Ref. [5]. In the other two cases $\beta = 1, 4$ the situation is not as simple. Nevertheless we showed how to circumvent unknown group integrals via the projection formula (3.35) in the supersymmetry method if one considers one-sided correlated rectangular random matrices drawn from an ensemble with a quartic potential. For the Gaussian case two of the authors already applied the supersymmetry method to correlated Wishart ensemble and derived a compact expression for the level density, see Refs. [13, 14]. The projection formula (3.35) opens a way to perform this calculation for other probability densities as well.

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Appendix A. The derivation of Eq. (4.17)

Considering the partially quenched partition function (4.16) with the probability density (4.15), the integral that has to be performed via the projection formula (3.35) is

$$
\begin{aligned}
Q(\widehat{U}) \propto & \frac{1}{\int d[W] \exp\left(-n \operatorname{tr} W^\dagger W/\widetilde{\gamma}\right) \prod_{j=1}^{N_\mathrm{f}} \det(WW^\dagger + m_j^2 \mathbb{1}_{\gamma n})} \\
& \times \int d[\widehat{W}_1] \int d[W_1] \exp\left(-\frac{n}{\widetilde{\widetilde{\gamma}}}[\operatorname{tr} \widehat{W}_1 \widehat{W}_1^\dagger + \operatorname{tr} W_1 W_1^\dagger + \operatorname{str} \widehat{U}]\right) \\
& \times \prod_{j=1}^{N_\mathrm{f}} \operatorname{sdet}\left(\begin{bmatrix} \widehat{W}_1 \widehat{W}_1^\dagger + W_1 W_1^\dagger & W_1 \sqrt{\widehat{U}} \\ \sqrt{\widehat{U}} W_1^\dagger & \widehat{U} \end{bmatrix} + m_j^2 \mathbb{1}_{\gamma n + \gamma\widetilde{\gamma} k_2 | \gamma\widetilde{\gamma} k_2}\right) \\
= & \frac{\exp\left[-n \operatorname{str} \widehat{U}/\widetilde{\gamma}\right] \prod_{j=1}^{N_\mathrm{f}} \operatorname{sdet}\left(\widehat{U} + m_j^2 \mathbb{1}_{\gamma\widetilde{\gamma} k_1 | \gamma\widetilde{\gamma} k_2}\right)}{\int d[W] \exp\left(-n \operatorname{tr} W^\dagger W/\widetilde{\gamma}\right) \prod_{j=1}^{N_\mathrm{f}} \det(WW^\dagger + m_j^2 \mathbb{1}_{\gamma n})} \\
& \times \int d[\widehat{W}_1] \int d[W_1] \exp\left(-\frac{n}{\widetilde{\widetilde{\gamma}}}[\operatorname{tr} \widehat{W}_1 \widehat{W}_1^\dagger + \operatorname{tr} W_1 W_1^\dagger]\right)
\end{aligned}
\tag{A.1}
$$

$$\times \prod_{j=1}^{N_{\rm f}} \det\left(\widehat{W}_1\widehat{W}_1^\dagger + m_j^2 W_1(\widehat{U}+m_j^2\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2})^{-1}W_1^\dagger + m_j^2\mathbb{1}_{\gamma n+\gamma\widetilde{\gamma}(k_2-k_1)}\right).$$

In the second step we pushed out the block matrices $\widehat{U}+m_j^2\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}$ for each mass m_j. The product of determinants can be rewritten as

$$\det\left(\widehat{W}_1\widehat{W}_1^\dagger + m_j^2 W_1(\widehat{U}+m_j^2\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2})^{-1}W_1^\dagger + m_j^2\mathbb{1}_{\gamma n+\gamma\widetilde{\gamma}(k_2-k_1)}\right) \tag{A.2}$$

$$= m_j^{-2\gamma\nu}\mathrm{sdet}^{-1}(\widehat{U}+m_j^2\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2})\,\mathrm{sdet}\left(\widetilde{W}_1^\dagger\widetilde{W}_1+\widehat{U}'+m_j^2\mathbb{1}_{\gamma(n+\nu)+\gamma\widetilde{\gamma}k_2|\gamma\widetilde{\gamma}k_2}\right)$$

with

$$\widetilde{W}_1 = \begin{bmatrix} \widehat{W}_1 & W_1 \end{bmatrix} \tag{A.3}$$

such that $\widetilde{W}_1^\dagger \in \mathrm{Gl}^{(\beta)}(n+\nu+\widetilde{\gamma}k_2|\widetilde{\gamma}k_2; n+\widetilde{\gamma}(k_2-k_1))$ and with the supermatrix

$$\widehat{U}' = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{U} \end{bmatrix}. \tag{A.4}$$

The superfunction Q reads

$$Q(\widehat{U}) \propto \frac{\exp\left[-n\mathrm{str}\,\widehat{U}/\widetilde{\gamma}\right]}{\int d[W]\exp\left(-n\,\mathrm{tr}\,W^\dagger W/\widetilde{\gamma}\right)\prod_{j=1}^{N_{\rm f}}\det(W^\dagger W+m_j^2\mathbb{1}_{\gamma(n+\nu)})}\int d[\widehat{W}_1] \tag{A.5}$$

$$\times\exp\left(-\frac{n}{\widetilde{\gamma}}\mathrm{str}\,\widetilde{W}_1^\dagger\widetilde{W}_1\right)\mathrm{sdet}^{1/(\gamma\widetilde{\gamma})}\left([\widetilde{W}_1^\dagger\widetilde{W}_1+\widehat{U}']\otimes\mathbb{1}_{\gamma\widetilde{\gamma}N_{\rm f}}+\mathbb{1}_{\gamma(n+\nu)+\gamma\widetilde{\gamma}k_2|\gamma\widetilde{\gamma}k_2}\otimes m^2\right).$$

The integral over the supermatrix $\widetilde{W}_1$ resembles the partition function (4.8) with an external matrix $\widehat{U}'$. One can easily show that the projection formula (3.35) can be generalized to a partition function with rotation invariant probability density in superspace. Thus we apply the projection formula for norm-dependent ensembles, see Eq. (4.9), to replace the dyadic supermatrix $\widetilde{W}_1^\dagger\widetilde{W}_1$ with a $\gamma\widetilde{\gamma}N_{\rm f}\times\gamma\widetilde{\gamma}N_{\rm f}$ unitary matrix $U_\pi \in \mathrm{Herm}_{\odot}^{(\beta)}(0|\gamma N_{\rm f}) = \mathrm{U}^{(4/\beta)}(\gamma N_{\rm f})$ in the second tensor space in the superdeterminant (A.5). We recall the definitions (2.5) and (2.18). The subscript "π" of the unitary matrix U refers to physical mesons as they indeed agree with the mesons (Goldstone bosons) in the microscopic limit. For $N_{\rm f}=2$! the mesons are the pions which are usually denoted by π.

Employing the projection formula (3.35) to the expression (A.5) the superfunction Q takes the form

$$\begin{aligned} Q(\widehat{U}) \propto{}& \frac{\exp\left[-n\mathrm{str}\,\widehat{U}/\widetilde{\gamma}\right]}{\int d\mu(U_\pi)\exp\left(n\,\mathrm{tr}\,U_\pi/\widetilde{\gamma}\right)\det^{(n+\nu)/\widetilde{\gamma}}(U_\pi+m^2)\det^{-n/\widetilde{\gamma}}U_\pi} \\ &\times\int d\mu(U_\pi)\exp\left(-\frac{n}{\widetilde{\gamma}}\,\mathrm{tr}\,U_\pi\right)\det{}^{-n/\widetilde{\gamma}-k_2+k_1}U_\pi \\ &\times\mathrm{sdet}^{1/(\gamma\widetilde{\gamma})}\left(\widehat{U}'\otimes\mathbb{1}_{\gamma\widetilde{\gamma}N_{\rm f}}+\mathbb{1}_{\gamma(n+\nu)+\gamma\widetilde{\gamma}k_2|\gamma\widetilde{\gamma}k_2}\otimes(U_\pi+m^2)\right) \\ ={}& \frac{\exp\left[-n\mathrm{str}\,\widehat{U}/\widetilde{\gamma}\right]}{\int d\mu(U_\pi)\exp\left(n\,\mathrm{tr}\,U_\pi/\widetilde{\gamma}\right)\det^{(n+\nu)/\widetilde{\gamma}}(U_\pi+m^2)\det^{-n/\widetilde{\gamma}}U_\pi} \\ &\times\int d\mu(U_\pi)\exp\left(\frac{n}{\widetilde{\gamma}}\,\mathrm{tr}\,U_\pi\right)\det{}^{\nu/\widetilde{\gamma}}U_\pi\det{}^{(n+\nu)/\widetilde{\gamma}+k_2-k_1}(\mathbb{1}_{N_{\rm f}}+m^2U_\pi^{-1}) \\ &\times\mathrm{sdet}^{1/(\gamma\widetilde{\gamma})}\left(\widehat{U}\otimes\mathbb{1}_{\gamma\widetilde{\gamma}N_{\rm f}}+\mathbb{1}_{\gamma\widetilde{\gamma}k_1|\gamma\widetilde{\gamma}k_2}\otimes(U_\pi+m^2)\right). \end{aligned} \tag{A.6}$$

We also replaced the integral in the denominator via the projection formula. The superfunction Q can be plugged into the partition function (4.16) and we find Eq. (4.17).

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Universal distribution of Lyapunov exponents for products of Ginibre matrices

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Abstract. Starting from exact analytical results on singular values and complex eigenvalues of products of independent Gaussian complex random $N \times N$ matrices also called Ginibre ensemble we rederive the Lyapunov exponents for an infinite product. We show that for a large number t of product matrices the distribution of each Lyapunov exponent is normal and compute its t-dependent variance as well as corrections in a large-t expansion. Originally Lyapunov exponents are defined for the singular values of the product matrix that represents a linear time evolution. Surprisingly a similar construction for the moduli of the complex eigenvalues yields the very same exponents and normal distributions to leading order. We discuss a general mechanism for 2×2 matrices why the singular values and the radii of complex eigenvalues collapse onto the same value in the large-t limit. Thereby we rederive Newman's triangular law which has a simple interpretation as the radial density of complex eigenvalues in the circular law and study the commutativity of the two limits $t \to \infty$ and $N \to \infty$ on the global and the local scale. As a mathematical byproduct we show that a particular asymptotic expansion of a Meijer G-function with large index leads to a Gaussian.

1. Introduction

Lyapunov exponents are useful to study the stability of dynamical systems, but they also play an important role in statistical mechanics of disordered systems, localization theory, hidden Markov models and many others areas of physics and engineering.

The problem of the determination of Lyapunov exponents is intimately related to the asymptotic properties of products of random matrices in the limit when the number of factors tends to infinity. The randomness encoded in these matrices depends on the details of the problem in question and it is usually very difficult to find the exact values of the exponents. There are however some general theorems that guide the calculations. For example it is known that the largest Lyapunov exponent of the product of a random sequence of matrices generated by a stochastic process converges

almost surely to a limiting deterministic value in the limit of infinite sequence length. For large but finite sequences the largest Lyapunov exponent is a normally distributed random variable with the variance inversely proportional to the sequence length [1].

The relevance of products of random matrices to dynamical systems and ergodic theory was realized in the sixties [2] and since then the study of matrix products has been an active field of research in probability theory [3], condensed matter physics, and statistical mechanics [4, 5, 6].

It was noticed long time ago [7, 8] that products of random matrices naturally arise in the analysis of disordered systems in statistical mechanics. As an example one can think of the transfer matrix formulation of random Ising chains [9, 10]. In this case the transfer matrices are random matrices. In the thermodynamic limit the free energy density is given by the largest Lyapunov exponent of the product of transfer matrices. Another important physical example is the localization phenomenon in electronic systems [11]. In this case the leading Lyapunov exponent is related to the inverse localization length [12, 13, 14]. Other solvable physical models can be found in Yang-Mills theories [16]. In this field unitary transfer matrices in the group U(N) find applications in calculations of the Wilson loop operator for $N \to \infty$ [15].

Products of random matrices have many practical applications in other fields as well. For instance they arise in calculations of the capacity of a sequence of multiple-input-multiple-output arrays in wireless telecommunication [17, 18, 19] and in hidden Markov models applied in stochastic inference [20], in time series analysis, speech recognition, biological sequence analysis. In hidden Markov models the Lyapunov exponents correspond to the entropy rates [21, 22]. Also in image processing [23] product matrices play an important role.

The spectrum of Lyapunov exponents gives important information on the stability and the complexity of dynamical systems [2] and their effective information dimension [24]. For this reason a great effort has been made to develop computational methods to determine Lyapunov exponents for given theoretical models or to estimate them from experimental data. Numerical methods are directly based on the analysis of the equation of motion or measurements of the expansion rates of phase space [25, 26]. Algorithms have been developed for the Lyapunov spectrum from sample time series [27]. Also analytical approximations include methods based on the weak disorder expansion [28] or properties of determinants associated with transfer matrices [29, 30].

There are only a few models where Lyapunov exponents can be calculated exactly. They usually involve products of 2×2 matrices with randomness controlled by a single random parameter where the exact expressions result from some model specific simplifications which occur during calculations. The examples include classical disordered harmonic chains [7, 31], the tight-binding Anderson model [32, 33], quantum spin chains [34, 35, 36] and random Schrödinger operators [37], see also [4, 5, 6] for reviews. Recently a general method has been worked out to derive a scaling form for the Lyapunov exponents in the continuum limit for products of 2×2 matrices close to the identity [38] based on the Iwasawa decomposition of SL(2,R) [39].

An important solvable case where one can calculate the Lyapunov exponents exactly is the product of identically distributed Gaussian random matrices with independent, identically distributed (i.i.d.) centered real entries [40]. Such matrices are usually called real Ginibre matrices. This is a special case, first of all because one can analytically derive the whole spectrum of Lyapunov exponents $\{\hat{\mu}_1, \ldots, \hat{\mu}_N\}$ for any system size N. Second, the calculation uncovers a deep connection between the spectrum and the law of large numbers [40]. The exponents are exclusively shaped by the statistics of matrix elements and not by the matrix structure. In other words the two effects do not mix. A second much more recent example where all Lyapunov exponents have been calculated are products of independent Ginibre matrices, where each factor is multiplied by a fixed positive definite matrix [41, 42]. When these constant matrices are equal to the identity the results for the real, complex, and quaternion Ginibre ensembles agree up to a scaling factor $\beta/2$ where $\beta = 1, 2, 4$ is the Dyson index.

The fact that one can derive the whole spectrum is very useful for practical purposes since the spectrum can be used to test numerical algorithms [25, 26, 27]. Moreover one can analytically calculate the limiting law for the distribution of Lyapunov exponents in the limit $N \to \infty$. For the numbers constructed from Lyapunov exponents, that we call in this paper incremental singular values, $\hat{\lambda}_n = \exp[\hat{\mu}_n]$, $n = 1, \ldots, N$, the distribution is given by the triangular law [40].

In the present work we further elaborate on the Lyapunov spectrum for the product of complex Ginibre matrices. We consider complex Ginibre matrices that are Gaussian matrices with i.i.d. complex elements. We derive an exact form of finite t corrections to the Lyapunov spectrum, where t is the number of matrices in the product. For finite t the Lyapunov exponents are random variables. We calculate the joint probability distribution for these variables. For large t it is asymptotically given by a permanent of the product of independent Gaussian functions centered at the limiting values. Thereby we determine the widths of the distributions. We also improve this Gaussian approximation by considering another approximation based on the saddle point approximation. The latter approach works even better for a product of a small number of matrices since it still incorporates asymmetric parts of the individual eigenvalue distributions and to a small extent the original level repulsion.

In addition to the Lyapunov exponents $\hat{\mu}_n$, which are related to the singular values of the product matrix, one can define the corresponding exponents $\hat{\nu}_n$ for the moduli of the complex eigenvalues. The complex eigenvalue distribution of the product of Ginibre matrices is rotationally invariant in the complex plane [43, 44]. We find that the moduli of the eigenvalues become uncorrelated random variables in the large-t limit and we determine the form of their joint probability distribution. Surprisingly, the spectrum and the joint probability distribution of these exponents is identical to that of the Lyapunov exponents, $\hat{\nu}_n = \hat{\mu}_n$ for $n = 1, \ldots, N$.

A further consequence of this observation is discussed in Section 5. The triangular law for Lyapunov exponents corresponding to the singular values found by Isopi and Newman [45] can be understood as the radial distribution of eigenvalues of the Ginibre

matrix. The fundamental reason behind this interpretation is twofold. First, our insight says that the Lyapunov exponents constructed from the singular values and from the moduli of the eigenvalues agree with each other. Second, Ginibre ensembles belong to the class of isotropic random matrix ensembles. For those ensembles the sometimes called self-averaging property of the product of isotropic matrices [46, 47] and the Haagerup-Larsen theorem [48] are known. These two properties imply that the spectral statistics of a product of independent random matrices is equal to the statistics of the power of a single matrix in the limit of large matrix dimension $N \to \infty$. After taking the root of the product matrix the level density is the one of an ordinary Ginibre matrix which is the circular law for the complex eigenvalues and is equal to the triangular law for the moduli of the eigenvalues.

The paper is organized as follows. In Section 2 we define the linear evolution given by the product of Ginibre matrices and define the corresponding Lyapunov exponents. In Section 3 we derive their joint probability density based on the singular value distribution of the product matrix for finite and large t, keeping N finite. In Section 4 we compute the joint probability density for exponents based on the moduli of complex eigenvalues for finite and large t. In Section 5 we discuss the limit $N \to \infty$ for Lyapunov exponents and show that this limit commutes with the limit $t \to \infty$ on the global scale while it does not commute on the local scale of the mean level spacing. In Section 6 we conjecture the collapse of singular and eigenvalues for general isotropic ensembles and exemplify this for $N = 2$. We conclude the paper in Section 7. In the appendices we recall some identities of Meijer G-functions, compute a particular kind of a Hankel determinant and present some further details of our calculations.

2. Linear time evolution with Ginibre matrices

Let us consider a linear discrete-time evolution of an N-dimensional system described by N complex degrees of freedom. The state of the system at time t is given by an N-dimensional vector $\vec{x}_t$. The state at $t+1$ is related to the state at time t by the following linear equation

$$\vec{x}_{t+1} = X_{t+1}\vec{x}_t, \tag{2.1}$$

with the evolution operator X_{t+1} represented by an $N \times N$ matrix. The total evolution from the initial state

$$\vec{x}_t = \Pi(t)\vec{x}_0 \tag{2.2}$$

is effectively driven by the product matrix

$$\Pi(t) \equiv X_t X_{t-1} \cdots X_1. \tag{2.3}$$

Here we are interested in X_j's being i.i.d. complex non-Hermitian random matrices. In particular we consider the case of Ginibre matrices which centered and Gaussian distributed,

$$d\mu(X_j) = dX_j \exp\left[-\operatorname{Tr} X_j^\dagger X_j\right] \tag{2.4}$$

for all $j = 1, \ldots, t$. The differential dX_j denotes the product of the differential of all independent matrix elements. Towards the end of the paper we comment on the evolution for general isotropic random matrices which are defined by the invariance of the probability measure $d\mu(X_j) = d\mu(UX_jV)$ where $U, V \in \mathrm{U(N)}$ are arbitrary unitary matrices. Isotropic matrices are sometimes called bi-unitarily invariant or rotational invariant. Ginibre matrices belong to this class.

We are interested in the large t behavior of the system, approximating a continuous time evolution. This behavior is controlled by the Lyapunov exponents which are related to the singular values of $\Pi(t)$. Let us denote the real eigenvalues of the positive matrix

$$S(t) \equiv \Pi^\dagger(t)\Pi(t) \tag{2.5}$$

by $\{s_n(t) \in \mathbb{R}_+, n = 1, \ldots, N\}$. Their square roots $\sqrt{s_n(t)}$ correspond to singular values of $\Pi(t)$. Then the Lyapunov exponents are defined as

$$\hat{\mu}_n = \lim_{t\to\infty} \frac{\ln \hat{s}_n(t)}{2t}, \tag{2.6}$$

where $\hat{s}_n(t)$ are the ordered eigenvalues of $S(t)$: $\hat{s}_1(t) \leq \hat{s}_2(t) \leq \ldots \leq \hat{s}_N(t)$. Throughout this paper we denote ordered (increasing) sequences like $\hat{s}_n$ or $\hat{\mu}_n$ by a hat.

In many physical situations the number of time steps in the evolution is large but finite. Hence it is interesting to study finite size corrections to the limiting values, and the rate of convergence to these values. Thus we want to address the question how this limit is realized when t tends to infinity ($t \gg 1$). Our focus lies on the corresponding quantities for finite t

$$\mu_n(t) \equiv \frac{\ln s_n(t)}{2t} \ , \tag{2.7}$$

which we call finite t Lyapunov exponents, $\mu_n(t) \in \mathbb{R}$ for $n = 1, \ldots, N$. In the limit $t \to \infty$, after ordering, they become the standard Lyapunov exponents: $\hat{\mu}_n = \lim_{t\to\infty} \hat{\mu}_n(t)$. We look for a probabilistic law that governs the distribution of the finite t Lyapunov exponents, or equivalently their joint probability density $P_N^{(t)}(\mu_1, \ldots, \mu_N)$ for finite t and N. Given the recent progress on the joint distribution of singular values (and complex eigenvalues) for a finite product of $N \times N$ Ginibre matrices for finite t and N this can be easily calculated, and the limits $t \to \infty$ and subsequently $N \to \infty$ can be taken.

3. Lyapunov exponents from singular values

The initial point of our calculations is an exact expression for the joint probability distribution of real eigenvalues of the matrix $S(t)$ (2.5) at finite N and t [18, 19],

$$P_N^{(t)}(s_1, \ldots, s_N) ds_1 \cdots ds_N = \frac{ds_1 \cdots ds_N}{N! \prod_{a=1}^N \Gamma^{t+1}(a)} \Delta_N(s) \det \left[G_{0,t}^{t,0} \left(\begin{matrix} - \\ 0,\ldots,0,a-1 \end{matrix} \middle| \ s_b \right) \right]_{1\leq a,b\leq N}, \tag{3.1}$$

where $\Delta_N(s)$ is the Vandermonde determinant

$$\Delta_N(s) = \det \left[s_a^{b-1} \right]_{1\leq a,b\leq N} = \prod_{1\leq a<b\leq N} (s_b - s_a). \tag{3.2}$$

The function $G^{t,0}_{0,t}\left(\begin{smallmatrix}-\\ 0,\ldots,0,a-1\end{smallmatrix}\middle|\, s\right)$ is a particular case of the Meijer G-function (A.1) whose properties and definition are recalled in Appendix A. As any special function, it possesses many helpful properties which facilitate calculations. For simplicity we drop the explicit t-dependence of the singular values and of the Lyapunov exponents in the ensuing discussions as it will be clear from the context if t is finite or infinite.

The road map to find the large t asymptotics is the following. In subsection 3.1 we find a determinantal representation of the joint probability distribution of Lyapunov exponents made of one-point probability distributions. We calculate the moments of these one-point distributions. The cumulant expansion yields an asymptotic expansion to any order in $1/t$. This result is discussed in detail for large t, in subsection 3.2. Moreover, we compare the cumulant expansion with a saddle point approximation which also incorporates a residual level repulsion as well as an asymmetric part of the distributions of the individual Lyapunov exponents. In subsection 3.3 we come back to the discussion of the corresponding singular values $\exp[\mu_j]$ which we call incremental singular values since they are the average contribution to the total singular value of each single random matrix in the product $\Pi(t)$.

3.1. Reduction to "decoupled" random variables

The joint probability distribution $P_N^{(t)}(\mu_1,\ldots,\mu_N)$ for Lyapunov exponents can be directly read off from eq. (3.1) by the change of variables $s_n \equiv \exp(2t\mu_n)$,

$$P_N^{(t)}(\mu_1,\ldots,\mu_N)d\mu_1\cdots d\mu_N = \frac{(2t)^N d\mu_1\cdots d\mu_N}{N!\prod_{a=1}^N \Gamma^{t+1}(a)} \det_{1\le a,b\le N}\left[\exp(2tb\mu_a)\right] \times \det_{1\le a,b\le N}\left[G^{t,0}_{0,t}\left(\begin{smallmatrix}-\\ 0,\ldots,0,a-1\end{smallmatrix}\middle|\, \exp(2t\mu_b)\right)\right]. \tag{3.3}$$

The change of variables introduces a Jacobian which yields for each variable μ_n the exponential factor $ds_n = 2te^{t\mu_n}d\mu_n$. These factors have been absorbed in the last equation in the Vandermonde determinant $\det\left[\exp((b-1)2t\mu_a)\right]$ by replacing $(b-1)\to b$. The first determinant in eq. (3.3) can be expanded as

$$(2t)^N \det_{1\le a,b\le N}\left[\exp(2tb\mu_a)\right] = \sum_{\omega\in S_N}\mathrm{sign}(\omega)\prod_{b=1}^N 2t\exp(2t\omega(b)\mu_b), \tag{3.4}$$

where S_N denotes the group of permutations of N elements and "sign" is the sign function which is $+1$ for even permutations and -1 for odd ones. The factors $2t\exp[2t\omega(b)\mu_b]$ can be absorbed into the second determinant

$$P_N^{(t)}(\mu_1,\ldots,\mu_N) = \frac{1}{N!\prod_{a=1}^N \Gamma^{t+1}(a)} \times \sum_{\omega\in S_N}\mathrm{sign}(\omega)\det_{1\le a,b\le N}\left[2t\exp(2t\omega(b)\mu_b)G^{t,0}_{0,t}\left(\begin{smallmatrix}-\\ 0,\ldots,0,a-1\end{smallmatrix}\middle|\, \exp(2t\mu_b)\right)\right]. \tag{3.5}$$

By virtue of eq. (A.3) the last expression can be cast into the form

$$P_N^{(t)}(\mu_1,\ldots,\mu_N) = \frac{1}{N!\prod_{a=1}^N \Gamma^{t+1}(a)} \tag{3.6}$$

$$\times \sum_{\omega\in S_N} \mathrm{sign}(\omega) \det_{1\le a,b\le N} \left[2tG^{t,0}_{0,t} \left({}^{-}_{\omega(b),\ldots,\omega(b),a+\omega(b)-1} \right| \exp(2t\mu_b) \right) \right].$$

The skew-symmetry of the determinant under permutations of its rows and columns allows us to absorb the prefactor $\mathrm{sign}(\omega)$ into the determinant via rearranging the rows. Hence we end up with

$$P_N^{(t)}(\mu_1,\ldots,\mu_N) = \frac{1}{N!\prod_{a=1}^N \Gamma^{t+1}(a)} \sum_{\omega\in S_N} \det_{1\le a,b\le N} \left[F_{ab}\left(\mu_{\omega(b)}\right) \right], \tag{3.7}$$

where

$$F_{ab}(\mu) \equiv 2tG^{t,0}_{0,t} \left({}^{-}_{b,\ldots,b,a+b-1} \right| \mathrm{e}^{2t\mu}). \tag{3.8}$$

Thus the problem is reduced to the analysis of the function $F_{ab}(\mu)$. By construction this function is positive semi-definite. With help of the integral identity (A.2), F_{ab} can be normalized such that the function

$$f_{ab}(\mu) \equiv \frac{F_{ab}(\mu)}{\int F_{ab}(\mu')d\mu'} = \frac{F_{ab}(\mu)}{\Gamma^{t-1}(b)\,\Gamma(a+b-1)} \tag{3.9}$$

can be interpreted as a probability density for a single random variable. Replacing $F_{ab}(\mu)$ with its normalized version $f_{ab}(\mu)$ the joint probability distribution reads

$$P_N^{(t)}(\mu_1,\ldots,\mu_N) = \frac{1}{N!\prod_{a=1}^N \Gamma^{2}(a)} \sum_{\omega\in S_N} \det_{1\le a,b\le N} \left[\Gamma(a+b-1) f_{ab}(\mu_{\omega(b)}) \right]. \tag{3.10}$$

In passing from eq. (3.7) to eq. (3.10) we have pulled the factor $\prod_{a=1}^N \Gamma^{t-1}(a)$ out of the determinant. This factor cancels the corresponding prefactor in eq. (3.7) leaving the product of the second powers in front of the determinant in eq. (3.10).

Using the cumulant expansion we argue in the next subsection that the probability densities $f_{ab}(\mu)$ can be approximated by Gaussian functions in the limit $t\to\infty$. Therefore let us define the moment generating function

$$M_{ab}(\vartheta) \equiv \int_{-\infty}^{+\infty} d\mu\ \exp(\mu\vartheta) f_{ab}(\mu) = \sum_{n=0}^{\infty} \frac{\vartheta^n}{n!} \langle \mu^n \rangle_{ab}\ . \tag{3.11}$$

where $\langle \mu^n \rangle_{ab} \equiv \int_{-\infty}^{+\infty} d\mu f_{ab}(\mu)\mu^n$ are the moments. This moment generating function can be calculated with help of eq. (A.2),

$$M_{ab}(\vartheta) = \frac{\Gamma^{t-1}\left(b+\vartheta/(2t)\right)\Gamma\left(a+b-1+\vartheta/(2t)\right)}{\Gamma^{t-1}(b)\,\Gamma(a+b-1)}. \tag{3.12}$$

The expansion in ϑ at $\vartheta=0$ yields the moments $\langle\mu^n\rangle_{ab}$. The logarithm of the moment generating function is the cumulant generating function

$$g_{ab}(\vartheta) \equiv \ln\left(M_{ab}(\vartheta)\right) = (t-1)\ln\left(\frac{\Gamma\left(b+\vartheta/(2t)\right)}{\Gamma(b)}\right) + \ln\left(\frac{\Gamma\left(a+b-1+\vartheta/(2t)\right)}{\Gamma(a+b-1)}\right). \tag{3.13}$$

The coefficients of the corresponding Taylor series of $g_{ab}(\vartheta)$ at $\vartheta=0$ are the cumulants $\kappa^{(n)}_{ab}$,

$$g_{ab}(\vartheta) \equiv \sum_{n=1}^{\infty} \frac{\vartheta^n}{n!}\kappa^{(n)}_{ab} = \sum_{n=1}^{\infty} \frac{\vartheta^n}{(2t)^{n-1}n!} \left(\frac{\psi^{(n-1)}(b)}{2} + \frac{\psi^{(n-1)}(a+b-1)-\psi^{(n-1)}(b)}{2t} \right). \tag{3.14}$$

Hereby we employed the definition of the digamma function and its derivatives,

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x), \quad \psi^{(n)}(x) = \frac{d^n}{dx^n}\psi(x) \quad \left(\psi^{(0)}(x) \equiv \psi(x),\ \psi^{(1)}(x) \equiv \psi'(x)\right). \tag{3.15}$$

The first cumulant (=first moment) corresponds to the mean value $\kappa_{ab}^{(1)} = \langle\mu\rangle_{ab} = \int d\mu f_{ab}(\mu)\mu$ and is equal to

$$m_{ab} \equiv \kappa_{ab}^{(1)} = \frac{\psi(b)}{2} + \frac{\psi(a+b-1)-\psi(b)}{2t}\ . \tag{3.16}$$

The second cumulant corresponds to the variance $\kappa_{ab}^{(2)} = \int d\mu f_{ab}(\mu)\,(\mu - m_{ab})^2$ and takes the value

$$(\sigma_{ab})^2 \equiv \kappa_{ab}^{(2)} = \frac{1}{2t}\left(\frac{\psi'(b)}{2} + \frac{\psi'(a+b-1)-\psi'(b)}{2t}\right). \tag{3.17}$$

We emphasize that so far all results are exact for finite t.

3.2. Large t limit

We apply the standard argument based on the analysis of the large-t behavior of cumulants to show that $f_{ab}(\mu)$ can be approximated by a Gaussian function for large t. Thereby we have first to center the distribution $f_{ab}(\mu)$ and normalize its second moment. The exact limit $t \to \infty$ will yield a Gaussian. This limit justifies to replace $f_{ab}(\mu)$ by a Gaussian centered at m_{ab} and with the standard deviation σ_{ab}.

For this purpose we define the standardized random variable $\mu_* \equiv (\mu - m_{ab})/\sigma_{ab}$. Thereby we denote standardized quantities by $*$ in this and the next section. The random variable μ_* is distributed as $f_{*ab}(\mu_*) \equiv \sigma_{ab} f(\mu_*\sigma_{ab} + m_{ab})$. The same notation is applied for cumulants. By construction, the standardized mean is $m_{*ab} = 0$ and the standardized variance is $\sigma_{*ab} = 1$. The higher standardized cumulants are

$$\kappa_{*ab}^{(n)} \equiv \frac{\kappa_{ab}^{(n)}}{(\sigma_{ab})^n} \sim t^{1-n/2} \longrightarrow 0\ ,\ n = 3, 4, \ldots \tag{3.18}$$

They tend to zero when t goes to infinity. Therefore the standardized cumulant generating function is in the limit $t \to \infty$,

$$\lim_{t\to\infty} g_{*ab}(\vartheta) = \frac{1}{2}\vartheta^2. \tag{3.19}$$

By analytic continuation to imaginary values $\vartheta = \imath\omega$ we get $\lim_{t\to\infty} g_{*ab}(\imath\omega) = -\omega^2/2$ and hence $\lim_{t\to\infty} M_{*ab}(\imath\omega) = \exp(-\omega^2/2)$. The inverse Fourier transform for the moment generating function yields the limit

$$\lim_{t\to\infty} f_{*ab}(\mu) = \frac{1}{\sqrt{2\pi}}\ \exp\left[-\frac{\mu^2}{2}\right]. \tag{3.20}$$

Inverting the process of standardization $f_{ab}(\mu) = \sigma_{ab}^{-1} f_{*ab}\left((\mu - m_{ab})/\sigma_{ab}\right)$ we get the following asymptotic expansion

$$f_{ab}(\mu) = \frac{2t\ G_{0,\,t}^{\,t,\,0}\left(\begin{smallmatrix}-\\ b,\ldots,b,a+b-1\end{smallmatrix}\middle|\ \exp(2t\mu)\right)}{\Gamma^{t-1}(b)\,\Gamma(a+b-1)} \overset{t\gg 1}{\approx} \frac{1}{\sqrt{2\pi(\sigma_{ab})^2}}\exp\left(-\frac{(\mu-m_{ab})^2}{2(\sigma_{ab})^2}\right)\ , \tag{3.21}$$

with m_{ab} and σ_{ab} given by eqs. (3.16) and (3.17). In other words, for large t we can replace $f_{ab}(\mu)$ in (3.10) by the Gaussian function eq. (3.21). Here we have also reinserted the definition of $f_{ab}(\mu)$ from eqs. (3.9) and (3.8) in order to stress that this is the first main result of this section, namely the asymptotic expansion of a Meijer G-function in the double scaling limit of large argument and large index. We are not aware of such a result in the literature. In particular it is different from the well-known large argument expansion, cf. [49].

The expression (3.10) can be further simplified for large $t \gg 1$ since the mean value $m_{ab} \longrightarrow m_b$, cf. eq. (3.16), and the variance $(\sigma_{ab})^2 \longrightarrow (\sigma_b)^2$, cf. eq. (3.17), asymptotically depend on a single index

$$m_b \equiv \frac{\psi(b)}{2} \ , \ \sigma_b^2 \equiv \frac{\psi'(b)}{4t} \tag{3.22}$$

and hence $f_{ab}(\mu) \longrightarrow f_b(\mu)$ with

$$f_b(\mu) \equiv \frac{1}{\sqrt{2\pi\sigma_b^2}} \exp\left(-\frac{(\mu - m_b)^2}{2\sigma_b^2}\right), \tag{3.23}$$

which was known for $b = N$ [3]. Since these functions are independent of the index a, after replacing $f_{ab}(\mu_{\varepsilon(b)})$ by $f_b(\mu_{\varepsilon(b)})$ we can pull the factors $f_b(\mu_{\varepsilon(b)})$ out the determinant in eq. (3.10). This yields

$$\begin{aligned} P_N^{(t)}(\mu_1, \ldots, \mu_N) &\overset{t\gg 1}{\approx} \frac{\det_{1\le a,b\le N}\left[\Gamma(a+b-1)\right]}{N!\prod_{a=1}^N \Gamma^2(a)} \sum_{\varepsilon\in S_N} \prod_{b=1}^N f_b\left(\mu_{\varepsilon(b)}\right) \\ &= \frac{1}{N!}\mathrm{per}_{1\le a,b\le N}\left[f_b(\mu_a)\right]. \end{aligned} \tag{3.24}$$

Here the sum over permutations without signs is equal to the definition of the permanent, $\mathrm{per}_{1\le a,b\le N}\left[f_b(\mu_a)\right]$. The prefactor simplifies to $1/N!$ since

$$\det_{1\le a,b\le N}\left[\Gamma(a+b-1)\right] = \prod_{a=1}^N \Gamma^2(a) \ , \tag{3.25}$$

as recalled in Appendix B.

Let us state the main result of this section in its explicit form which is the joint probability distribution for large t,

$$\begin{aligned} P_N^{(t)}(\mu_1, \ldots, \mu_N) &\overset{t\gg 1}{\approx} \frac{1}{N!}\mathrm{per}_{1\le a,b\le N}\left[\sqrt{\frac{2t}{\pi\psi'(b)}} \exp\left(-t\frac{(2\mu_a - \psi(b))^2}{2\psi'(b)}\right)\right] \\ &\equiv P_N(\mu_1, \ldots, \mu_N). \end{aligned} \tag{3.26}$$

The limiting joint probability distribution sustains its invariance under permutations of the indices, $P_N(\mu_1, \ldots, \mu_N) = P_N(\mu_{\omega(1)}, \ldots, \mu_{\omega(N)})$. More explicitly, the joint probability density is a symmetrized product of one-point functions or densities, which means in physical language that it describes a system of N independent, non-interacting, indistinguishable bosons. Starting from the determinantal process of the singular values the appearance of a permanent is somewhat surprising, whereas it quite naturally arises

for complex eigenvalues after integrating over the angles, see e.g. in [50, 51]. We will come back to this point at the end of section 4.

Note that the dependence of $P_N^{(t)}$ on t appears only through the widths of the Gaussian peaks. Their positions are independent of t in this approximation.

The density defined as

$$\rho_N(\mu) \equiv \int d\mu_2 \dots d\mu_N P_N(\mu, \mu_2, \dots, \mu_N) \tag{3.27}$$

is in our case

$$\rho_N(\mu) = \frac{1}{N}\sum_{b=1}^{N} f_b(\mu) = \frac{1}{N}\sum_{b=1}^{N} \frac{1}{\sqrt{2\pi\sigma_b^2}} \exp\left(-\frac{(\mu - m_b)^2}{2\sigma_b^2}\right). \tag{3.28}$$

When t increases the peaks become more narrow and, eventually in the limit $t \to \infty$, the Gaussian peaks turn into Dirac delta functions and we recover the deterministic laws [40, 41] for the Lyapunov exponents $\hat{\mu}_b = \psi(b)/2$,

$$\lim_{t\to\infty} \rho_N(\mu) = \frac{1}{N}\sum_{b=1}^{N} \delta\left(\mu - \frac{\psi(b)}{2}\right). \tag{3.29}$$

Employing Newman's argument [40] one can show that the positions of the peaks for general Dyson index $\beta = 1, 2, 4$ are given by $\psi(\beta b/2)/2$ with $b \in \mathbb{N}$. Thus the positions we calculated fit into the results obtained for products of real Ginibre matrices by Newman [40] and agree with the more general recent result by Forrester [41] who considered complex Ginibre matrices multiplied by a fixed positive definitive matrix. Forrester's work was extended by Kargin [42] to $\beta = 1, 4$. Let us emphasize that our result (3.28) gives finite-t corrections to this deterministic law. Moreover we stress that the same limit has a corresponding consequence for the Meijer G-functions for the individual peaks, when taking the limit $t \to \infty$,

$$\lim_{t\to\infty} \frac{2tG_{0,\,t}^{t,\,0}\left(\begin{smallmatrix}-\\ b,\dots,b,a+b-1\end{smallmatrix}\middle|\, \exp(2t\mu)\right)}{\Gamma^{t-1}(b)\,\Gamma(a+b-1)} = \delta\left(\mu - \frac{\psi(b)}{2}\right) \tag{3.30}$$

and

$$\lim_{t\to\infty} \frac{2t\sigma_{ab}G_{0,\,t}^{t,\,0}\left(\begin{smallmatrix}-\\ b,\dots,b,a+b-1\end{smallmatrix}\middle|\, \exp(2t(\sigma_{ab}\mu_* + m_{ab}))\right)}{\Gamma^{t-1}(b)\,\Gamma(a+b-1)} = \frac{1}{\sqrt{2\pi}}\, \exp\left[-\frac{\mu^2}{2}\right]. \tag{3.31}$$

Already for finite but sufficiently large t when the peaks cease to overlap, each Gaussian peak $f_b(\mu)$, see eq. (3.23), can be identified as a finite size distribution of the $(N-b+1)$-th largest Lyapunov exponent $\hat{\mu}_b$. Due to the recursion $\psi(b+1) = \psi(b) + 1/b$ the distance between neighboring peaks is $m_{b+1} - m_b = 1/(2b)$ and the sum of their widths is $\sigma_{b+1} + \sigma_b \approx 1/\sqrt{bt}$. So the peaks separate when $(m_{b+1} - m_b) \gg (\sigma_{b+1} + \sigma_b)$ implying $t \gg 4b$. Thus, for the system with N degrees of freedom all peaks get separated for $t \gg 4N$. Note that the positions m_b and the widths σ_b are independent of N. When N increases, just new peaks appear in the distribution while the old ones neither change in shape nor shift their positions.

Let us study the quality of the approximation that has led us to eq. (3.26). In the derivation of the asymptotic form (3.26) for large t we used the fact that the functions

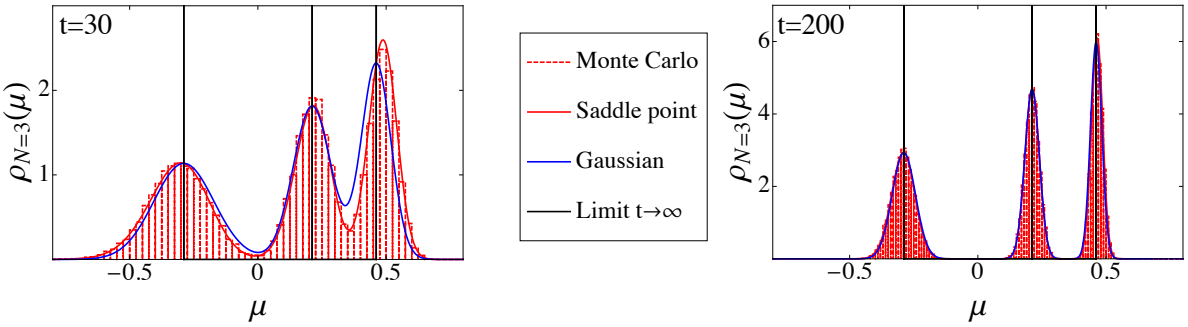

Figure 1. Comparison of the density of Lyapunov exponents $\rho_{N=3}(\mu)$ given in the Gaussian approximation (3.28) (blue curve), in the saddle point approximation (3.34) (red curve) and generated by Monte Carlo simulations (red histogram, ensemble size = 10000 product matrices). We consider products of $t = 30$ (left plot) and of $t = 200$ (right plot) matrices. The peaks (black vertical lines) are located at $\mu = \psi(b)/2$ for $b = 1, 2, 3$, which are approximately equal to $\{-0.29, 0.21, 0.46\}$. Note that the Gaussian approximation yields a good agreement only if t is large enough. But even then the deviations become visible for larger Lyapunov exponents. The saddle point approximation works better since it incorporates lower order corrections. Nevertheless also the saddle point approximation has its limits explaining the small, but remaining deviations from the numerics.

$f_{ab}(\mu)$ can be approximated by Gaussian functions (3.21) and that their mean values m_{ab} and their variances $(\sigma_{ab})^2$ asymptotically only depend on a single index b, see eq. (3.22), if one neglects $1/t$ terms. The 1/t terms have a twofold effect on the shape of the density. First, the positions and widths of the peaks solely resulting from the single random variable distributions $f_b(\mu)$ weakly dependent on t. Second a repulsion between peaks is introduced due to the determinant in eq. (3.7). We illustrate these two effects in Fig. 1 for the level density where we compare the asymptotic formula (3.28) and a saddle point approximation of $f_{ab}(\mu)$ for the inverse Fourier transform of the moment generating function (3.11),

$$f_{ab}(\mu) \overset{t\gg 1}{\approx} \sqrt{\frac{2t}{\pi\psi'(\vartheta_0(\mu))}} \frac{\Gamma^{t-1}(\vartheta_0(\mu))\Gamma(a-1+\vartheta_0(\mu))}{\Gamma^{t-1}(b)\Gamma(a+b-1)} \exp[-2t\mu(\vartheta_0(\mu)-b)]$$
$$\equiv \frac{h_{ab}(\mu)}{\Gamma(a+b-1)}, \tag{3.32}$$

with

$$\vartheta_0(\mu) = \int_0^\infty dy \Theta(2\mu - \psi(y)) \tag{3.33}$$

and Θ being the Heaviside function. This approximation is derived in Appendix C. Note that in the large t limit the distribution h_{ab} indeed becomes the Gaussian (3.23) and independent of the index a. The level density in the approximation (3.32) is

$$\rho_N^{(t,\text{Saddle})}(\mu) \equiv \frac{1}{N\prod_{a=1}^N \Gamma(a)} \sum_{j,l=1}^N (-1)^{l+j} \det_{\substack{1\le a,b\le N \\ a\neq j, b\neq l}} [\Gamma(a+b-1)]\, h_{jl}(\mu)$$

$$= \frac{1}{N}\sum_{j,l=1}^{N}(-1)^{l+j}\left(\sum_{k=0}^{N-1}\frac{(k!)^2}{\Gamma(k-j+2)\Gamma(k-l+2)}\right)\frac{h_{jl}(\mu)}{[(j-1)!(l-1)!]^2}. \quad (3.34)$$

Hereby we integrated over all but one Lyapunov exponents, $\mu_1, \ldots, \mu_{N-1}$, and we expanded the determinant (3.10) in the columns and rows where the remaining distribution $f_{ab}(\mu) \approx h_{ab}(\mu)$ stands. Note that f_{ab} as well as h_{ab} are normalized. The cofactor of the Hankel determinant (3.25) is calculated in Appendix B.

The main conclusion from the comparison in Fig. 1 is that the corrections do not have any significant effect on the shape of the distribution when the peaks are separated. In particular for the smallest singular values this requirement is often satisfied. Nevertheless the corrections can become quite important for $t \approx N$ up to $10N$ in which case the saddle point approximation (3.32) is better suited. For the largest eigenvalues the effect of these corrections is the strongest.

In Fig. 1 we compare our analytical results with Monte-Carlo simulations for 3×3 product matrices, too. Within the numerical accuracy the agreement is quite good for the Gaussian approximation (3.28) for $t = 200$ and becomes better for the saddle point approximation (3.34) already at $t = 30$.

3.3. Incremental singular values

We close this section by going back to the singular values because in some physical situations it is more convenient to use them rather than Lyapunov exponents. Consider the t-th root of the matrix $S(t)$,

$$\Lambda(t) = \left(\Pi^\dagger(t)\Pi(t)\right)^{1/(2t)}, \quad (3.35)$$

in contrast to eq. (2.5). We define incremental singular values as

$$\lambda_n(t) \equiv \exp(\mu_n(t)) = s_n^{1/(2t)}(t)\ , \quad (3.36)$$

which correspond to the real positive eigenvalues of the matrix $\Lambda(t)$. Intuitively, the incremental singular values $\lambda_n(t)$ give the typical incremental contraction or expansion factors for the configuration space under a single average time step of the evolution. Of course they contain exactly the same information as the Lyapunov exponents. Their joint probability distribution is obtained from that for the Lyapunov exponents by the simple change of variables in eq. (3.36) inserted in eq. (3.3). Using eq. (3.10) this gives

$$\begin{aligned} P_N^{(t)}(\lambda_1, \ldots, \lambda_N)d\lambda_1 \cdots d\lambda_N &= \lambda_1^{-1} \cdots \lambda_N^{-1} P_N^{(t)}\left(\mu_1 = \ln\lambda_1, \ldots, \mu_n = \ln\lambda_N\right) d\mu_1 \cdots d\mu_N \\ &= \frac{1}{N!\prod_{a=1}^{N}\Gamma^2(a)}\sum_{\varepsilon\in S_N}\det_{1\le a,b\le N}\left[\Gamma(a+b-1)\Phi_{ab}(\lambda_{\varepsilon(b)})d\lambda_{\varepsilon(b)}\right], \end{aligned} \quad (3.37)$$

where

$$\Phi_{ab}(\lambda) = \frac{1}{\lambda}f_{ab}(\ln\lambda)\ . \quad (3.38)$$

For large t when $f_{ab}(\mu)$ is approximated by normal distributions, $\Phi_{ab}(\lambda)$ can be approximated by log-normal distributions. Otherwise everything works exactly in the

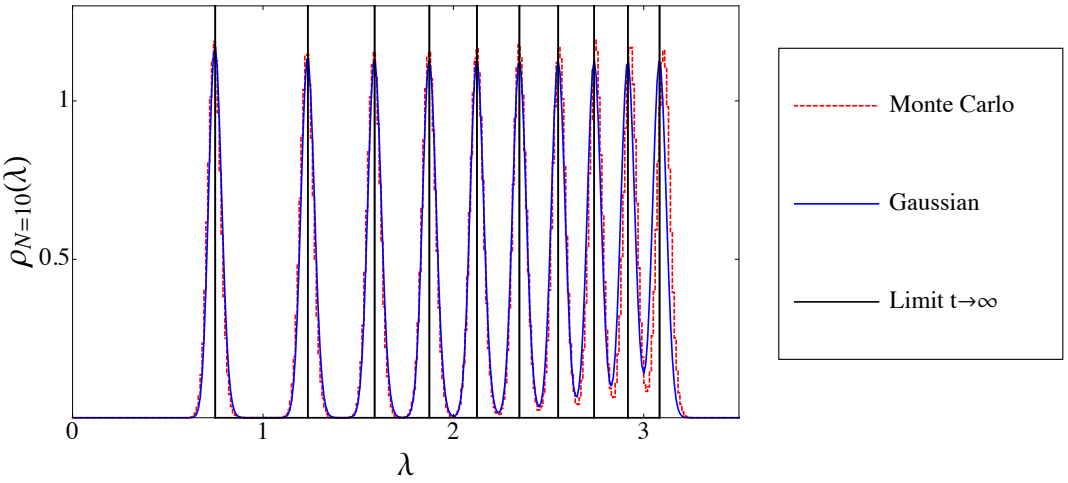

Figure 2. Shown is the comparison of the analytical prediction (3.41) (blue curve) and Monte-Carlo simulations (red dashed histogram, ensemble size = 1000 product matrices) of the density of incremental singular values $\rho_{N=10}(\lambda)$. The number of matrices multiplied is $t = 200$. The sharp peaks appearing for $t \to \infty$ are shown by black vertical lines at the positions $\exp[\psi(b)/2]$, $b = 1, \ldots, 10$. The deviation increases for larger singular values as expected since the overlap of the peaks becomes stronger.

same way as for Lyapunov exponents. In particular, when t is large enough to neglect the $1/t$ corrections, we obtain the counterpart of eq. (3.26)

$$P_N^{(t)}(\lambda_1, \ldots, \lambda_N) \overset{t\gg 1}{\approx} \frac{1}{N!} \mathrm{per}_{1\le a,b\le N} \left[\Phi_b(\mu_a)\right] \tag{3.39}$$

with

$$\Phi_b(\lambda) \equiv \frac{1}{\sqrt{2\pi\sigma_b^2}\,\lambda} \exp\left(-\frac{(\ln\lambda - m_b)^2}{2\sigma_b^2}\right) \tag{3.40}$$

and m_b, σ_b^2 are given by eq. (3.22). The functions $\Phi_b(\lambda)$ have maxima at $\exp[\psi(b)/2]$. The density of incremental singular values is given by the normalized sum

$$\rho_N(\lambda) = \lambda^{-1}\rho_N(\mu = \ln(\lambda)) = \frac{1}{N}\sum_{b=1}^{N} \Phi_b(\lambda) = \frac{1}{N}\sum_{b=1}^{N} \frac{1}{\sqrt{2\pi\sigma_b^2}\,\lambda} \exp\left(-\frac{(\ln\lambda - m_b)^2}{2\sigma_b^2}\right), \tag{3.41}$$

in analogy to eq. (3.28). Again this turns into a sum of delta functions in the limit $t \to \infty$,

$$\lim_{t\to\infty} \rho_N(\lambda) = \frac{1}{N}\sum_{b=1}^{N} \delta\left(\lambda - \mathrm{e}^{\psi(b)/2}\right). \tag{3.42}$$

We have tested this prediction against Monte-Carlo simulations for finite size systems. In Fig. 2 we show histograms of incremental singular values calculated analytically and numerically. We see that the log-normal functions provide a very good approximation to the actual shapes.

4. Lyapunov exponents from the moduli of complex eigenvalues

Rather than using singular values, the complex eigenvalues, $Z_n(t) = R_n(t)\mathrm{e}^{\imath\varphi_n(t)}$, $n = 1, \ldots, N$, are an alternative way to characterize the spectral properties of the matrix $\Pi(t)$, see eq. (2.3). In general the singular values and the moduli of complex eigenvalues are unrelated, apart from their product which is equal to $|\det \Pi(t)|$ and bounds on their Euclidean norm which result from the trace $\mathrm{Tr}\,\Pi^\dagger(t)\Pi(t)$ (see eq. (6.23)), respectively. However in the large t limit, the moduli $R_n(t)$ of the complex eigenvalues will behave exactly in the same way as the singular values $\sqrt{s_n(t)}$. In fact repeating the same construction as in section 3, taking the t-th root of $R_n(t)$ will lead to the very same normal distribution, frozen at identical positions as the limiting singular values. For that reason we will use the same term Lyapunov exponent which is otherwise reserved for the singular values, only.

We pursue a calculation similar to section 3. Thereby we first show that all complex eigenvalues $Z_n(t)$ can be traced back to decoupled random variables apart from a trivial determinantal coupling, see subsection 4.1. In the second step we employ the cumulant expansion to find Dirac delta functions in the leading order and Gaussian (for the corresponding Lyapunov exponents) and log-normal (for the moduli of eigenvalues) distributions in the next-to-leading order, see subsection 4.2. In subsection 4.3, we present an alternative approach by first integrating over the angles $\varphi_n(t)$ and then taking the limit $t \to \infty$. This alternative construction is also applied to the case $\beta = 4$ since the analytical result for the joint probability density of the complex eigenvalues is known [53, 54, 51] for this case as well.

4.1. Reduction to "decoupled" random variables

The definition (2.6) of Lyapunov exponents requires to take the t-th root and the logarithm of the positive singular values. However, for complex variables this is not a unique procedure. If one takes for example the root $Z^{1/t}$, the question arises which of the t roots we have to take. When choosing the primary root the resulting spectrum will be mapped onto a circular sector of the angle $2\pi/t$ which eventually shrinks to the positive semi-axis in the limit $t \to \infty$. Another alternative choice is to take the root of the moduli of the eigenvalues only, i.e.

$$Z_n(t) = R_n(t)\mathrm{e}^{\imath\varphi_n(t)} \;\longrightarrow\; R_n^{1/t}(t)\mathrm{e}^{\imath\varphi_n(t)}. \tag{4.1}$$

Indeed this choice seems to be a more natural construction. When multiplying the product $\Pi(t)$ by new matrices, the angular parts $\varphi_n(t)$ of the eigenvalues will run around on circles while the radial part $R_n(t)$ will either exponentially contract or expand. Thus it is not the angular part we have to worry about in the large t limit since it stays in a compact set. It is the radial part of the eigenvalues which has to be rescaled such that the support stays fixed. Therefore we decide for the rooting (4.1). We emphasize that the kind of rooting is crucial to find our results which may change for other constructions.

The definition of the Lyapunov exponents at finite and infinite t starting from the moduli of complex eigenvalues are

$$\nu_n(t) \equiv \frac{\ln R_n(t)}{t} \tag{4.2}$$

and

$$\nu_n \equiv \lim_{t\to\infty} \frac{\ln R_n(t)}{t}. \tag{4.3}$$

These definitions are the analog of those for the Lyapunov exponents corresponding to the singular values, see Eqs. (2.6) and (2.7). Hereby recall that the variables $s_n(t)$ are the squared singular values which results in an additional prefactor 1/2.

The initial point of our calculation is an exact expression for the joint probability distribution of the complex eigenvalues of the product matrix $\Pi(t)$ eq. (2.3) at finite N and t, see [55, 56],

$$P_N^{(t)}(Z_1,\ldots,Z_N)d^2Z_1\cdots d^2Z_N = \frac{d^2Z_1\cdots d^2Z_N}{N!\pi^N\prod_{a=1}^N\Gamma^t(a)}|\Delta_N(Z)|^2\prod_{b=1}^N G_{0,\,t}^{t,\,0}\left(\begin{matrix}-\\0,\ldots,0\end{matrix}\Big|\ |Z_b|^2\right), \tag{4.4}$$

where d^2Z_n is the flat measure in the complex plane. As in the previous section we again drop the explicit t-dependence of all quantities. We change to polar coordinates and employ the variables (4.2) such that the joint-probability distribution reads

$$P_N^{(t)}\left(\nu_1,\varphi_1,\ldots,\nu_N,\varphi_N\right)\prod_{n=1}^N d\nu_n d\varphi_n \tag{4.5}$$

$$= \frac{t^N\prod_{a=1}^N\exp[2t\nu_a]}{N!\pi^N\prod_{a=1}^N\Gamma^t(a)}|\Delta_N(\exp[t\nu+\imath\varphi])|^2\prod_{b=1}^N G_{0,\,t}^{t,\,0}\left(\begin{matrix}-\\0,\ldots,0\end{matrix}\Big|\ \exp[2t\nu_b]\right)d\nu_b d\varphi_b.$$

We extend the first product by the identity $1 = \mathrm{e}^{\imath\varphi_a}\mathrm{e}^{-\imath\varphi_a}$. With help of the identity $\left(\prod_{a=1}^N x_a\right)\Delta_N(x) = \det_{1\le a,b\le N}[x_a^b]$ we get

$$\left(\prod_{a=1}^N \exp[2t\nu_a+\imath\varphi_a-\imath\varphi_a]\right)|\Delta_N(\exp[t\nu+\imath\varphi])|^2 \tag{4.6}$$

$$= \det_{1\le a,b\le N}\Big[\exp[b(t\nu_a+\imath\varphi_a)]\Big]\det_{1\le a,b\le N}\Big[\exp[b(t\nu_a-\imath\varphi_a)]\Big]. \tag{4.7}$$

We expand one of these determinants and repeat all steps which have led us from eq. (3.3) to eq. (3.7). Thus we end up with

$$P_N^{(t)}\left(\nu_1,\varphi_1,\ldots,\nu_N,\varphi_N\right) = \frac{1}{N!(2\pi)^N}\sum_{\varepsilon\in S_N}\det_{1\le a,b\le N}\left[\frac{\mathrm{e}^{\imath(a-b)\varphi_{\varepsilon(b)}}}{[\Gamma(a)\Gamma(b)]^{t/2}}\tilde{F}_{ab}(\nu_{\varepsilon(b)})\right], \tag{4.8}$$

where

$$\tilde{F}_{ab}(\nu) = 2tG_{0,\,t}^{t,\,0}\left(\begin{matrix}-\\(a+b)/2,\ldots,(a+b)/2\end{matrix}\Big|\ \exp[2t\nu]\right). \tag{4.9}$$

This function is angle-independent and positive semi-definite. It is the counterpart of $F_{ab}(\mu)$, cf. eq. (3.8). This function can be normalized with help of eq. (A.2),

$$\tilde{f}_{ab}(\nu) \equiv \frac{\tilde{F}_{ab}(\nu)}{\int\tilde{F}_{ab}(\nu')d\nu'} = \frac{\tilde{F}_{ab}(\nu)}{\Gamma^t\left[(a+b)/2\right]}\ , \tag{4.10}$$

which has again the interpretation of a probability density function. Then the joint probability density takes the form

$$P_N^{(t)}\left(\nu_1,\varphi_1,\ldots,\nu_N,\varphi_N\right)=\frac{1}{N!(2\pi)^N}\sum_{\varepsilon\in S_N}\det_{1\le a,b\le N}\left[\left(\frac{\Gamma\left((a+b)/2\right)}{\sqrt{\Gamma(a)\Gamma(b)}}\right)^t \mathrm{e}^{\imath(a-b)\varphi_{\varepsilon(b)}}\tilde{f}_{ab}(\nu_{\varepsilon(b)})\right]. \tag{4.11}$$

This is an exact expression for the joint probability distribution of the Lyapunov exponents constructed from the moduli of the complex eigenvalues for any $t\in\mathbb{N}$.

Skipping the definition of the moment generating function we directly turn to the cumulant generating function,

$$\tilde{g}_{ab}(\vartheta)\equiv\ln\left(\int_{-\infty}^{+\infty}d\nu\tilde{f}_{ab}(\nu)\exp(\nu\vartheta)\right)=t\ln\left(\frac{\Gamma\left[(a+b)/2+\vartheta/(2t)\right]}{\Gamma\left[(a+b)/2\right]}\right), \tag{4.12}$$

in analogy to eq. (3.13). The Taylor series of $\tilde{g}_{ab}$ at $\vartheta=0$ is

$$\tilde{g}_{ab}(\vartheta)\equiv\sum_{n=1}^{\infty}\frac{\vartheta^n}{n!}\tilde{\kappa}_{ab}^{(n)}=\sum_{n=1}^{\infty}\frac{\vartheta^n}{n!}\frac{1}{2(2t)^{n-1}}\psi^{(n-1)}\left(\frac{a+b}{2}\right). \tag{4.13}$$

The cumulants can be simply read off. In particular, the first two are equal to

$$\tilde{m}_{ab}\equiv\int d\nu\tilde{f}_{ab}(\nu)\nu=\tilde{\kappa}_{ab}^{(1)}=\frac{1}{2}\psi\left(\frac{a+b}{2}\right) \tag{4.14}$$

and

$$\tilde{\sigma}_{ab}^2\equiv\int d\nu\tilde{f}_{ab}(\nu)\left(\nu-\tilde{m}_{ab}\right)^2=\tilde{\kappa}_{ab}^{(2)}=\frac{1}{4t}\psi'\left(\frac{a+b}{2}\right). \tag{4.15}$$

Again we underline that these results are exact for any $t\in\mathbb{N}$.

4.2. Large t limit

The cumulant expansion (4.13) determines the asymptotic large t behavior of $\tilde{f}_{ab}(\nu)$. Therefore we pursue the same idea as in subsection 3.2 and center the single-variable distribution $\tilde{f}_{ab}(\nu)$ and normalize its variance. After finding the Gaussian behavior in the large t limit we go back to the non-standardized variables in the original problem.

We standardize the random variable ν by subtracting the mean and normalizing the variance to unity, which is again denoted by an asterisk. Consequently the higher order standardized cumulants scale as $\tilde{\kappa}_{*ab}^{(n)}=\tilde{\kappa}_{ab}^{(n)}/(\tilde{\sigma}_{ab})^n\sim t^{1-n/2}$ for large t and $n\ge 2$. Eventually they vanish in the limit $t\to\infty$ and as a consequence, following the same argument as leading to eq. (3.21), the distributions $\tilde{f}_{ab}(\nu)$ asymptotically become normal, i.e.

$$\tilde{f}_{ab}(\nu)\overset{t\gg 1}{\approx}\frac{1}{\sqrt{2\pi\tilde{\sigma}_{ab}^2}}\exp\left(-\frac{(\nu-\tilde{m}_{ab})^2}{2\tilde{\sigma}_{ab}^2}\right) \tag{4.16}$$

with $\tilde{m}_{ab}$ from eq. (4.14) and $\tilde{\sigma}_{ab}$ from eq. (4.15). This function is identical to the distribution of Lyapunov exponents corresponding to the singular values (3.21), with the difference that the mean and the variance still depend on both matrix indices a and

b in the leading order of the $1/t$ expansion. Note, that for the diagonal elements $a = b$ and for large t the functions $\tilde{f}_{bb}(\nu)$ are identical to $f_b(\nu)$, i.e. $\tilde{f}_{bb}(\nu) \approx f_b(\nu)$ for $t \gg 1$. Especially we have $\tilde{m}_{bb} = m_b$ and $\tilde{\sigma}_{bb} = \sigma_b$, cf. eqs. (3.22), (4.14), and (4.15).

Let us discuss the prefactors in the determinant (4.11),

$$D_{ab}(t) \equiv \left(\frac{\Gamma\left[(a+b)/2\right]}{\sqrt{\Gamma(a)\Gamma(b)}} \right)^t \tag{4.17}$$

which become Kronecker symbols. For $a = b \geq 1$ these prefactors are indeed equal to unity. For $a \neq b \geq 1$ we use the fact that the geometric average is larger than the arithmetic one, $[(a+b+2j)/\sqrt{4(a+j)(b+j)}]^t > 1$ for all $j = 0, 1, \ldots$ We have

$$\begin{aligned} D_{ab}(t) < & \left(\frac{\Gamma\left[(a+b)/2\right]}{\sqrt{\Gamma(a)\Gamma(b)}} \right)^t \left(\frac{a+b}{\sqrt{4ab}} \right)^t = \left(\frac{\Gamma\left[(a+b+2)/2\right]}{\sqrt{\Gamma(a+1)\Gamma(b+1)}} \right)^t \\ & < \ldots < \lim_{j\to\infty} \left(\frac{\Gamma\left[(a+b+2j)/2\right]}{\sqrt{\Gamma(a+j)\Gamma(b+j)}} \right)^t = 1. \end{aligned} \tag{4.18}$$

The limit can be done via Stirling's formula. Therefore the determinant eq. (4.11) reduces to the product of diagonal elements in the large t limit. As a consequence the dependence on the angles φ_n completely disappears. Therefore we arrive at

$$P_N^{(t)}\left(\nu_1, \phi_1, \ldots, \nu_N, \phi_N\right) \overset{t\gg 1}{\approx} \frac{1}{N!(2\pi)^N} \sum_{\varepsilon\in S_N} \prod_{b=1}^{N} \tilde{f}_{bb}\left(\nu_{\varepsilon(b)}\right) = \frac{1}{N!(2\pi)^N} \mathrm{per}_{1\leq a,b\leq N}\left[f_b(\nu_a)\right]. \tag{4.19}$$

Note that we employed the Gaussian approximation $f_b(\nu)$, see eq. (3.23), since the means, $\tilde{m}_{bb}$, and the variances, $\tilde{\sigma}_{bb}$, agree with those for the Lyapunov exponents constructed from the singular values. This is in hindsight our justification for giving them the same names.

Because the result (4.19) is independent of the angles φ_n, integrating over them yields a trivial factor $(2\pi)^N$,

$$\begin{aligned} \int_0^{2\pi} \ldots \int_0^{2\pi} d\varphi_1 \ldots d\varphi_N P_N^{(t)}\left(\nu_1, \varphi_1, \ldots, \nu_N, \varphi_N\right) & \overset{t\gg 1}{\approx} P_N(\nu_1, \ldots, \nu_N) \\ & = \frac{1}{N!} \mathrm{per}_{1\leq a,b\leq N}\left[f_b(\nu_a)\right]. \end{aligned} \tag{4.20}$$

The resulting distribution is identical to the distribution for the Lyapunov exponents corresponding to the singular values, see eq. (3.26). Consequently the same results apply to the density of the Lyapunov exponents obtained from the moduli of the complex eigenvalues, eq. (3.28) and its limit as a sum of delta functions eq. (3.29).

It is straightforward to transform the joint probability density eq. (4.19) back to the incremental radii $r_n \equiv \mathrm{e}^{\nu_n}$,

$$P_N\left(\nu_1 = \ln r_1, \ldots, \nu_N = \ln r_N\right) = \frac{1}{N!} \mathrm{per}_{1\leq a,b\leq N}\left[\frac{1}{\sqrt{2\pi\sigma_b^2 r}} \exp\left(-\frac{(\ln r - m_b)^2}{2\sigma_b^2} \right) \right]. \tag{4.21}$$

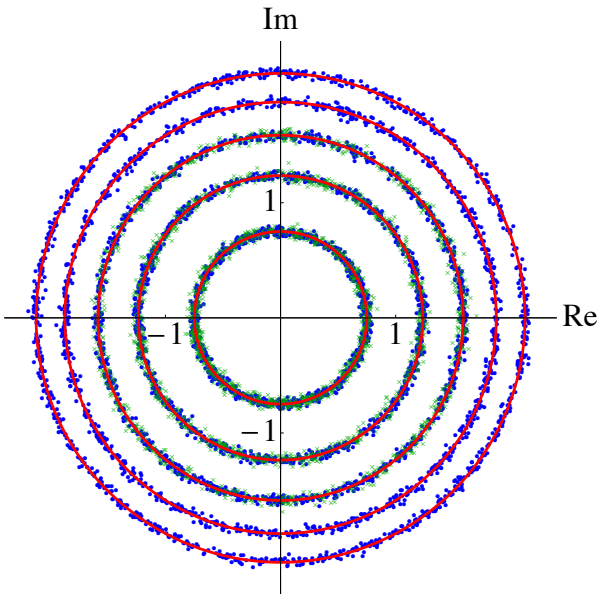

Figure 3. Scatter plot of the complex eigenvalues of the product matrices $\Pi(N = 3, t = 300)$ (green crosses) and $\Pi(N = 5, t = 500)$ (blue dots) derived by the rooting procedure (4.1). The plot was generated by Monte-Carlo simulations of 1000 product matrices for each setting. The solid red lines represent rings with radii $\exp[\psi(b)/2]$, $b = 1, \ldots, 5$, given by the analytical result in the limit $t \to \infty$.

Their joint probability density is a combination of log-normal distributions with exactly the same parameters as for the singular values (3.39). The result (4.21) implies that for large t the radii r_b describe narrow rings centered around the origin with their maxima at $\exp[\psi(b)/2]$, $b = 1, \ldots, N$, cf. Fig. 3. In particular the moduli r_b have log-normal distributions and the phases φ_b are independent and uniformly distributed. The determinantal repulsion between complex eigenvalues is completely lost since they are radially separated. As a consequence the angular degrees of freedom cease to interact and become independent of each other in the limit $t \to \infty$.

Indeed also for the results (4.19) and (4.21) we can investigate the $1/t$ correction, in particular we can apply a saddle point approximation similar to eq. (3.34). However the Monte Carlo simulations performed show already a perfect agreement with the Gaussian approximation, see Fig. 4. The reason is the prefactor (4.17) in front of the single variable distributions $\tilde{f}_{ab}(\nu)$ which additionally suppresses the level repulsion. This behavior is much stronger than for the incremental singular values. Nevertheless, both distributions, the one for the radii and the singular values, will eventually agree, as it can be seen for the smallest radii and singular values in Fig. 4.

4.3. An alternative approach

We close this section by offering a short-cut from the joint density eq. (4.5) to the final result eq. (4.19). Once all angles are integrated out the moduli of the complex eigenvalues Z_n of the product of Ginibre matrices immediately become independent

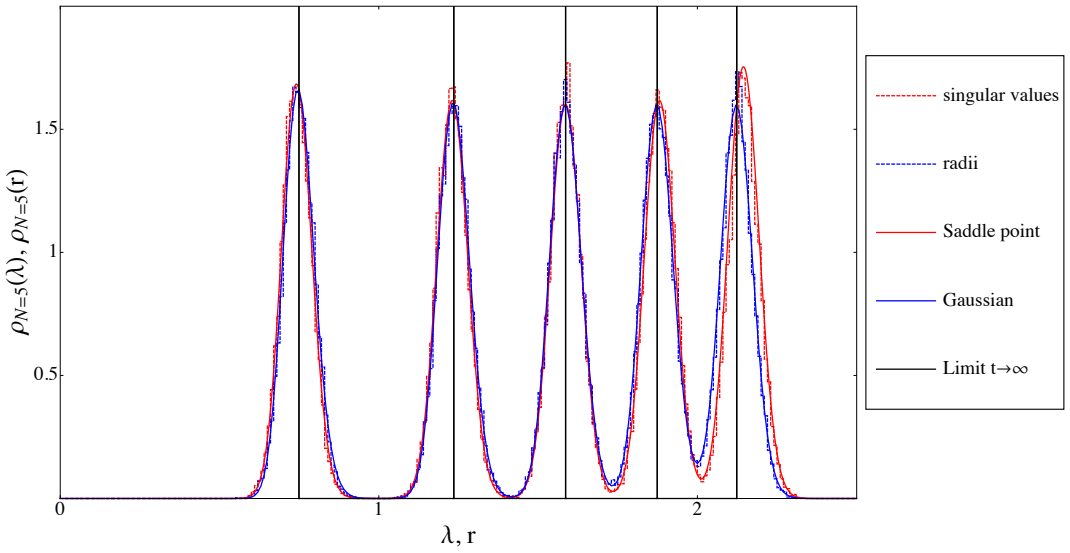

Figure 4. The histograms show the distribution of the incremental singular values (red dashed histogram) and of the incremental radii of the complex eigenvalues (blue dashed histogram) of products of $t = 100$ 5×5 complex Ginibre matrices generated by Monte Carlo simulations (ensemble size = 10000 product matrices). The distribution of the radii are well approximated by the analytical result (4.21) (blue curve) while the corresponding saddle point approximation (3.34) for the incremental singular values (red curve) is needed for a better agreement for higher singular values. For the smallest radii and singular values all distributions perfectly agree. The positions of the limiting result $\exp[\psi(b)/2]$, $b = 1, \ldots, 5$, are shown by vertical lines.

random variables, see Refs. [50, 51] and for a general discussion Ref. [52]. These integrations can be already performed for the distribution (4.5) such that we immediately arrive at

$$\int_0^{2\pi} \prod_{n=1}^{N} d\phi_n P_N^{(t)}(\nu_1, \phi_1, \ldots, \nu_N, \phi_N) = \frac{1}{N!} \mathrm{per}_{1\le a,b\le N} \left[\frac{2t G_{0,t}^{t,0}\left(\begin{smallmatrix} - \\ a,\ldots,a \end{smallmatrix} \middle| \exp[2t\nu_b] \right)}{\Gamma^t(b)} \right]. \quad (4.22)$$

This result is still exact for finite t. The application of the asymptotic limit of the Meijer G-function (3.21) immediately leads to the following answer

$$\int_0^{2\pi} \prod_{n=1}^{N} d\phi_n P_N^{(t)}(\nu_1, \phi_1, \ldots, \nu_N, \phi_N) \overset{t\gg 1}{\approx} \frac{1}{N!} \mathrm{per}_{1\le a,b\le N} \left[\frac{1}{\sqrt{2\pi\sigma_b^2}} \exp\left(-\frac{(\nu_a - m_b)^2}{2\sigma_b^2} \right) \right], \quad (4.23)$$

which is identical to eq. (4.20). The parameters for the mean and variance are given in eq. (3.22). Let us emphasize again that the loss of angular dependence also directly results from the large t limit.

Let us ask at this point about the situation for general Dyson index $\beta = 1, 2, 4$. The integration over all angles is a non-trivial task in the case $\beta = 1$ though it was shown in Ref. [57] that in the large t limit all eigenvalues become real with probability $+1$, and in Ref. [54] an expression for the joint probability density was derived for an arbitrary isotropic weight.

For $\beta = 4$ the situation is much easier. Not only explicit expressions for the joint probability densities of quaternionic Ginibre matrices [53] and of general isotropic weight [54] were derived, also the integral over the angles was done [51]. Performing these integrals also leads to a permanent, which reads for Ginibre matrices

$$\int_0^{2\pi} \prod_{n=1}^{N} d\phi_n P_N^{(t,\ \beta=4)}(\nu_1, \phi_1, \ldots, \nu_N, \phi_N) = \frac{1}{N!} \mathrm{per}_{1\le c,d\le N} \left[\frac{2t G_{0,\,t}^{t,\,0}\left(\begin{matrix} - \\ 2c,\ldots,2c \end{matrix} \middle| \exp[2t\nu_d] \right)}{\Gamma^t(2b)} \right]. \quad (4.24)$$

The asymptotic limit (3.21) of the Meijer G-function still applies, one has to set $a = 1$, $d = b$ and $b = 2c$ in eq. (3.21). This yields for the Lyapunov exponents constructed from the moduli of the complex eigenvalues

$$\begin{aligned} &\int_0^{2\pi} \prod_{n=1}^{N} d\phi_n P_N^{(t,\ \beta=4)}(\nu_1, \phi_1, \ldots, \nu_N, \phi_N) \\ &\overset{t\gg 1}{\approx} \frac{1}{N!} \mathrm{per}_{1\le c,d\le N} \left[\frac{1}{\sqrt{2\pi\sigma_{2c}^2}} \exp\left(-\frac{(\nu_d - m_{2c})^2}{2\sigma_{2c}^2} \right) \right]. \end{aligned} \quad (4.25)$$

Note the similarity to eq. (4.20) although the product now consists of quaternion matrices, only. Nevertheless, we have to be careful when interpreting this result as a hint that the final level statistics for $\beta = 4$ become, apart from a factor 2 in the indices, identical to the ones for $\beta = 2$. The scatter plots in Fig. 6 show that the eigenvalues are by far not uniformly distributed along the rings. Thus the angular distribution will be non-trivial for $\beta = 4$.

When taking the exact limit $t \to \infty$ of eq. (4.25) the Gaussian functions convert to into Dirac delta functions at the positions $\nu = \psi(2c)/2$, $c = 1, \ldots, N$. These positions were already found by Kargin [42] for the Lyapunov exponents from singular values for the product of quaternionic Ginibre matrices.

Indeed it would be nice to find also the finite t corrections to this limit for the singular values for $\beta = 1, 4$. However the group integrals involved in this problem prevent an explicit expression for the joint probability density, see [18, 19] for comparison to the approach applied to the case $\beta = 2$. Nonetheless we conjecture that the Lyapunov exponents from singular values and moduli of complex eigenvalues should again coincide as for $\beta = 2$. This conjecture is at least confirmed by Monte Carlo simulations, see Fig. 6, as well as by a direct analysis of 2×2 matrices, see subsection 6.2.

5. Large N limit

Let us take the limit $N \to \infty$, too. In particular, we ask the question whether the limits $t \to \infty$ and $N \to \infty$ commute. This question is at the heart of understanding both kinds of limits. In particular one can consider the local spectral statistics as well as the global one.

Let us stick first to the global statistics and the situation where we take $t \to \infty$ first. For this purpose two important remarks concerning the limit $N \to \infty$ are in order. The complex eigenvalues of an $N \times N$ Ginibre matrix X_j are scattered on a disk of radius

which grows approximately as $\sqrt{N}$. Therefore we have to fix the support by rescaling the matrices,

$$X_{*j} = \frac{X_j}{\sqrt{N}},\ j = 1, \ldots, t, \tag{5.1}$$

to find a proper limit for the macroscopic level density in the limit $N \to \infty$. Then the spacing between the complex eigenvalues as well as between the singular values tends to zero and the spectral distributions become continuous functions for $N \to \infty$. In particular, the limiting eigenvalue distribution of rescaled Ginibre matrices is given by a uniform density on the unit disk centered at the origin of the complex plane which is the so-called circular law. Exactly this circular law is also found for a product of complex Ginibre matrices after taking the root of the radii for t fixed and $N \to \infty$, cf. Refs. [42, 43].

After rescaling the moduli of the complex eigenvalues are on average smaller or equal to unity. Thereby the corresponding evolution $\vec{x}_{t+1} = X_{*t}\vec{x}_t$ is contractive and hence the Lyapunov exponents are expected to be non-positive. Because the evolution is linear the incremental singular values (or radii) rescale as $\lambda_{*n} = \lambda_n/\sqrt{N}$. Quantities corresponding to this normalization are denoted by an asterisk in this section.

The rescaling results in a trivial shift for the Lyapunov exponents, i.e.

$$\hat{\mu}_{*b} = \frac{1}{2}\left(\psi(b) - \ln N\right),\ b = 1, \ldots, N. \tag{5.2}$$

The smallest Lyapunov exponent is approximately equal to $\hat{\mu}_{*1} \approx -1/2 \ln N$ for $N \gg 1$ and the largest one is

$$\hat{\mu}_{*N} = \frac{1}{2}\left(\psi(N) - \ln N\right) \overset{N\gg 1}{\approx} -\frac{1}{4N}. \tag{5.3}$$

Therefore all Lyapunov exponents are negative and for $N \to \infty$ the spectrum extends from $-\infty$ to 0. The probability that a randomly chosen exponent μ'_* is less or equal to $\hat{\mu}_{*b}$ is

$$\mathrm{Prob}(\mu'_* \leq \hat{\mu}_{*b}) = \frac{b}{N}. \tag{5.4}$$

Choosing the rescaled variable $x = b/N \in]0, 1]$ this probability reads

$$\mathrm{Prob}\left(\mu'_* \leq \frac{1}{2}\left(\psi(Nx) - \ln N\right)\right) = x. \tag{5.5}$$

In the limit $N \to \infty$ this variable becomes a continuous variable $x \in]0, 1]$. Moreover, for any fixed μ_* we can approximate $\psi(Nx) \approx \ln(Nx) + O(1/N)$ such that we have

$$\mathrm{Prob}\left(\mu'_* \leq \frac{\ln(x)}{2}\right) \overset{N\gg 1}{\approx} \int_{-\infty}^{\ln(x)/2} \rho_*(\mu'_*) d\mu'_* = x. \tag{5.6}$$

Here

$$\rho_*(\mu_*) \equiv \lim_{N\to\infty} \rho_{*N}(\mu_*) \tag{5.7}$$

is the limiting density of Lyapunov exponents for the product of independent normalized Ginibre matrices X_{*j} from eq. (3.29). The last equation can be easily solved for $\rho_*(\mu_*)$,

$$\rho_{\mu*}(\mu_*) = 2\mathrm{e}^{2\mu_*}, \quad \mu_* \ \leq 0. \tag{5.8}$$

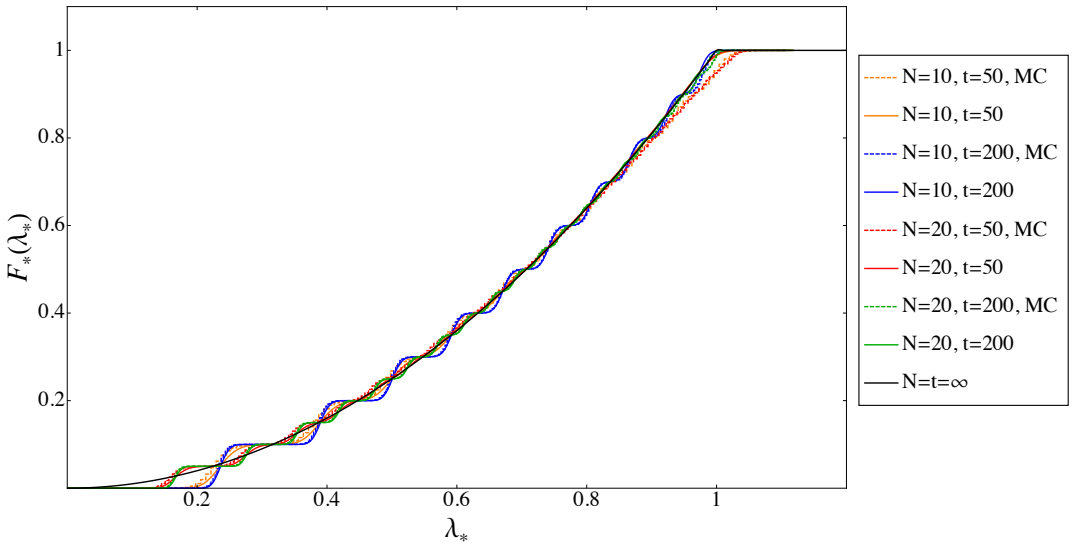

Figure 5. The analytical results (solid curves) for the cumulative distribution for the incremental singular values $F_*(\lambda_*)$ are compared to Monte Carlo simulations (dashed histograms) for varying matrix dimension N and varying numbers of matrices t in the product $\Pi(t)$. The black solid curve is the $N, t \to \infty$ result.

Changing from Lyapunov exponents to incremental singular values $\lambda_{*b} = \mathrm{e}^{\mu_{*b}}$, we obtain

$$\rho_*(\mu_*)d\mu_* = \rho_*(\lambda_*)d\lambda_* = 2\lambda_* d\lambda_*, \quad \lambda_* \in [0,1]. \tag{5.9}$$

This is the celebrated triangular law first derived by Newman [40, 45].

Obviously one can repeat exactly the same calculations starting from the moduli of complex eigenvalues and obtains the same results, replacing $\mu_* \to \nu_*$ and $\lambda_* \to r_*$. We note in passing that the triangular distribution of incremental radii is identical to the limiting radial distribution of the complex eigenvalues of normalized Ginibre matrices $X_*/\sqrt{N}$, $N \to \infty$, which is given by the uniform distribution on the complex unit disk. Here the linear behavior is nothing more than the Jacobian resulting from the choice of polar coordinates.

It is instructive to examine the convergence of the finite N distribution to the limiting triangular law. The cumulative distribution for the triangular law defined as

$$F_*(\lambda_*) = \begin{cases} \int_0^{\lambda_*} d\lambda'_* \rho_*(\lambda'_*) = \lambda_*^2 & , \ \lambda_* \in [0,1], \\ 1 & , \ \lambda_* \geq 1 \end{cases} \tag{5.10}$$

is trivially obtained. It is the probability to find a singular value smaller than λ_*. For finite N (and $t \to \infty$) the cumulative distribution is just the counting function

$$F_{*N}(\lambda_*) = \frac{1}{N} \sum_{n=1}^{N} \Theta\left(\lambda_* - \frac{\exp[\psi(n)/2]}{\sqrt{N}}\right), \tag{5.11}$$

with $\lim_{N\to\infty} F_{*N}(x) = F_*(x)$. We show the evolution of the shape of this staircase function in N and t in Fig. 5.

Let us study if the limits $t \to \infty$ and $N \to \infty$ commute. Therefore we consider the moments of the density of the singular values which are for the triangular law

$$\lim_{N\to\infty} \lim_{t\to\infty} \langle \lambda_*^n \rangle_* = \int_0^1 d\lambda_* \rho_*(\lambda_*) \lambda_*^n = \frac{2}{n+2} \text{ for all } n > -1. \tag{5.12}$$

Recall that this law is obtained by taking first the limit $t \to \infty$ and then $N \to \infty$. Let us invert this order. The moments of the singular value distribution of the product of t normalized Ginibre for $N \to \infty$ is equal to the Fuss-Catalan numbers [58]

$$\lim_{N\to\infty} \langle s_*^k(t) \rangle_* = \frac{1}{tk+1} \begin{pmatrix} (t+1)k \\ k \end{pmatrix} \text{ for all } k > -\frac{1}{t+1}. \tag{5.13}$$

We choose $k = n/(2t)$ while keeping n fixed and sending t to infinity. Changing the integration variable from singular values to their roots, $\lambda = s_*^{1/(2t)}$, we get $s_*^k(t) = \lambda_*^n(t)$. The binomial symbol on the right hand side tends to unity for $t \to \infty$, and the prefactor to $2/(n+2)$. Combining everything we have

$$\lim_{t\to\infty} \lim_{N\to\infty} \langle \lambda_*^n(t) \rangle_* = \lim_{t\to\infty} \lim_{N\to\infty} \langle s_*^k(t) \rangle_* = \frac{2}{n+2} = \lim_{N\to\infty} \lim_{t\to\infty} \langle \lambda_*^n \rangle_*. \tag{5.14}$$

We see that indeed the limits $t \to \infty$ and $N \to \infty$ commute. To have an idea how the limiting shape of the distribution is approached when t and N increase we plot in Fig. 5 the cumulative distribution for a collection of systems with finite t and N, showing both analytic and Monte-Carlo results.

To conclude this section we can ask if the commutativity of the two limits carries over to the local statistics as well. When taking first the limit $N \to \infty$ it was shown [55] that the level statistics in the bulk and at the soft edge follow the universal results [61] for complex Ginibre matrices. Especially the level spacing distribution in the bulk behaves for small spacing Δr as $P(\Delta r)dr \approx \Delta r^3 d\Delta r \propto \Delta r^2 d\Delta r^2$, see Refs. [59, 60]. These results are independent of t and, hence, will also not change when taking the limit $t \to \infty$ afterwards. When reversing the two limits, in particular when first taking the limit $t \to \infty$ and then the limit $N \to \infty$, we will find the statistics of the harmonic oscillator. This can be realized after unfolding the level spacing distribution of the incremental radii, i.e. $r_* \to r_*^2$, the level spacing distribution at finite N but $t = \infty$ is

$$\begin{aligned} P_N(\Delta r_*^2) &= \frac{1}{N-1} \sum_{j=1}^{N-1} \delta\left(\Delta r_*^2 - \exp\left[\psi(j+1)\right] + \exp\left[\psi(j)\right]\right) \\ &= \frac{1}{N-1} \sum_{j=1}^{N-1} \delta\left(\Delta r_*^2 - \exp\left[\psi(j)\right]\left(\exp\left[\frac{1}{j}\right] - 1\right)\right). \end{aligned} \tag{5.15}$$

In the limit $N \to \infty$ the variable $x = j/N$ becomes continuous and the sum can be approximated by an integral such that

$$\begin{aligned} P(\Delta r_*^2) &\equiv \lim_{N\to\infty} P_N(\Delta r_*^2) \\ &= \lim_{N\to\infty} \int_0^1 dx \delta\left(\Delta r_*^2 - \exp\left[\psi(Nx)\right]\left(\exp\left[\frac{1}{Nx}\right] - 1\right)\right) = \delta(\Delta r_*^2 - 1) \end{aligned} \tag{5.16}$$

which is the one of an harmonic oscillator. This result is far away from the unfolded level spacing distribution of the Ginibre ensemble which has a linear slope in Δr^2, $P(\Delta r)dr \equiv P(\Delta r^2)d\Delta r^2 \approx \Delta r^2 d\Delta r^2$, for small spacing $\Delta r \ll 1$, see Refs. [59, 60]. Therefore on the local scale the two limits do not commute in contrast to the global scale, cf. eq. (5.14). The same argument is expected to be true for the local statistics of the incremental singular values.

Since the two limits commute on the global scale while they do not commute on the local one, we claim that there should be a non-trivial double scaling limit where new results should show up. In particular we expect a mesoscopic scale of the spectrum which may also show a new kind of universal statistics.

6. Isotropic evolution with arbitrary weights

So far we have discussed the evolution (2.1) driven by independent Ginibre matrices. An important property of this random matrix ensemble is its isotropic nature, meaning that it is invariant under bi-unitary transformations, $d\mu(X_*) = d\mu(UX_*V^{-1})$, with respect to the right and left multiplication of any unitary matrices $U, V \in \mathrm{U(N)}$. We want to generalize our discussion to more general isotropic random matrix ensembles, particularly to non-Gaussian weights. For this purpose we recall Newman's argument [40] to find the Lyapunov exponents constructed from the singular values in subsection 6.1. In subsection 6.2 we discuss why the Lyapunov exponents corresponding to the radii of the complex eigenvalues agree with those of the singular values for products of 2×2 complex matrices identically drawn from an arbitrary isotropic weight. Moreover we briefly discuss the extension of this argument to arbitrary dimension N and arbitrary Dyson index $\beta = 1, 2, 4$.

6.1. Newman's argument for the singular values

Let us recall a general argument given by Newman [40], which can be applied to an arbitrary isotropic evolution. It says that in the large t limit the Lyapunov exponents become deterministic. This behavior is related to some kind of self-averaging different from the one discussed in [46, 47].

Newman's argument is based on a particular definition of the Lyapunov exponents constructed from the singular values. The sum of k largest Lyapunov exponents is given by

$$\Sigma_k(t) \equiv \hat{\mu}_N(t) + \ldots + \hat{\mu}_{N-k+1}(t) = \max_{A \in \mathbb{C}^{N\times k}} \frac{1}{2t} \ln \frac{\det A^\dagger \Pi^\dagger(t)\Pi(t)A}{\det A^\dagger A}, \quad (6.1)$$

where the maximum is taken over all complex $N \times k$ matrices A whose singular values do not vanish, i.e. $\det A^\dagger A \neq 0$. We denote the average of an observable $O(\Pi(t))$ by

$$\langle O(\Pi(t))\rangle_t = \int d\mu(X_1) \cdots d\mu(X_t) O(\Pi(t)). \quad (6.2)$$

Then Newman's argument is equivalent to the fact that for any integrable test function f depending on the vector $\Sigma(t) = (\Sigma_1(t), \ldots, \Sigma_N(t))$ we have

$$\lim_{t\to\infty} \langle f(\Sigma(t))\rangle_t = f(\langle \Sigma(1)\rangle_1), \tag{6.3}$$

where on the right hand side we average over a single matrix ($t = 1$), only.

The idea to prove the claim (6.3) is to introduce a telescopic product in the definition (6.1),

$$\begin{aligned} \Sigma_k(t) &= \max_{A\in\mathbb{C}^{N\times k}} \left\{ \frac{1}{2t} \ln \left(\prod_{j=1}^{t} \frac{\det A^\dagger \Pi^\dagger(j)\Pi(j)A}{\det A^\dagger \Pi^\dagger(j-1)\Pi(j-1)A} \right) \right\} \\ &= \max_{A\in\mathbb{C}^{N\times k}} \left\{ \frac{1}{2t} \sum_{j=1}^{t} \ln \left(\frac{\det A_j^\dagger X_j^\dagger X_j A_j}{\det A_j^\dagger A_j} \right) \right\} \end{aligned} \tag{6.4}$$

with $A_j = \Pi(j-1)A$ and $\Pi(0)$ the N-dimensional identity matrix. Note that the sum cannot be simply pushed through the operation "max" since the matrices A_j, $j = 1, \ldots, N$, depend on each other. Exactly at this point the isotropy of the weight becomes important. With the help of the average one can show that

$$\langle f(\Sigma(t))\rangle_t = \int d\mu(X_1)\cdots d\mu(X_t) f\left(\max_{A\in\mathbb{C}^{N\times k}} \left\{ \frac{1}{2t} \sum_{j=1}^{t} \ln \left(\frac{\det A_j^\dagger X_j^\dagger X_j A_j}{\det A_j^\dagger A_j} \right) \right\} \right) \tag{6.5}$$

$$= \int d\mu(X_1)\cdots d\mu(X_t) f\left(\frac{1}{2t} \sum_{j=1}^{t} \ln \left(\det P_k^\dagger X_j^\dagger X_j P_k \right) \right) \tag{6.6}$$

$$= \int d\mu(X_1)\cdots d\mu(X_t) f\left(\frac{1}{2t} \sum_{j=1}^{t} \max_{A\in\mathbb{C}^{N\times k}} \left\{ \ln \left(\frac{\det A^\dagger X_j^\dagger X_j A}{\det A^\dagger A} \right) \right\} \right).$$

The reason is that A_j has the singular value decomposition $A_j = U_j P_k \Lambda_j V_j$, with $U_j \in \mathrm{U(N)}$, $V_j \in \mathrm{U(k)}$, $\Lambda_j = \mathrm{diag}(\lambda_{1j}, \ldots, \lambda_{kj}) \in \mathbb{R}_+^k$, and P_k the matrix mapping k-dimensional vectors as $v = (v_1, \ldots, v_k) \in \mathbb{C}^k$ to the trivially embedded N-dimensional vectors $(v_1, \ldots, v_k, 0, 0, \ldots, 0) \in \mathbb{C}^N$. The matrix V_j as well as the diagonal matrix Λ_j trivially drop out of the ratios of determinants. The matrix U_j can be readily absorbed in the measure of X_j due to the substitution $X_j \to X_j U_j^\dagger$ and the isotropy of the measure $d\mu(X_j)$. Thus everything only depends on the matrices X_j and on the embedding (projection) matrix P_k which is independent of A_j. Therefore we can completely omit taking the maximum of $A = U_1\Lambda_1 V_1$, cf. the second line of eq. (6.5), and exchange the sum with the maximum. To restore the dependence on A_j we substitute $X_j \to X_j U_1$ anew. Hence we find the identity (6.5).

In the limit $t \to \infty$ the sum is equal to the average of a single random matrix because of the law of large numbers. In particular we have

$$\hat{\mu}_N + \ldots + \hat{\mu}_{N-k+1} = \left\langle \max_{A\in\mathbb{C}^{N\times k}} \frac{1}{2} \ln \frac{\det A^\dagger \Pi^\dagger(1)\Pi(1)A}{\det A^\dagger A} \right\rangle_1 \tag{6.7}$$

From this equation one can also simply determine the incremental singular values $\hat{\lambda}_n = \exp[\hat{\mu}_n]$. In the case of complex Ginibre ensembles the result (6.7) yields $\hat{\mu}_n = \psi(n)/2$. In Ref. [40] this proof was given for $\beta = 1$, only.

We stress that the whole line of argument also applies in the case of general Dyson index $\beta = 1, 2, 4$. One only has to assume that the weight is invariant under right multiplication with the groups O(N), U(N) and USp($2N$), respectively, and that the first moment of the Lyapunov exponents exists. Note that we only need the invariance under right multiplication. This is the reason why introducing fixed covariance matrices in the product of matrices did not cause any problems as it was considered in Ref. [41] for $\beta = 2$ and in Ref. [42] for $\beta = 1, 2, 4$.

6.2. Lyapunov exponents of general isotropic 2×2 *random matrices*

The question arises if products of random matrices drawn from any isotropic ensemble lead to a collapse of the Lyapunov exponents from the singular values and from the moduli of the complex eigenvalues to one and the same distribution as it was shown in sections 3 and 4. For a product of 2×2 random matrices this question can be answered positively. For this purpose we consider the product matrix

$$\Pi(t) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = X_t X_{t-1} \dots X_1 \quad \text{with} \quad X_j = \begin{bmatrix} x_{11}^{(j)} & x_{12}^{(j)} \\ x_{21}^{(j)} & x_{22}^{(j)} \end{bmatrix} \in \mathbb{C}^{2\times 2}, \tag{6.8}$$

whose random matrices are drawn from the same isotropic weight $P(X)dX = d\mu(X) = d\mu(UXV^{-1})$ with $U, V \in \mathrm{U}(2)$.

Let us denote the two t-dependent Lyapunov exponents of the singular values by $\mu_1(t)$ and $\mu_2(t)$ as defined in eq. (2.7). Then Newman's argument tells us that for any integrable test function f depending on $\hat{\mu}_2(t) = \max\{\mu_1(t), \mu_2(t)\}$ and $(\ln|\det \Pi(t)|)/t = \mu_1(t) + \mu_2(t)$ we have

$$\lim_{t\to\infty} \langle f(\hat{\mu}_2(t), \mu_1(t) + \mu_2(t))\rangle_t = f(\langle \hat{\mu}_2(1)\rangle_1, \langle \mu_1(1) + \mu_2(1)\rangle_1). \tag{6.9}$$

Note that on the right hand side the average is only over a single random matrix, $\Pi(1) = X_1$.

The aim is to show that the Lyapunov exponents of the moduli of the eigenvalues $\nu_1(t)$ and $\nu_2(t)$ agree with $\mu_1(t)$ and $\mu_2(t)$ in the large t-limit, i.e.

$$\lim_{t\to\infty} \langle f(\hat{\mu}_2(t), \mu_1(t) + \mu_2(t))\rangle_t = f(\langle \hat{\nu}_2(1)\rangle_1, \langle \nu_1(1) + \nu_2(1)\rangle_1) \tag{6.10}$$

with $\hat{\nu}_2(t) = \max\{\nu_1(t), \nu_2(t)\}$ for all integrable test functions. For this purpose we first construct an analytical relation between $\nu_{1,2}(t)$ and $\mu_{1,2}(t)$.

The isotropy allows us to absorb the unitary matrices U_j resulting from the generalized Schur decomposition [54, 55, 56],

$$\Pi(t) = U_t \begin{bmatrix} z_1 & \Delta \\ 0 & z_2 \end{bmatrix} U_t^\dagger \text{ with } X_j = U_j \begin{bmatrix} z_{1j} & \Delta_j \\ 0 & z_{2j} \end{bmatrix} U_{j-1}^\dagger \text{ and } U_0 = U_t. \tag{6.11}$$

The variables $z_1, z_2, \Delta \in \mathbb{C}$ depend on $z_{1j}, z_{2j}, \Delta_j \in \mathbb{C}$ via the relations

$$z_1 = \prod_{j=1}^{t} z_{1j}, \;\; z_2 = \prod_{j=1}^{t} z_{2j}, \;\; \Delta = \sum_{j=1}^{t} \left(\prod_{l=1}^{j-1} z_{1l} \right) \Delta_j \left(\prod_{l=j+1}^{t} z_{2l} \right). \tag{6.12}$$

The quantities $\hat{\mu}_2(t)$ and $\mu_1(t)+\mu_2(t)$ read in terms of the variables $z_{1/2}$ and Δ as

$$\hat{\mu}_2(t) = \frac{1}{2t}\ln\left(\frac{|z_1|^2+|z_2|^2+|\Delta|^2+\sqrt{(|z_1|^2+|z_2|^2+|\Delta|^2)^2-4|z_1z_2|^2}}{2}\right), \quad (6.13)$$

$$\mu_1(t)+\mu_2(t) = \frac{1}{t}\ln|z_1z_2| = \frac{1}{t}\sum_{j=1}^{t}(\ln|z_{1j}|+\ln|z_{2j}|) = \nu_1(t)+\nu_2(t). \quad (6.14)$$

Note that these quantities only depend on $|z_{1,2}|$ and $|\Delta|$. After plugging these relations into the finite t average over the test function f and decomposing the variables $z_{1j}=R_{1j}e^{\imath\varphi_{1j}}$ and $z_{2j}=R_{2j}e^{\imath\varphi_{2j}}$ into radial and angular parts we obtain

$$\begin{aligned}&\langle f(\hat{\mu}_2(t),\mu_1(t)+\mu_2(t))\rangle_t\\ &=\prod_{j=1}^{t}\left(4\int_0^\infty dR_{1j}dR_{2j}\int_0^{2\pi}d\varphi_{1j}d\varphi_{2j}\int_{\mathbb{C}}d^2\Delta_j\int_{\mathrm{U}(2)/\mathrm{U}^2(1)}d\chi(U_j)R_{1j}R_{2j}P\left(|z_{1j}|,|z_{2j}|,\Delta_j\right)\right)\\ &\quad\times\frac{\left|\prod_{j=1}^{t}z_{1j}-\prod_{j=1}^{t}z_{2j}\right|^2}{2}f\left(\hat{\mu}_2(t),\frac{1}{t}\sum_{j=1}^{t}(\ln|z_{1j}|+\ln|z_{2j}|)\right),\end{aligned} \quad (6.15)$$

see Ref. [54]. The factor 1/2 results from the ordering of z_1 and z_2 which is originally included in the generalized Schur decomposition and can be lifted by taking this factor into account. The Haar measure of the co-set $\mathrm{U}(2)/\mathrm{U}^2(1)$ is denoted as $d\chi(U_j)$, $j=1,\ldots,N$. Let us stress that the isotropy of the probability density P indeed allows us to absorb the dependence of P on the angles of the two eigenvalues z_1 and z_2 in the integral over Δ.

The integration over the phases $\mathrm{e}^{\imath\varphi_{1j}}$ and $\mathrm{e}^{\imath\varphi_{2j}}$ simplifies the integral (6.15) to

$$\begin{aligned}&\langle f(\hat{\mu}_2(t),\mu_1(t)+\mu_2(t))\rangle_t\\ &=\prod_{j=1}^{t}\left(4\int_0^\infty dR_{1j}dR_{2j}\int_0^{2\pi}d\varphi_{1j}d\varphi_{2j}\int_{\mathbb{C}}d^2\Delta_j\int_{\mathrm{U}(2)/\mathrm{U}^2(1)}d\chi(U_j)R_{1j}^3R_{2j}P\left(|z_{1j}|,|z_{2j}|,\Delta_j\right)\right)\\ &\quad\times f\left(\hat{\mu}_2(t),\frac{1}{t}\sum_{j=1}^{t}(\ln|z_{1j}|+\ln|z_{2j}|)\right),\end{aligned} \quad (6.16)$$

The collective permutation $z_{1j}\leftrightarrow z_{2j}$ employed here is legitimate. Therefore the single probability densities of the set of variables $\{z_{1j},z_{2j},\Delta_j\}$ factorize and become statistically independent. Interestingly the average over a single set of variables $\{z_{1j},z_{2j},\Delta_j\}$ with a fixed index j is equal to the original integral over a single matrix X_j, i.e. eq. (6.16) also holds for $t=1$, which is quite important to find the right hand side of eq. (6.10).

In the next step we calculate upper and lower bounds for the maximal Lyapunov exponent $\hat{\mu}_2(t)$. Looking at the relation (6.13) it is immediate that $\hat{\mu}_2(t)$ is monotonously increasing in $|\Delta|$. Hence it is certainly true that

$$\hat{\mu}_2(t) \geq \frac{1}{2t}\ln\left(\frac{|z_1|^2+|z_2|^2+\sqrt{(|z_1|^2+|z_2|^2)^2-4|z_1z_2|^2}}{2}\right) \quad (6.17)$$

$$= \frac{1}{t}\ln\max\{|z_1|,|z_2|\} = \max\left\{\frac{1}{t}\sum_{j=1}^{t}\ln|z_{1j}|, \frac{1}{t}\sum_{j=1}^{t}\ln|z_{2j}|\right\}.$$

Note that the sum cannot be pushed through the operation "max". The upper bound can be found by estimating $|\Delta|$, i.e.

$$|\Delta| \leq \sum_{j=1}^{t}\left(\prod_{l=1}^{j-1}|z_{1l}|\right)|\Delta_j|\left(\prod_{l=j+1}^{t}|z_{2l}|\right) \leq \max_{k=1,\ldots,t}\left\{\left(\prod_{l=1}^{k-1}|z_{1l}|\right)\left(\prod_{l=k+1}^{t}|z_{2l}|\right)\right\}\sum_{j=1}^{t}|\Delta_j|. \tag{6.18}$$

Because of the statistical independence of the matrices with a fixed j this inequality becomes

$$\begin{aligned}\frac{1}{t}\ln|\Delta| \leq & \max_{k=1,\ldots,t}\left\{\frac{1}{t}\left(\sum_{l=1}^{k-1}\ln|z_{1l}| + \sum_{l=k+1}^{t}\ln|z_{2l}|\right)\right\} + \frac{1}{t}\ln\left(\sum_{j=1}^{t}|\Delta_j|\right) \qquad (6.19)\\ \overset{t\gg 1}{\approx} & \sup_{p\in]0,1[}\{p\langle\ln|z_{11}|\rangle_1 + (1-p)\langle\ln|z_{21}|\rangle_1\} + \frac{1}{t}\ln\left(t\langle|\Delta_1|\rangle_1\right)\\ = & \langle\hat{\nu}_1(1)\rangle_1 + \frac{1}{t}\ln\left(t\langle|\Delta_1|\rangle_1\right)\end{aligned}$$

in the large t-limit. The latter equation results from the fact that the supremum is reached at the boundary of the interval $p \in]0,1[$ and that the moment of $|\Delta_1|$ is bounded. Therefore there is a constant $0 < c < \infty$ such that

$$|\Delta| \leq ct\exp[t\langle\hat{\nu}_1(1)\rangle_1] \tag{6.20}$$

for all $t \in \mathbb{N}$. This inequality together with $0 \leq |z_{1,2}| \leq \widetilde{c}\exp[t\langle\hat{\nu}_1(1)\rangle_1]$, where $0 < \widetilde{c} < \infty$ is a second constant, yields the upper bound

$$\hat{\mu}_1 \leq \frac{1}{2t}\ln\left(2\widetilde{c}^2 + c^2t^2\right) + \langle\hat{\nu}_1(1)\rangle_1 \tag{6.21}$$

for all $t \in \mathbb{N}$.

Collecting everything the bounds tell us that the large t limit is

$$\lim_{t\to\infty}\frac{1}{t}\hat{\mu}_1 = \langle\hat{\nu}_1(1)\rangle_1. \tag{6.22}$$

Equation (6.22) together with eq. (6.14) prove that eq. (6.10) is indeed true. In particular it shows two things. First, the two Lyapunov exponents constructed from the moduli of the complex eigenvalues of a product of 2×2 matrices independently and isotropically distributed take deterministic values in the large t limit. Second, the deterministic values of those Lyapunov exponents agree with those constructed from the singular values. Both properties are true for quite general random matrix ensembles. The only additional condition apart from the isotropy is the existence of the first moments of the random variables $|\Delta_j|$ and $\ln|z_{1j,2j}|$. The existence of these moments guarantees the existence of the limits and the correctness of the calculation presented above.

Note that despite the general inequality

$$\mathrm{Tr}\,\Pi(t)\Pi^\dagger(t) = \sum_{j=1}^{N}s_j(t) = \sum_{j=1}^{N}R_j^2(t) + \sum_{1\leq l<k\leq N}|\Delta_{lk}|^2 \geq \sum_{j=1}^{N}R_j^2(t) \tag{6.23}$$

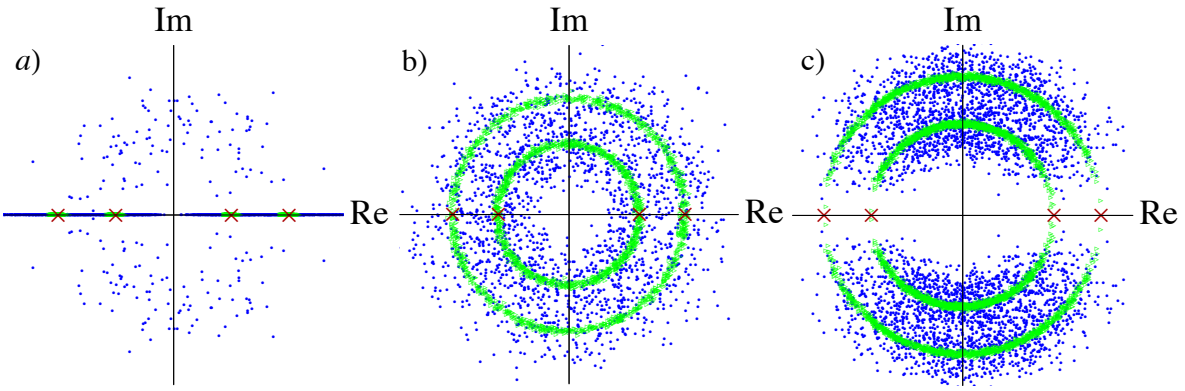

Figure 6. Scatter plots for product matrices of all three Dyson indices $\beta = 1$ (a), $\beta = 2$ (b), and $\beta = 4$ (c). The large red crosses are the positions $(\pm \exp[\psi(\beta n/2)/2]$ with $n = 1, 2)$ of the incremental singular values at $t \to \infty$. All three plots were generated by Monte Carlo simulations of products of Ginibre matrices for $N = 2$ at $t = 5$ (dark blue dots) and $t = 500$ (light green triangles) drawn from an ensemble size 1000. Note that only the case $\beta = 2$ develops an angular independent spectral density while for $\beta = 1$ all eigenvalues will be eventually real as proved by Forrester [57]. For $\beta = 4$ the dependence on the angle becomes non-trivial which we conjecture to be $\sin^2 \varphi$.

(which is equal if and only if the matrix is normal) the agreement of both kinds of Lyapunov exponents does not immediately result in the statement that the matrix $\Pi(t)$ becomes normal in the large t limit. Considering the bound (6.20) we notice that the off-diagonal element $|\Delta|$ may become exponentially large. Indeed one can easily construct such a situation by setting $|z_{1j}|, |z_{2i}| > 1$ for all i, j. Therefore the way how we root the matrices is crucial in the large t limit.

Two questions arise from our result. First, can we generalize our argument to arbitrary matrix dimension N? To answer this we emphasize that our calculation relies on the explicit, known relation between singular values and the components of the generalized Schur decomposition, see eqs. (6.13) and (6.14), which can be indeed extended to the cases $N = 3, 4$. Nevertheless we expect that there is a general argument. Therefore we conjecture that the Lyapunov exponents of the moduli of the complex eigenvalues and of the singular values are deterministic and agree with each other for general isotropic ensembles.

Second, can we generalize our argument to the Dyson indices $\beta = 1, 4$, i.e. to the product of real and quaternion Ginibre matrices? In the case $\beta = 4$ and $N = 2$ one can show that we find a factorization of the probability densities similar to eq. (6.16) and the same calculation can be done analogously. Therefore one can answer the question positively in this case. The situation for general N is much more involved but we expect that also there the Lyapunov exponents qualitatively behave the same as in the case $\beta = 2$, only their positions may change and the angles of the complex eigenvalues will not be uniformly distributed, see Fig. 6.c. Regarding the distribution of the angles we expect that the density behaves as $\sin^2 \varphi$. The reason is the macroscopic distance of the

complex eigenvalues in the large t limit such that the repulsion between the eigenvalues is suppressed. Only the repulsion of a complex conjugate pair will survive since the two eigenvalues lie on the same circle.

The case $\beta = 1$ is as usual non-trivial. The matrices may have real eigenvalues as well as complex conjugate pairs, see [54]. In the case $N = 2$ the situation with a complex conjugate pair immediately yields that the eigenvalues condense on a fixed ring equal to the square root of the determinant of the product matrix. Newman's argument for the singular values applies to all three Dyson indices $\beta = 1, 2, 4$ such that the modulus of the determinant becomes deterministic (it is the product of the singular values) and thus also the the moduli of the complex eigenvalue pairs. However Forrester already showed [57] that in the large t limit almost all eigenvalues will be real. The statistics of these real eigenvalues is still unclear because of the modulus of the Vandermonde determinant. Hence, the probability densities of the single matrices always remain coupled. Therefore we can conclude that the case $\beta = 1$ will not yield the same result as $\beta = 2$ in the angular part of the distribution. But Fig. 6.a shows that the radii still seem to condense at the positions of the singular values.

7. Conclusions

We presented a solvable case of an isotropic time evolution with evolution operators being independent complex $N \times N$ Ginibre matrices. The entire spectrum of Lyapunov exponents, traditionally defined in terms of the singular values, was computed including their positions (which are in agreement with [40, 41, 62, 42]), individual and the joint probability distributions with their $1/t$ corrections in the large t limit. Surprisingly the Lyapunov exponents which can analogously be constructed for the moduli of the complex eigenvalues show exactly the same large t behavior. Thereby they do not only condense on the same values as the Lyapunov exponents for the singular values but also share the same variance and normal distribution around this value. Therefore we understand this behavior as a universal property which is also expected for general isotropic weights and general Dyson index $\beta = 1, 2, 4$.

The normal distributions with means $\psi(n)/2$ and variances $\psi'(n)/(4t)$ are the non-perturbative leading order correction to the deterministic values of the Lyapunov exponents for $t \to \infty$. They agree very well with finite $t \approx 10N$ Monte Carlo simulation for the moduli of the complex eigenvalues while for the singular values we showed that the saddle point approximation of the inverse Fourier transform of the moment generating function yields a better agreement for finite t. The reason is the underlying structure involved in this problem. The joint probability distributions of the singular values and of the complex eigenvalues are given by determinantal point processes reflecting the level repulsion. In the large t limit this repulsion is suppressed and a permanent remains in both cases. The convergence to this result is enhanced for the eigenvalues by prefactors which are absent for the singular values. This shows that the mechanism how the singular values and the eigenvalues approach their deterministic

values $\psi(n)/2$ is different. Nonetheless they share a particular asymptotic expansion of the Meijer G-function with large index and argument which is still at the heart of taking the limit $t \to \infty$.

The limiting angular dependence is uniform for $\beta = 2$. This behavior is in contrast to the case for the product of real and quaternion Ginibre matrices. In the real case all eigenvalues become real [57] while in the quaternion case the level density exhibits a non-trivial angular dependence. Nevertheless we claim that the radii of the eigenvalues will approach the same values as the singular values for all three Dyson indices and general isotropic random matrix ensembles in the limit $t \to \infty$. This is supported by our numerical simulations as well as by a discussion of the case $N = 2$. We also considered the case $\beta = 4$ for Ginibre matrices and found that the Lyapunov exponents constructed from the moduli of the complex eigenvalues indeed take the limit $\psi(2n)/2$ derived for the Lyapunov exponents corresponding to the singular values [42].

Moreover, we showed that the triangular law for $N \to \infty$ can be simply interpreted as the radial distribution of the Ginibre ensemble of the limiting circular law. Thereby we proved that the two limits $t \to \infty$ and $N \to \infty$ commute on the global scale of the spectrum of the product matrix. This commutativity is not valid anymore on the local scale. On the scale of the mean level spacing of the complex eigenvalues the limits by taking $N \to \infty$ first and then $t \to \infty$ yield a level repulsion as found for a complex Ginibre ensemble, i.e. $P(\Delta r)d\Delta r \approx \Delta r^3 d\Delta r$ for $\Delta r \ll 1$, see Refs. [59, 60]. Reversing this order we find the level statistics of the harmonic oscillator for the radii squared. Therefore one has to be careful on which scale of the spectrum one takes both limits. We conjecture the existence of a non-trivial scale of a double scaling limit due to this insight.

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Appendix A. Some identities for Meijer G-functions

Meijer G-functions are a broad class of special functions comprising most of the known special functions. They are defined as the inverse Mellin transform of certain quotients of products of gamma functions. We do not give their general definition, but we restrict ourselves to a small subclass of Meijer G-functions which are used in our calculations.

We consider Meijer G-functions of the following form given by an integral [63]

$$G_{0,\,t}^{t,\,0}\left(\begin{smallmatrix}-\\ a_1,\ldots,a_t\end{smallmatrix}\middle|\, s\right) = \int_{\mathcal{C}} \Gamma(a_1-u)\ldots\Gamma(a_t-u)s^u\frac{du}{2\pi\imath}, \tag{A.1}$$

over a contour $\mathcal{C}$ that goes from $-\imath\infty$ to $+\imath\infty$ leaving all poles of the Gamma functions on the right hand side. The Mellin transform of this function is

$$\int_0^\infty ds s^{u-1} G_{0,\,t}^{t,\,0}\left(\begin{smallmatrix}-\\ a_1,\ldots,a_t\end{smallmatrix}\middle|\, s\right) = \Gamma(a_1-u)\ldots\Gamma(a_k-u)\ . \tag{A.2}$$

Moreover Meijer G-functions fulfill the simple but useful identity

$$s^b G_{0,\,t}^{t,\,0}\left(\begin{smallmatrix}-\\ a_1,\ldots,a_t\end{smallmatrix}\middle|\, s\right) = G_{0,\,t}^{t,\,0}\left(\begin{smallmatrix}-\\ b+a_1,\ldots,b+a_t\end{smallmatrix}\middle|\, s\right). \tag{A.3}$$

which is needed several times in our calculations. This identity is a consequence of the shift $s^u \to s^{u+b}$ in the power in the integrand (A.3) which can be compensated by the substitution $u \to u-b$.

Appendix B. Computation of the normalizing Hankel determinant

In order to be self contained we calculate the Hankel determinant appearing in eq. (3.25),

$$\det_{1\le a,b\le N}\left[\Gamma(a+b-1)\right] = \prod_{a=1}^{N}\Gamma^2(a), \tag{B.1}$$

which is a special case of a results by Normand [64]. We do this by applying Andreief's formula [65]

$$\det_{1\le a,b\le N}\left[\int dx\,\Phi_a(x)\Psi_b(x)\right] = \frac{1}{N!}\int dx_1\ldots dx_N \det_{1\le a,b\le N}\left[\Phi_a(x_b)\right]\det_{1\le a,b\le N}\left[\Psi_a(x_b)\right]. \tag{B.2}$$

Here $\{\Phi_a(x)\}$ and $\{\Psi_a(x)\}$, $a=1,\ldots,N$ are two sets of integrable functions of a real variable.

The Gamma functions on the left hand side of (B.1) can be written as

$$\Gamma(a+b-1) = \int_0^\infty dx\, x^{a+b-2}\exp(-x) = \int_0^\infty dx\,\Phi_a(x)\Psi_b(x), \tag{B.3}$$

such that we identify $\Phi_a(x) = \Psi_a(x) = x^{a-1}\exp(-x/2)$ for $x \ge 0$, $a = 1,\ldots,N$. Andréief's formula then yields

$$\det_{1\le a,b\le N}\left[\Gamma(a+b-1)\right] = \frac{1}{N!}\int dx_1\ldots dx_N\left(\det_{1\le a,b\le N}\left[x_b^{a-1}\exp\left(-\frac{x_b}{2}\right)\right]\right)^2. \tag{B.4}$$

Due to the skew-symmetry of the determinant under permutations as well as its multilinearity the rows can be linearly combined without changing its value. The idea is to combine them in such a way that after applying the Andréief integral again we have to take a determinant of diagonal elements, only. The Laguerre polynomials in monic normalization, denoted by

$$L_n(x) = \sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\frac{(-1)^{n-j}n!}{j!}x^j, \tag{B.5}$$

will do the job. They are orthogonal with respect to the weight $\exp[-x]dx$, i.e.

$$\int_0^\infty dx \exp[-x] L_a(x) L_b(x) = (a!)^2 \delta_{ab}. \tag{B.6}$$

Therefore we have

$$\begin{aligned} \det_{1\le a,b\le N} [\Gamma(a+b-1)] &= \frac{1}{N!} \int dx_1 \dots dx_N \left(\det_{1\le a,b\le N} \left[L_{a-1}(x_b) \exp\left(-\frac{x_b}{2}\right)\right]\right)^2 \\ &= \det_{1\le a,b\le N} \left[\int_0^\infty L_{a-1}(x) L_{b-1}(x) \exp(-x) \right] \\ &= \prod_{a=0}^{N-1} (a!)^2. \end{aligned} \tag{B.7}$$

In the second line we employed eq. (B.2) and in the third line eq. (B.6). The last line is nothing else than the claim (B.1).

In a similar way we want to calculate the cofactor of the Hankel determinant (3.25),

$$C_{jl} = (-1)^{j+l} \det_{\substack{1\le a,b\le N \\ a\neq j, b\neq l}} [\Gamma(a+b-1)], \tag{B.8}$$

which appears in eq. (3.34). Also this determinant can be calculated via the Andréief integral. For this purpose we introduce two integrals over the angles φ_1 and φ_2,

$$C_{jl} = -\int_0^{2\pi} \frac{d\varphi_1}{2\pi} \int_0^{2\pi} \frac{d\varphi_2}{2\pi} \det \begin{bmatrix} \left\{ \int_0^\infty dx\, x^{a+b-2} \exp(-x) \right\}_{1\le a,b\le N} & \left\{ e^{\imath(a-j)\varphi_1} \right\}_{1\le a\le N} \\ \left\{ e^{\imath(b-l)\varphi_2} \right\}_{1\le b\le N} & 0 \end{bmatrix}. \tag{B.9}$$

We use the same trick again by rearranging the columns and rows such that we have in the upper left block integrals over two Laguerre polynomials and thus a diagonal matrix. An expansion in this diagonal matrix yields

$$C_{jl} = \prod_{a=0}^{N-1} (a!)^2 \int_0^{2\pi} \frac{d\varphi_1}{2\pi} \int_0^{2\pi} \frac{d\varphi_2}{2\pi} \exp[\imath([1-j]\varphi_1 + [1-l]\varphi_2)] \sum_{k=0}^{N-1} \frac{L_k(e^{\imath\varphi_1}) L_k(e^{\imath\varphi_2})}{(k!)^2}. \tag{B.10}$$

In the last step the two integrals, which factorize, can be performed and we find

$$C_{jl} = (-1)^{j+l} \prod_{a=0}^{N-1} (a!)^2 \sum_{k=0}^{N-1} \left(\frac{k!}{(j-1)!(l-1)!} \right)^2 \frac{1}{\Gamma(k-j+2)\Gamma(k-l+2)}. \tag{B.11}$$

Note that the function $1/\Gamma(z)$ is an entire function which is zero for negative semi-definite integers. Therefore the sum is usually smaller than the boundary shown here, i.e. its range is $k = \max\{j,l\} - 1, \dots, N-1$.

Appendix C. Saddle point approximation of $f_{ab}(\mu)$

We consider the saddle point approximation of the inverse Fourier transform of the moment generating function (3.11),

$$\begin{aligned} f_{ab}(\mu) &= \int_{-\imath\infty}^{+\imath\infty} \frac{d\vartheta}{2\pi\imath} \exp[-\mu\vartheta] M_{ab}(\vartheta) \\ &= \int_{-\imath\infty}^{+\imath\infty} \frac{d\vartheta}{2\pi\imath} \exp[-\mu\vartheta] \frac{\Gamma^{t-1}(b+\vartheta/(2t))\Gamma(a+b-1+\vartheta/(2t))}{\Gamma^{t-1}(b)\Gamma(a+b-1)}. \end{aligned} \tag{C.1}$$

After rescaling $\vartheta \to 2t\vartheta$ the saddle point equation and its solution are

$$\psi(b+\vartheta_b(\mu)) = 2\mu \Rightarrow \vartheta_b(\mu) = \int_0^\infty dy \Theta(2\mu - \psi(y)) - b = \vartheta_0(\mu) - b, \tag{C.2}$$

where Θ is the Heaviside function. In fact there are also other saddle points. However only the solution $\vartheta_b(\mu) = \vartheta_0(\mu) - b$ can be reached in the limit $t \to \infty$. We perform the saddle point expansion $\vartheta = \vartheta_0(\mu) - b + \imath\delta\vartheta/\sqrt{t}$ and find

$$\begin{aligned} f_{ab}(\mu) &\overset{t\gg 1}{\approx} \frac{2\sqrt{t}\Gamma^{t-1}(\vartheta_0(\mu))\Gamma(a-1+\vartheta_0(\mu))\exp[-2t\mu(\vartheta_0(\mu)-b)]}{\Gamma^{t-1}(b)\Gamma(a+b-1)} \\ &\quad \times \int_{-\infty}^{+\infty} \frac{d\delta\vartheta}{2\pi} \exp\left[-\frac{\psi'(\vartheta_0(\mu))\delta\vartheta^2}{2}\right] \\ &= \sqrt{\frac{2t}{\pi\psi'(\vartheta_0(\mu))}} \frac{\Gamma^{t-1}(\vartheta_0(\mu))\Gamma(a-1+\vartheta_0(\mu))}{\Gamma^{t-1}(b)\Gamma(a+b-1)} \exp[-2t\mu(\vartheta_0(\mu)-b)]. \end{aligned} \tag{C.3}$$

This expression seems to factorize in a b and an a dependent part apart from the constant prefactor $1/\Gamma(a+b-1)$ but this is a misleading conclusion. The argument μ also depends on the index b in the determinant (3.10). Therefore the level repulsion corresponding to the determinant is still present in this particular approximation.

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RELATING THE BURES MEASURE TO THE CAUCHY TWO-MATRIX MODEL

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Abstract. The Bures metric is a natural choice in measuring the distance of density operators representing states in quantum mechanics. In the past few years a random matrix ensemble and the corresponding joint probability density function of its eigenvalues was identified. Moreover a relation with the Cauchy two-matrix model was discovered but never thoroughly investigated, leaving open in particular the following question: How are the kernels of the Pfaffian point process of the Bures random matrix ensemble related to the ones of the determinantal point process of the Cauchy two-matrix model and moreover, how can it be possible that a Pfaffian point process derives from a determinantal point process? We give a very explicit answer to this question. The aim of our work has a quite practical origin since the calculation of the level statistics of the Bures ensemble is highly mathematically involved while we know the statistics of the Cauchy two-matrix ensemble. Therefore we solve the whole level statistics of a density operator drawn from the Bures prior.

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1. Introduction

The measuring and estimation of the distance between a density operator $\rho_{\rm exp}$ assumed after a finite number of experiments and the true quantum state $\rho_{\rm true}$ is an intrinsically hard task [63]. It becomes even harder since the metric on the set of quantum states is not uniquely determined by the standard conditions that a density operator ρ has to be positive definite, Hermitian and the trace is equal to one ($\mathrm{tr}\,\rho = 1$). Imposing additional conditions onto the metric restricts this ambiguity. The Bures metric [27],

$$D(\rho_1, \rho_2) = \sqrt{2 - 2\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}} = \sqrt{2 - 2\mathrm{tr}\sqrt{\sqrt{\rho_2}\rho_1\sqrt{\rho_2}}}, \tag{1.1}$$

plays a distinguished role since it is the only metric which is also monotone, Fisher-adjusted, Fubini-Study-adjusted, and Riemannian, see Ref. [67] for a clear explanation of these notions. One important application of the Bures metric is to define a geometric quantum discord measuring the strength of quantum correlations. Some recent works on this topic are Refs. [1, 21, 28, 36, 38, 56, 58, 62, 69].

The property that the Bures metric is Riemannian is particularly important for the application of random matrix theory. The Riemannian length element can be obtained by considering the distance between an $N \times N$ dimensional density operator ρ with eigenvalues $z = \mathrm{diag}(z_1, \ldots, z_N) \in [0,1]^N$ and its infinitesimal

neighbour $\rho + d\rho$ yielding [39, 40]

$$ds^2 = [D(\rho, \rho + d\rho)]^2 = \frac{1}{2} \sum_{i,j=1}^{N} \frac{d\rho_{ij}^2}{z_i + z_j}. \tag{1.2}$$

The joint probability density of the eigenvalues of ρ is then [37]

$$p^{(N,a,\text{fixed})}(z) \propto \delta \left(1 - \sum_{j=1}^{N} z_j \right) \prod_{j=1}^{N} z_j^a \prod_{1 \le i < j \le N} \frac{(z_j - z_i)^2}{z_j + z_i}, \tag{1.3}$$

where the case $a = -1/2$ corresponds to a full rank. When considering density operators of rank $M \le N$ the exponent is $a = -1/2 + N - M$, see Ref. [67]. The latter case becomes important when measuring the separability of qubits and qutrits (systems of two and three quantum states) on hyperareas in the set of quantum states [64, 65].

We remark that the joint probability density (1.3) can be interpreted as a log-gas with the pair-interaction $\exp[2\ln|z_i - z_j| - \ln|z_i + z_j|]$. This interpretation has been successfully applied in [19] to calculate the distribution of the purity $\Sigma_2 = \operatorname{tr} \rho^2 = \sum_{j=1}^{N} z_j^2$ in the limit of large N.

The authors of Ref. [57] showed that the ensemble of density operators of dimension $N \times N$ with rank $M \le N$ distributed via the Bures measure can be generated by the random matrix[1]

$$\rho = \frac{A(1+U)(1+U^\dagger)A^\dagger}{\operatorname{tr}(1+U)AA^\dagger(1+U^\dagger)}, \tag{1.4}$$

where the complex $N \times M$ matrix A is distributed by a Gaussian and the unitary $M \times M$ matrix U by the Haar-measure on the unitary group $\mathrm{U}(M)$ with the prefactor $|\det(1+U)|^{2(N-M)}$. This approach connects the Bures measure with another topic in random matrix theory, namely product matrices. Quite recently there is a revival of interest in product matrices due to breakthroughs in the approach of asymptotic freeness and of integrable structures, with the latter applying at finite matrix dimension N and finite number of matrices multiplied. This allowed the study of the singular values and the eigenvalues for finite matrix size as well as in particular limits for all three Dyson indices $\beta = 1, 2, 4$ corresponding to real, complex and quaternion matrices, respectively, see Refs. [8, 5, 32, 33, 48, 49, 55, 4, 42, 72, 70, 22, 25, 3, 26, 24, 23, 41, 6, 59, 54]. Even the convergence properties of Lyapunov exponents for a product of infinitely many matrices can now be studied in detail exhibiting interesting scaling limits [31, 9]. Additionally, interesting relations to the distributions which have the combinatorial series of the Fuß-Catalan numbers, the Raney numbers and their generalizations as their moments [68, 59, 34, 9, 54] have arisen. Exactly this relation was recently employed to calculate the level density for a generalized version of the Bures measure [53].

One particular limit is on the local scale of the mean level spacing at the hard edge, which is also known as the microscopic limit since it is the scale of the smallest eigenvalues around the origin. It was shown that product matrices show a different

[1]Note that in Ref. [57] the order of the product is the other way around since the non-zero eigenvalues do not depend on this order. However in their order the density operator has full rank because the generic zero modes are chopped off.

universal behaviour than the chiral Gaussian random matrix ensembles. The eigenvalues and singular values are no longer described by the Bessel kernel but by Meijer G-functions which seem to also hold for a larger class of random matrix ensembles [3, 4, 33, 48, 49, 55]. Exactly the same questions about the level statistics can be asked for density operators distributed via the Bures measure. Are those operators also in the universality class given by Meijer G-functions? Indeed this question is legitimate since another random matrix ensemble, namely the Cauchy two-matrix model [14], also exhibits a Meijer G-kernel at the hard edge [16], and is also known to involve products of random matrices [33]. Interestingly, [14] contains a relation between a particular kind of a Cauchy two-matrix model and the Bures ensemble. This relation was not worked out in detail since only the normalization constants of both ensembles were considered. Thus the question is if the Cauchy two-matrix model and the Bures prior have more in common than the normalization. This was raised in [14]: *"The relationship between the two model does not seem to go much further in the sense that there is no direct and simple relationship between the correlation functions of the two models. It seems, however, that some connection should be present and is worth exploring. We leave it as an open problem to establish a connection between these two models on the level of the correlation functions."* Here we solve this open problem by expressing the correlation kernels for Bures measure in terms of those for the Cauchy two-matrix model. Thus we have an analytical formula for all eigenvalue statistics, and in particular for the eigenvalue density of the density operators at finite matrix dimension N. One consequence, to be explored in a subsequent work, is thus corresponding analytical formulae for moments of the eigenvalue density. Existing studies relating to this topic are restricted to numerical computations in the $N = 3$ case [66], with a number of exact results conjectured.

The joint probability distribution of the eigenvalues we are considering is not exactly the one of the Bures measure (1.3) but the Fourier-Laplace transform of it,

$$p^{(N,a,\mathrm{B})}(z) \propto \prod_{j=1}^{N} z_j^a \mathrm{e}^{-z_j} \prod_{1\le i<j\le N} \frac{(z_j - z_i)^2}{z_j + z_i}, \tag{1.5}$$

where $a > -1$ and each $z_\nu \geq 0$. Despite this fact we call the random matrix ensemble corresponding to this distribution the Bures ensemble, too. It is exactly the distribution (1.5) which is directly related to the Cauchy two-matrix ensemble. This relation is remarkable since the Cauchy two-matrix model is a determinantal point process corresponding to bi-orthogonal polynomials [15] (all important facts will be recall in section 2) while the joint probability distribution (1.5) forms a Pfaffian point process (see e.g. [30, Ch. 6]) due to Schur Pfaffian identity (see e.g. [43]). As is well known from the study of the Hilbert-Schmidt measure in the context of quantum density matrices, the imposition of the Dirac delta function in mapping from the correlations for Eq. (1.5) to those for the original Bures measure (1.3) requires coupling of the Pfaffian structure at finite N to an auxiliary scaling variable multiplying each eigenvalue, which in turn is subject to a Fourier-Laplace transform (see e.g. [50, Eq. (2.28)]). Although not pursued in the present work, again results for the Hilbert-Schmidt measure [50] lead us to expect that the local spectral statistics should be the same due to universality. Furthermore, the joint probability density (1.5) is interesting on its own since it can be also found in a completely different topic. In two-dimensional quantum gravity $p^{(N,a,\mathrm{B})}(z)$ is related to the $\mathrm{O}(N)$ vector model, see [47].

The determinantal point process of the Cauchy two-matrix model is recalled in section 2. Thereby we also calculate the average of ratios and products of characteristic polynomials of the two matrices which was not done before. This derivation is analogous to the computations in [11, 18, 45, 71]. On first sight it seems to be impossible to marry the kernels of the Bures ensemble with the Cauchy two-matrix model. We show in section 3 that this is nonetheless possible because of the particular form of the joint probability density. Pursing this approach we are able to express all kernels of the Bures ensemble in terms of the kernels for the Cauchy two-matrix model which were already derived in Refs. [14, 15, 16].

In section 5 we summarize and briefly discuss our results. In the appendices we present the details of our proofs.

2. Recalling the Cauchy two-matrix model

Since the Cauchy two-matrix model, introduced in Ref. [14], plays a crucial role we want to recall relevant known facts. The general Cauchy two-matrix ensemble is a measure on two $N \times N$ positive definite Hermitian matrices M_1, M_2 distributed by

$$P^{(\mathrm{C})}(M_1, M_2) = \frac{\exp[-N(V_1(M_1) + V_2(M_2) + \mathrm{tr}\ln(M_1 + M_2))]}{\int d[M_1] \int d[M_2] \exp[-N(V_1(M_1) + V_2(M_2)) - N\mathrm{tr}\ln(M_1 + M_2)]}, \tag{2.1}$$

where $d[M_1]$ and $d[M_2]$ are the products of differentials of the real independent matrix entries of M_1 and M_2, respectively. The functions V_1, V_2 are referred to as potentials. For the special choice $\exp[-NV_1(M)] = \det^a M \exp[-\mathrm{tr}\, M]$, $\exp[-NV_2(M)] = \det^b M \ \exp[-\mathrm{tr}\, M]$, the corresponding joint probability density of the eigenvalues of M_1 and M_2 takes the form

$$p^{(N,a,b,\mathrm{C})}(x, y) \propto \frac{\prod_{j=1}^N x_j^a \mathrm{e}^{-x_j} y_j^b \mathrm{e}^{-y_j} \prod_{1 \le j < k \le N} (x_k - x_j)^2 (y_k - y_j)^2}{\prod_{j=1}^N \prod_{k=1}^N (x_j + y_k)}. \tag{2.2}$$

It enjoys a number of special integrability properties [16], culminating in the explicit evaluation of the joint hard edge correlation function.

In subsection 2.1, we introduce general partition functions for the Cauchy two-matrix ensemble as the integral over products and ratios of characteristic polynomials weighted by the joint probability density (2.2). Starting from these partition functions we recall the bi-orthogonal polynomials, their Cauchy transforms, and some other transforms employed in Ref. [14, 15, 16], in subsection 2.2. These polynomials together with partition functions comprising of only two characteristic polynomials (presented in subsection 2.3) build the kernels of the determinantal point process which the Cauchy two-matrix model obeys. Eventually we exhibit the explicit form of the (k, l)-point correlation function of the joint probability density (2.2) in subsection 2.4. When presenting these known results we briefly sketch their derivation to make our work self-contained, particularly introducing our notation and the normalization which slightly differs from the one in Ref. [14, 15, 16].

2.1. Setting of the Cauchy two-matrix model.

Generalised partition functions with ratios of characteristic polynomials play a crucial role when analyzing spectral statistics, see [11, 18, 44, 45, 46, 71] for the computations of those averages for some

ensembles. This is also true for the Cauchy two-matrix ensemble whose partition function may consist of four sets of characteristic polynomials

$$Z^{(N,a,b,\mathrm{C})}_{k_1|l_1;k_2|l_2}(\kappa_1,\lambda_1;\kappa_2,\lambda_2) := \frac{1}{(N!)^2}\int_{\mathbb{R}_+^{2N}} \frac{\Delta_N^2(x)\Delta_N^2(y)\,d[x]d[y]}{\prod_{i,j=1}^N(x_i+y_j)} \times \prod_{j=1}^N \left(x_j^a y_j^b \mathrm{e}^{-x_j-y_j} \frac{\prod_{i=1}^{l_1}(x_j-\lambda_{1,i})\prod_{i=1}^{l_2}(y_j-\lambda_{2,i})}{\prod_{i=1}^{k_1}(x_j-\kappa_{1,i})\prod_{i=1}^{k_2}(y_j-\kappa_{2,i})} \right). \tag{2.3}$$

Here the variables $\kappa_1 = \mathrm{diag}(\kappa_{1,1},\ldots,\kappa_{1,k_1})$ and $\kappa_2 = \mathrm{diag}(\kappa_{2,1},\ldots,\kappa_{2,k_2})$ do not lie on the positive real line while $\lambda_1 = \mathrm{diag}(\lambda_{1,1},\ldots,\lambda_{1,l_1})$ and $\lambda_2 = \mathrm{diag}(\lambda_{2,1},\ldots,\lambda_{2,l_2})$ can be arbitrary complex numbers. Moreover we have used the Vandermonde determinant

$$\Delta_N(u) = \prod_{1\le i<j\le N}(u_j-u_i) = \det[u_a^{b-1}]_{1\le a,b,\le N} \tag{2.4}$$

which is one part of the Jacobian from diagonalizing the matrices M_1 and M_2. The notation of the indices of the partition function (2.3) is reminiscent of the one used in the supersymmetry approach applied to random matrix theory, see Ref. [13, 45], namely this kind of partition function is intimately related to supersymmetry. Moreover the prefactor $1/(N!)^2$ normalizes the partition function such that it is equivalent to one for an ordered set of variables $x_1 \le x_2 \le \ldots \le x_N$ and $y_1 \le y_2 \le \ldots \le y_N$. However such an ordering is often quite inconvenient for calculations so we consider the integral without an ordering. The measures $d[x] = dx_1dx_2\cdots dx_N$ and $d[y] = dy_1dy_2\cdots dy_N$ are the products of the differentials of the variables x and y.

Let us consider some particular cases of these partition functions. The case $k_1 = k_2 = l_1 = l_2 = 0$ yields the normalization constant of the joint probability density (2.2),

$$\begin{aligned} Z^{(N,a,b,\mathrm{C})}_{0|0;0|0} &= \det\left[\int_{\mathbb{R}_+^2} dxdy \frac{x^{a+i-1}y^{b+j-1}\mathrm{e}^{-x-y}}{x+y}\right]_{1\le i,j\le N} \\ &= \prod_{j=1}^N \frac{[(j-1)!]^2\Gamma[a+j]\Gamma[b+j]\Gamma[a+b+j]}{\Gamma[a+b+N+j]}, \end{aligned} \tag{2.5}$$

see Ref. [14]. This result can be obtained by applying Andréief's integration theorem [10] and evaluating the determinant.

2.2. **Bi-orthogonal polynomials of the Cauchy two-matrix model.** The two kinds of bi-orthogonal polynomials can be calculated with the help of these partition functions, since they correspond to the averaged characteristic polynomials for the variables $\{x_i\}$ and $\{y_j\}$. These are the cases $(k_1|l_1;k_2|l_2) = (0|1;0|0)$ and

$(k_1|l_1;k_2|l_2) = (0|0;0|1)$ and yield the polynomials in monic normalization [17]

$$\begin{aligned} p_n^{(a,b)}(x) &= (-1)^n \frac{Z_{0|1;0|0}^{(n,a,b,\mathrm{C})}(x)}{Z_{0|0;0|0}^{(n,a,b,\mathrm{C})}} \\ &= \frac{\det\left[\int_{\mathbb{R}_+^2} dx'dy \frac{x'^{a+i-1}y^{b+j-1}\mathrm{e}^{-x'-y}}{x'+y} \,\middle|\, x^{i-1} \right]_{\substack{1\le i\le n+1 \\ 1\le j\le n}}}{\det\left[\int_{\mathbb{R}_+^2} dxdy \frac{x^{a+i-1}y^{b+j-1}\mathrm{e}^{-x-y}}{x+y}\right]_{1\le i,j\le n}} \\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{\Gamma(a+b+n+j+1)\Gamma(a+b+n+1)\Gamma(a+n+1)}{\Gamma(a+b+2n+1)\Gamma(a+b+j+1)\Gamma(a+j+1)}x^j \\ &= (-1)^n \frac{(a+b+1)_n(a+1)_n}{(a+b+n+1)_n}\, {}_2F_2\left(\begin{matrix} -n,\ a+b+n+1 \\ a+b+1,\ a+1 \end{matrix}\,\middle|\, x\right), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \widetilde{p}_n^{(a,b)}(y) &= (-1)^n \frac{Z_{0|0;0|1}^{(n,a,b,\mathrm{C})}(y)}{Z_{0|0;0|0}^{(n,a,b,\mathrm{C})}} \\ &= (-1)^n \frac{(a+b+1)_n(b+1)_n}{(a+b+n+1)_n}\, {}_2F_2\left(\begin{matrix} -n,\ a+b+n+1 \\ a+b+1,\ b+1 \end{matrix}\,\middle|\, y\right). \end{aligned} \tag{2.7}$$

Both polynomials can be derived by applying a generalized version of Andréief's integration theorem [45, 10] and then expanding and evaluating the determinant. These steps are illustrated in lines two and three of (2.6). This calculation was performed in Ref. [16]. The functions ${}_pF_q$ are the generalized hypergeometric functions given by [35]

$${}_pF_q\left(\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}\,\middle|\, z\right) = \sum_{j=0}^{\infty} \frac{\prod_{l=1}^{p}(a_l)_j}{\prod_{l=1}^{q}(b_l)_j}\frac{z^j}{j!} \tag{2.8}$$

with

$$(a)_j = \prod_{l=0}^{j-1}(a+l) = \frac{\Gamma(a+j)}{\Gamma(a)} \tag{2.9}$$

the Pochhammer symbol. The polynomials (2.6) and (2.7) are bi-orthogonal with respect to the weight $g^{(a,b,\mathrm{C})}(x,y) = x^a y^b \exp[-x-y]/(x+y)$,

$$\begin{aligned} \int_{\mathbb{R}_+^2} dxdy\, g^{(a,b,\mathrm{C})}(x,y) p_n^{(a,b)}(x)\widetilde{p}_l^{(a,b)}(y) &= \frac{Z_{0|0;0|0}^{(n+1,a,b,\mathrm{C})}}{Z_{0|0;0|0}^{(n,a,b,\mathrm{C})}}\delta_{nl} \\ &= \frac{[n!]^2\Gamma[a+n+1]\Gamma[b+n+1](\Gamma[a+b+n+1])^2}{\Gamma[a+b+2n+2]\Gamma[a+b+2n+1]}\delta_{nl}. \end{aligned} \tag{2.10}$$

Another particularly helpful representation of the bi-orthogonal polynomials is in terms of Meijer G-functions [35],

$$G^{m,n}_{p,q}\left(\left.\begin{matrix}a_1,\ldots,a_n;a_{n+1},\ldots,a_p\\ b_1,\ldots,b_m;b_{m+1},\ldots,b_q\end{matrix}\right|z\right)=\int_C\frac{ds}{2\pi\imath}\frac{z^s\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^p\Gamma(a_j-s)},\tag{2.11}$$

where the contour C goes from $-\imath\infty$ to $+\imath\infty$ and lets the poles of $\Gamma(b_j-s)$ on the right side of the path while the poles of $\Gamma(1-a_j+s)$ are on the left side. Since each generalized hypergeometric function is a Meijer G-function the polynomials (2.6) and (2.7) are

$$\begin{aligned}p_n^{(a,b)}(x)&=(-1)^n\frac{n!\Gamma(a+b+n+1)\Gamma(a+n+1)}{\Gamma(a+b+2n+1)}G^{1,1}_{2,3}\left(\left.\begin{matrix}-a-b-n;n+1\\0;-a,-a-b\end{matrix}\right|x\right),\\ \widetilde{p}_n^{(a,b)}(y)&=(-1)^n\frac{n!\Gamma(a+b+n+1)\Gamma(b+n+1)}{\Gamma(a+b+2n+1)}G^{1,1}_{2,3}\left(\left.\begin{matrix}-a-b-n;n+1\\0;-b,-a-b\end{matrix}\right|y\right),\end{aligned}\tag{2.12}$$

cf. Ref. [16].

The Cauchy-transform of these polynomials frequently appear, too. They are proportional to the partition functions with the indices $(k_1|l_1;k_2|l_2)=(1|0;0|0)$ and $(k_1|l_1;k_2|l_2)=(0|0;1|0)$,

$$\begin{aligned}&\mathcal{C}[\widetilde{p}]_n^{(a,b)}(x)=(-1)^n\frac{Z^{(n,a,b,\mathrm{C})}_{1|0;0|0}(x)}{Z^{(n,a,b,\mathrm{C})}_{0|0;0|0}}\\ &=\frac{\det\left[\int_{\mathbb{R}_+^2}dx'dy'\frac{x'^{a+j-1}y'^{b+i-1}\mathrm{e}^{-x'-y'}}{x'+y'}\;\middle|\;\int_{\mathbb{R}_+^2}dx'dy'\frac{x'^{a}y'^{b+i-1}\mathrm{e}^{-x'-y'}}{(x-x')(x'+y')}\right]_{\substack{1\le i\le n\\1\le j\le n-1}}}{\det\left[\int_{\mathbb{R}_+^2}dx'dy'\frac{x'^{a+i-1}y'^{b+j-1}\mathrm{e}^{-x'-y'}}{x'+y'}\right]_{1\le i,j\le n}}\\ &=\frac{\Gamma[a+b+2n]}{[(n-1)!]^2\Gamma[a+n]\Gamma[b+n]\Gamma[a+b+n]}\int_{\mathbb{R}_+^2}dx'dy'\frac{x'^{a}y'^{b}\mathrm{e}^{-x'-y'}}{(x-x')(x'+y')}\widetilde{p}_{n-1}^{(a,b)}(y'),\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}&\mathcal{C}[p]_n^{(a,b)}(y)=(-1)^n\frac{Z^{(n,a,b,\mathrm{C})}_{0|0;1|0}(y)}{Z^{(n,a,b,\mathrm{C})}_{0|0;0|0}}\\ &=\frac{\Gamma[a+b+2n]}{[(n-1)!]^2\Gamma[a+n]\Gamma[b+n]\Gamma[a+b+n]}\int_{\mathbb{R}_+^2}dx'dy'\frac{x'^{a}y'^{b}\mathrm{e}^{-x'-y'}}{(y-y')(x'+y')}p_{n-1}^{(a,b)}(x'),\end{aligned}\tag{2.14}$$

with $x, y \notin \mathbb{R}_+$. In Ref. [16] it was shown that the integral over y' for $\mathcal{C}[\widetilde{p}]_n^{(a,b)}$ and x' for $\mathcal{C}[p]_n^{(a,b)}$ is equal to a Meijer G-function, too, such that

$$\begin{aligned}\mathcal{C}[\widetilde{p}]_n^{(a,b)}(x) &= \frac{(-1)^n(a+b+2n-1)}{(n-1)!\Gamma[a+n]}\int_{\mathbb{R}_+} dx' \frac{x'^a}{x'-x} G^{2,1}_{2,3}\left(\begin{matrix}-a-n+1; n+b\\ 0, b; -a\end{matrix}\middle| x'\right),\\ \mathcal{C}[p]_n^{(a,b)}(y) &= \frac{(-1)^n(a+b+2n-1)}{(n-1)!\Gamma[b+n]}\int_{\mathbb{R}_+} dy' \frac{y'^b}{y'-y} G^{2,1}_{2,3}\left(\begin{matrix}-b-n+1; n+a\\ 0, a; -b\end{matrix}\middle| y'\right).\end{aligned} \tag{2.15}$$

Also the remaining integrals can be performed by using the following four remarkable properties of Meijer G-functions [60]

$$z^\gamma G^{m,n}_{p,q}\left(\begin{matrix}a_1,\ldots,a_n; a_{n+1},\ldots,a_p\\ b_1,\ldots,b_m; b_{m+1},\ldots,b_q\end{matrix}\middle| z\right) = G^{m,n}_{p,q}\left(\begin{matrix}a_1+\gamma,\ldots,a_n+\gamma; a_{n+1}+\gamma,\ldots,a_p+\gamma\\ b_1+\gamma,\ldots,b_m+\gamma; b_{m+1}+\gamma,\ldots,b_q+\gamma\end{matrix}\middle| z\right),$$

$$G^{m,n}_{p,q}\left(\begin{matrix}a_1,\ldots,a_n; a_{n+1},\ldots,a_p\\ b_1,\ldots,b_m; b_{m+1},\ldots,b_q\end{matrix}\middle| z\right) = G^{n,m}_{q,p}\left(\begin{matrix}1-b_1,\ldots,1-b_m; 1-b_{m+1},\ldots,1-b_q\\ 1-a_1,\ldots,1-a_n; 1-a_{n+1},\ldots,1-a_p\end{matrix}\middle| \frac{1}{z}\right),$$

$$\begin{aligned}&\int_0^\infty \frac{dz'}{z'} G^{m,n}_{p,q}\left(\begin{matrix}a_1,\ldots,a_n; a_{n+1},\ldots,a_p\\ b_1,\ldots,b_m; b_{m+1},\ldots,b_q\end{matrix}\middle| \frac{z}{z'}\right) G^{m',n'}_{p',q'}\left(\begin{matrix}a'_1,\ldots,a'_{n'}; a'_{n'+1},\ldots,a'_{p'}\\ b'_1,\ldots,b'_{m'}; b'_{m'+1},\ldots,b'_{q'}\end{matrix}\middle| z'\right)\\ &\qquad = G^{m+m',n+n'}_{p+p',q+q'}\left(\begin{matrix}a_1,\ldots,a_n, a'_1,\ldots,a'_{n'}; a_{n+1},\ldots,a_p, a'_{n'+1},\ldots,a'_{p'}\\ b_1,\ldots,b_m, b'_1,\ldots,b'_{m'}; b_{m+1},\ldots,b_q, b'_{m'+1},\ldots,b'_{q'}\end{matrix}\middle| z\right),\end{aligned}$$

$$G^{m,n+1}_{p+1,q+1}\left(\begin{matrix}a_1,\ldots,a_n, c; a_{n+1},\ldots,a_p\\ b_1,\ldots,b_m; b_{m+1},\ldots,b_q, c\end{matrix}\middle| z\right) = G^{m,n}_{p,q}\left(\begin{matrix}a_1,\ldots,a_n; a_{n+1},\ldots,a_p\\ b_1,\ldots,b_m; b_{m+1},\ldots,b_q\end{matrix}\middle| z\right) \tag{2.16}$$

and the identification for a special case of a Meijer G-function

$$G^{1,1}_{1,1}\left(\begin{matrix}0\\ 0\end{matrix}\middle| z\right) = \frac{1}{1+z}. \tag{2.17}$$

Then the Cauchy transforms of the polynomials are also Meijer G-functions

$$\begin{aligned}\mathcal{C}[\widetilde{p}]_n^{(a,b)}(x) &= \frac{(-1)^{n+1}(a+b+2n-1)}{(n-1)!\Gamma[a+n]} G^{3,1}_{2,3}\left(\begin{matrix}-n; n+a+b-1\\ -1, a-1, a+b-1\end{matrix}\middle| -x\right),\\ \mathcal{C}[p]_n^{(a,b)}(y) &= \frac{(-1)^{n+1}(a+b+2n-1)}{(n-1)!\Gamma[b+n]} G^{3,1}_{2,3}\left(\begin{matrix}-n; n+a+b-1\\ -1, b-1, a+b-1\end{matrix}\middle| -y\right).\end{aligned} \tag{2.18}$$

When replacing $x \to x + \imath\varepsilon$ and $y \to y + \imath\varepsilon$ with $x, y \in \mathbb{R}_+$ and taking the imaginary part of the Cauchy transform in the limit $\varepsilon \to 0$ we recover the result of Ref. [16],

$$\begin{aligned}&\frac{1}{\pi}\lim_{\varepsilon\to 0}\operatorname{Im}\mathcal{C}[\widetilde{p}]_n^{(a,b)}(x+\imath\varepsilon)\\ &\quad = -\frac{\Gamma[a+b+2n]}{[(n-1)!]^2\Gamma[a+n]\Gamma[b+n]\Gamma[a+b+n]}\int_{\mathbb{R}_+} dy' \frac{x^a y'^b \mathrm{e}^{-x-y'}}{x+y'}\widetilde{p}^{(a,b)}_{n-1}(y')\\ &\quad = \frac{(-1)^n(a+b+2n-1)}{(n-1)!\Gamma[a+n]} x^a G^{2,1}_{2,3}\left(\begin{matrix}-a-n+1; n+b\\ 0, b; -a\end{matrix}\middle| x\right),\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\pi}\lim_{\varepsilon\to 0}\operatorname{Im}\mathcal{C}[p]_n^{(a,b)}(y+\imath\varepsilon)\\
&\quad=-\frac{\Gamma[a+b+2n]}{[(n-1)!]^2\Gamma[a+n]\Gamma[b+n]\Gamma[a+b+n]}\int_{\mathbb{R}_+}dx'\frac{x'^a y^b \mathrm{e}^{-x'-y}}{x'+y}p_{n-1}^{(a,b)}(x')\\
&\quad=\frac{(-1)^n(a+b+2n-1)}{(n-1)!\Gamma[b+n]}y^b G_{2,3}^{2,1}\left(\begin{matrix}-b-n+1;n+a\\0,a;-b\end{matrix}\middle|\, y\right).
\end{aligned}\tag{2.19}$$

We show in subsection 3.2 that the expressions in terms of Meijer G-functions will carry over to the Bures ensemble as well.

2.3. **Determinantal point process of the Cauchy two-matrix model.** In general the partition function (2.3) gives rise to a determinantal point process (see Refs. [17, 45] for general ensembles corresponding to bi-orthogonal polynomials) and can be expressed in terms of partition functions with one and two characteristic polynomials only. Assuming that $N+l_1-k_1=N+l_2-k_2=\widetilde{N}>1$, and $\kappa_1,\kappa_2,\lambda_1,\lambda_2$ pairwise different then we have

$$\begin{aligned}
Z^{(N,a,b,\mathrm{C})}_{k_1|l_1;k_2|l_2}(\kappa_1,\lambda_1;\kappa_2,\lambda_2)&=\frac{(-1)^{k_1(k_1-1)/2+k_2(k_2-1)/2+l_1l_2}Z^{(\widetilde{N},a,b,\mathrm{C})}_{0|0;0|0}}{\mathrm{B}_{k_1|l_1}(\kappa_1;\lambda_1)\mathrm{B}_{k_2|l_2}(\kappa_2;\lambda_2)}\\
\times\det\left[\begin{array}{c|c}\dfrac{Z^{(\widetilde{N}+1,a,b,\mathrm{C})}_{1|0;1|0}(\kappa_{1,i};\kappa_{2,j})}{Z^{(\widetilde{N},a,b,\mathrm{C})}_{0|0;0|0}} & \dfrac{1}{\kappa_{1,i}-\lambda_{1,j}}\dfrac{Z^{(\widetilde{N},a,b,\mathrm{C})}_{1|1;0|0}(\kappa_{1,i},\lambda_{1,j})}{Z^{(\widetilde{N},a,b,\mathrm{C})}_{0|0;0|0}}\\[2ex]\hline\\[-2ex]\dfrac{1}{\kappa_{2,j}-\lambda_{2,i}}\dfrac{Z^{(\widetilde{N},a,b,\mathrm{C})}_{0|0;1|1}(\kappa_{2,j},\lambda_{2,i})}{Z^{(\widetilde{N},a,b,\mathrm{C})}_{0|0;0|0}} & -\dfrac{Z^{(\widetilde{N}-1,a,b,\mathrm{C})}_{1|0;0|1}(\lambda_{1,j};\lambda_{2,i})}{Z^{(\widetilde{N},a,b,\mathrm{C})}_{0|0;0|0}}\end{array}\right],
\end{aligned}\tag{2.20}$$

where the indices are $1\le i\le k_1$ in the first rows and $1\le i\le l_2$ in the last rows and $1\le j\le k_2$ in the first columns and $1\le j\le l_1$ in the last ones. Recall that $\kappa_{1,j},\kappa_{2,j}\in\mathbb{C}\setminus\mathbb{R}_+$ for each $j=1,2,\dots$ We employed the mixed Cauchy-Vandermonde determinant [12, 45]

$$\begin{aligned}
\mathrm{B}_{k|l}(\kappa;\lambda)&=\frac{\Delta_k(\kappa)\Delta_l(\lambda)}{\prod_{i=1}^k\prod_{j=1}^l(\kappa_i-\lambda_j)}\\
&=\begin{cases}(-1)^{l(l-1)/2}\det\left[\begin{array}{cc}\left\{\dfrac{1}{\kappa_a-\lambda_b}\right\}_{\substack{1\le a\le k\\1\le b\le l}} & \left\{\kappa_a^{b-1}\right\}_{\substack{1\le a\le k\\1\le b\le k-l}}\end{array}\right], & k\ge l,\\[2ex](-1)^{k(k-1)/2}\det\left[\begin{array}{cc}\left\{\lambda_a^{b-1}\right\}_{\substack{1\le a\le l\\1\le b\le l-k}} & \left\{\dfrac{1}{\kappa_b-\lambda_a}\right\}_{\substack{1\le a\le l\\1\le b\le k}}\end{array}\right], & k\le l.\end{cases}
\end{aligned}\tag{2.21}$$

This determinant plays a crucial role in the theory of supermatrices [13].

We underline that the case $k_1=l_1=k$ and $k_2=l_2=l$ is important when calculating the (k,l)-point correlation function of the Cauchy two-matrix model, see subsection 2.4. Another important case of the partition function is when $k_1=k_2=k$ and $l_1=l_2=l$ which is needed in subsection 3.2 to invert the relation between the Bures ensemble and the Cauchy two-matrix ensemble.

The general case of k_1, k_2, l_1 and l_2 arbitrary can be obtained by sending some of the variables $\kappa_1,\kappa_2,\lambda_1,\lambda_2$ to infinity. Then one also finds the bi-orthogonal

polynomials (2.6) and (2.7) and their Cauchy transform (2.13) and (2.14) in the determinant after applying l'Hospital's rule and making use of the skew-symmetry of the determinant under permutation of rows and columns.

Let us look at the two-point partition functions in the kernels of the determinant (2.20) in detail. All four kernels can be expressed in terms of integrals over the bi-orthogonal polynomials and their Cauchy-transform and thus in terms of Meijer G-functions,

$$\begin{aligned}
\frac{Z^{(N-1,a,b,\mathrm{C})}_{0|1;0|1}(\lambda_1;\lambda_2)}{Z^{(N,a,b,\mathrm{C})}_{0|0;0|0}} &= -\frac{\det\left[\begin{array}{c|c}\displaystyle\int_{\mathbb{R}_+^2} dxdy \frac{x^{a+i-1}y^{b+j-1}\mathrm{e}^{-x-y}}{x+y} & \lambda_1^{i-1} \\ \hline \lambda_2^{j-1} & 0\end{array}\right]_{1\le i,j\le N}}{\det\left[\int_{\mathbb{R}_+^2} dxdy \frac{x^{a+i-1}y^{b+j-1}\mathrm{e}^{-x-y}}{x+y}\right]_{1\le i,j\le N}} \\
&= \sum_{i,j=0}^{N-1} \frac{\Gamma[a+b+i+N+1](-\lambda_1)^i}{i!(N-i-1)!\Gamma[a+b+i+1]\Gamma[a+i+1]} \\
&\quad \times \frac{\Gamma[a+b+j+N+1](-\lambda_2)^j}{j!(N-j-1)!\Gamma[a+b+j+1]\Gamma[b+j+1]} \frac{1}{a+b+j+i+1} \\
&= \int_0^1 dt t^{a+b} G^{1,1}_{2,3}\left(\begin{array}{c} -N-a-b;N \\ 0;-a,-a-b\end{array}\middle| t\lambda_1\right) G^{1,1}_{2,3}\left(\begin{array}{c} -N-a-b;N \\ 0;-b,-a-b\end{array}\middle| t\lambda_2\right)
\end{aligned} \tag{2.22}$$

for the average of two characteristic polynomials in the numerator (cf. Ref. [16]),

$$\begin{aligned}
\frac{1}{\kappa-\lambda}\frac{Z^{(N,a,b,\mathrm{C})}_{1|1;0|0}(\kappa,\lambda)}{Z^{(N,a,b,\mathrm{C})}_{0|0;0|0}} &= \frac{\det\left[\begin{array}{c|c}\displaystyle\int_{\mathbb{R}_+^2} dxdy \frac{x^{a+i-1}y^{b+j-1}\mathrm{e}^{-x-y}}{x+y} & \lambda^{i-1} \\ \hline \displaystyle\int_{\mathbb{R}_+^2} dxdy \frac{x^{a}y^{b+j-1}\mathrm{e}^{-x-y}}{(\kappa-x)(x+y)} & \frac{1}{\kappa-\lambda}\end{array}\right]_{1\le i,j\le N}}{\det\left[\int_{\mathbb{R}_+^2} dxdy \frac{x^{a+i-1}y^{b+j-1}\mathrm{e}^{-x-y}}{x+y}\right]_{1\le i,j\le N}} \\
&= \frac{1}{\kappa-\lambda} - \int_{\mathbb{R}_+^2} dxdy \frac{x^a y^b \mathrm{e}^{-x-y}}{(\kappa-x)(x+y)} \frac{Z^{(N-1,a,b,\mathrm{C})}_{0|1;0|1}(\lambda;y)}{Z^{(N,a,b,\mathrm{C})}_{0|0;0|0}} \\
&= \frac{1}{\kappa-\lambda} + \int_0^1 dt G^{1,1}_{2,3}\left(\begin{array}{c} -N-a-b;N \\ 0;-a,-a-b\end{array}\middle| t\lambda\right) G^{3,1}_{2,3}\left(\begin{array}{c} -N;N+a+b \\ 0,a,a+b\end{array}\middle| -t\kappa\right)
\end{aligned} \tag{2.23}$$

and analogously

$$\begin{aligned}
&\frac{1}{\kappa-\lambda}\frac{Z^{(N,a,b,\mathrm{C})}_{0|1;0|1}(\kappa,\lambda)}{Z^{(N,a,b,\mathrm{C})}_{0|0;0|0}} \\
&= \frac{1}{\kappa-\lambda} + \int_0^1 dt G^{3,1}_{2,3}\left(\begin{array}{c} -N;N+a+b \\ 0,b,a+b\end{array}\middle| -t\kappa\right) G^{1,1}_{2,3}\left(\begin{array}{c} -N-a-b;N \\ 0;-b,-a-b\end{array}\middle| t\lambda\right)
\end{aligned} \tag{2.24}$$

for the partition function with one characteristic polynomial in the numerator and one in the denominator, and

$$\begin{aligned}
&\frac{Z^{(N+1,a,b,\mathrm{C})}_{1|0;1|0}(\kappa_1;\kappa_2)}{Z^{(N,a,b,\mathrm{C})}_{0|0;0|0}} \\
&= \frac{\det\left[\begin{array}{c|c} \int_{\mathbb{R}^2_+} dxdy \frac{x^{a+i-1}y^{b+j-1}\mathrm{e}^{-x-y}}{x+y} & \int_{\mathbb{R}^2_+} dxdy \frac{x^{a+i-1}y^{b}\mathrm{e}^{-x-y}}{(\kappa_2-y)(x+y)} \\ \hline \int_{\mathbb{R}^2_+} dxdy \frac{x^{a}y^{b+j-1}\mathrm{e}^{-x-y}}{(\kappa_1-x)(x+y)} & \int_{\mathbb{R}^2_+} dxdy \frac{x^{a}y^{b}\mathrm{e}^{-x-y}}{(\kappa_1-x)(\kappa_2-y)(x+y)} \end{array}\right]_{1\le i,j\le N}}{\det\left[\int_{\mathbb{R}^2_+} dxdy \frac{x^{a+i-1}y^{b+j-1}\mathrm{e}^{-x-y}}{x+y}\right]_{1\le i,j\le N}} \\
&= \int_{\mathbb{R}^2_+} dxdy \frac{x^a y^b \mathrm{e}^{-x-y}}{(\kappa_1-x)(\kappa_2-y)(x+y)} - \sum_{j=0}^{N-1}(a+b+2j+1) \\
&\quad \times G^{3,1}_{2,3}\left(\begin{array}{c} -j-1; j+a+b \\ -1, a-1, a+b-1 \end{array}\middle| -\kappa_1\right) G^{3,1}_{2,3}\left(\begin{array}{c} -j-1; j+a+b-1 \\ -1, b-1, a+b-1 \end{array}\middle| -\kappa_2\right) \\
&= \int_{\mathbb{R}^2_+} dxdy \frac{x^a y^b \mathrm{e}^{-x-y}}{(\kappa_1-x)(\kappa_2-y)(x+y)} - (-1)^{a+b}\kappa_1^a\kappa_2^b \\
&\quad \times \int_0^1 dt \left[G^{3,1}_{2,3}\left(\begin{array}{c} -a-N; N+b \\ 0, -a, b \end{array}\middle| -t\kappa_1\right) G^{3,1}_{2,3}\left(\begin{array}{c} -b-N; N+a \\ 0, -b, a \end{array}\middle| -t\kappa_2\right)\right. \\
&\quad \left. -G^{3,1}_{2,3}\left(\begin{array}{c} -a; b \\ 0, -a, b \end{array}\middle| -t\kappa_1\right) G^{3,1}_{2,3}\left(\begin{array}{c} -b; a \\ 0, -b, a \end{array}\middle| -t\kappa_2\right)\right]
\end{aligned} \tag{2.25}$$

for two characteristic polynomials in the denominator. The last term can be derived by writing the Meijer G-function as contour integrals, see Eq. (2.11), and employing the identity

$$\begin{aligned}
&\sum_{j=0}^{N-1}(a+b+2j+1)\frac{\Gamma[j+u+2]\Gamma[j+v+2]}{\Gamma[j+a+b-u]\Gamma[j+a+b-v]} = \frac{1}{3-a-b+u+v} \\
&\times\left[\frac{\Gamma[N+u+2]\Gamma[N+v+2]}{\Gamma[N+a+b-u-1]\Gamma[N+a+b-v-1]} - \frac{\Gamma[u+2]\Gamma[v+2]}{\Gamma[a+b-u-1]\Gamma[a+b-v-1]}\right],
\end{aligned} \tag{2.26}$$

which is based on Lemma 4.1 of Ref. [16] and can be proven by taking the difference of the sum for $N=k$ and $N=k-1$ and showing that both sides are the same. For the other results we used the same integral identities (2.16) as for the polynomials.

We emphasize that the results (2.22-2.25) are similar to but not exactly the same as the ones in Ref. [16] where the Cauchy-transforms were not calculated. The transforms presented in Ref. [16] can be found by choosing $\kappa_j \to \kappa_j + L_j \imath\varepsilon$ $(j=1,2)$ in the limit $\varepsilon \to 0$ with $\kappa_j \in \mathbb{R}_+$ and $L_j = \pm 1$. Taking the differences of

the results for $L_1 = +1$ and $L_1 = -1$ and the same for for L_2 we find

$$-\frac{1}{2\pi\imath}\sum_{L_1=\pm 1}\frac{L_1}{\kappa_1 + L_1\imath\varepsilon - \lambda}\frac{Z^{(N,a,b,\mathrm{C})}_{1|1;0|0}(\kappa_1 + L_1\imath\varepsilon, \lambda)}{Z^{(N,a,b,\mathrm{C})}_{0|0;0|0}}$$
$$= \delta(\kappa_1 - \lambda) - \kappa_1^{a+b}\int_0^1 dt t^{a+b} G^{1,1}_{2,3}\left(\begin{matrix} -N-a-b; N \\ 0; -a, -a-b\end{matrix}\middle| t\lambda\right)$$
$$\times G^{3,1}_{2,2}\left(\begin{matrix} -N-a-b; N \\ 0, -b; -a-b\end{matrix}\middle| t\kappa_1\right), \tag{2.27}$$

and

$$-\frac{1}{2\pi\imath}\sum_{L_2=\pm 1}\frac{L_1}{\kappa_2 + L_2\imath\varepsilon - \lambda}\frac{Z^{(N,a,b,\mathrm{C})}_{0|0;1|1}(\kappa_2 + L_2\imath\varepsilon, \lambda)}{Z^{(N,a,b,\mathrm{C})}_{0|0;0|0}}$$
$$= \delta(\kappa_2 - \lambda) - \kappa_2^{a+b}\int_0^1 dt t^{a+b} G^{3,1}_{2,2}\left(\begin{matrix} -N-a-b; N \\ 0, -a; -a-b\end{matrix}\middle| t\kappa_2\right)$$
$$\times G^{1,1}_{2,3}\left(\begin{matrix} -N-a-b; N \\ 0; -b, -a-b\end{matrix}\middle| t\lambda\right), \tag{2.28}$$

for the two kernels of the off-diagonal blocks and

$$-\frac{1}{(2\pi)^2}\sum_{L_1,L_2=\pm 1} L_1L_2\frac{Z^{(N+1,a,b,\mathrm{C})}_{1|0;1|0}(\kappa_1 + L_1\imath\varepsilon; \kappa_2 + L_2\imath\varepsilon)}{Z^{(N,a,b,\mathrm{C})}_{0|0;0|0}}$$
$$= \frac{\kappa_1^a\kappa_2^b \mathrm{e}^{-\kappa_1-\kappa_2}}{\kappa_1+\kappa_2} - \kappa_1^a\kappa_2^b\sum_{j=0}^{N-1}(a+b+2j+1)$$
$$\times G^{2,1}_{2,3}\left(\begin{matrix} -a-j; j+b+1 \\ 0, b; -a\end{matrix}\middle| \kappa_1\right) G^{2,1}_{2,3}\left(\begin{matrix} -b-j; j+a+1 \\ 0, a; -b\end{matrix}\middle| \kappa_2\right)$$
$$= \frac{\kappa_1^a\kappa_2^b}{\kappa_1+\kappa_2} - \kappa_1^a\kappa_2^b$$
$$\times \int_0^1 dt G^{2,1}_{2,3}\left(\begin{matrix} -a-N; N+b \\ 0, b; -a\end{matrix}\middle| t\kappa_1\right) G^{2,1}_{2,3}\left(\begin{matrix} -b-N; N+a \\ 0, a; -b\end{matrix}\middle| t\kappa_2\right) \tag{2.29}$$

which is up to the prefactor $-\kappa_1^a\kappa_2^b$ the kernel $K_{11}^{(N)}$ in Ref. [16]. To obtain the equality of Eq. (2.29) we have to notice that the second term of Eq. (2.26) yields the Meijer G-function $G^{1,0}_{0,1}\left(\begin{matrix} - \\ 0; - \end{matrix}\middle| \kappa_1 t\right) = e^{-\kappa_1 t}$ and the same for κ_2. The integral over t yields two terms such that it is correct that the first term of the second equality in Eq. (2.29) does not contain an exponential function.

2.4. **Correlation functions of the Cauchy two-matrix model.** We now come to the eigenvalue correlation functions of the Cauchy two-matrix model after having recalled the results for the partition functions. Here, we emphasize that there are two definitions of the (k,l)-point correlation function at the positions $x = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$ and $y = (y_1, \ldots, y_l) \in \mathbb{R}_+^l$, namely (see Refs. [51, 30, 2] and

references therein)

$$\begin{aligned}\widehat{R}_{k,l}^{(N,a,b,\mathrm{C})}(x;y) &:= \frac{1}{Z_{0|0;0|0}^{(N,a,b,\mathrm{C})}}\frac{1}{(N!)^2}\int_{\mathbb{R}_+^{2N}}\frac{\Delta_N^2(x')\Delta_N^2(y')\,d[x']d[y']}{\prod_{i,j=1}^N(x_i'+y_j')}\\ &\times\prod_{j=1}^N\left({x_j'}^a{y_j'}^b\mathrm{e}^{-x_j'-y_j'}\right)\prod_{j=1}^k\left(\frac{1}{N}\sum_{i=1}^N\delta(x_j-x_i')\right)\prod_{j=1}^l\left(\frac{1}{N}\sum_{i=1}^N\delta(y_j-y_i')\right)\\ &= \frac{1}{Z_{0|0;0|0}^{(N,a,b,\mathrm{C})}}\lim_{\varepsilon\to0}\sum_{L_j,L_i'=\pm}\prod_{j=1}^k\left(\frac{L_j}{2\pi\imath N}\frac{\partial}{\partial\widetilde{x}_j}\right)\prod_{i=1}^l\left(\frac{L_i}{2\pi\imath N}\frac{\partial}{\partial\widetilde{y}_i}\right)\\ &\times Z_{k|k;l|l}^{(N,a,b,\mathrm{C})}(\widetilde{x}+\imath L\varepsilon,x;\widetilde{y}+\imath L'\varepsilon,y)\Big|_{\widetilde{x}=x,\widetilde{y}=y},\end{aligned}\tag{2.30}$$

where $L=(L_1,\dots,L_k)$ and $L'=(L_1',\dots,L_l')$ and

$$\begin{aligned}R_{k,l}^{(N,a,b,\mathrm{C})}(x;y) &:= \frac{1}{Z_{0|0;0|0}^{(N,a,b,\mathrm{C})}}\frac{1}{(N!)^2}\int_{\mathbb{R}_+^{2N-k-l}}\prod_{j=k+1}^N dx_j\prod_{j=l+1}^N dy_j\frac{\Delta_N^2(x)\Delta_N^2(y)}{\prod_{i,j=1}^N(x_i+y_j)}\\ &\times\prod_{j=1}^N\left(x_j^a y_j^b\mathrm{e}^{-x_j-y_j}\right).\end{aligned}\tag{2.31}$$

These two definitions are not equivalent. Nonetheless they are related in a simple way. The correlation function $\widehat{R}_{k,l}^{(N,a,b,\mathrm{C})}(x;y)$ consists not only of the correlation function $R_{k,l}^{(N,a,b,\mathrm{C})}(x;y)$ but also of the lower order terms like $R_{k-1,l}^{(N,a,b,\mathrm{C})}(x_1,\dots,x_{k-1};y)$ or $R_{k,l-1}^{(N,a,b,\mathrm{C})}(x;y_1,\dots,y_{l-1})$. The reason is that $\widehat{R}_{k,l}^{(N,a,b,\mathrm{C})}(x;y)$ comprises "self-energy" terms proportional to Dirac delta-functions as $\delta(x_i-x_j)$ with $i\neq j$. Omitting these "self-energy" terms we find $R_{k,l}^{(N,a,b,\mathrm{C})}(x;y)$, i.e.

$$\widehat{R}_{k,l}^{(N,a,b,\mathrm{C})}(x;y)=\frac{(N!)^2}{(N-k)!(N-l)!N^{k+l}}R_{k,l}^{(N,a,b,\mathrm{C})}(x;y)+\text{lower order terms}.\tag{2.32}$$

We remark that the definition for $\widehat{R}_{k,l}^{(N,a,b,\mathrm{C})}$ is based on the partition function (2.20) with $k_1=l_1=k$ and $k_2=l_2=l$ for which we already calculated a simplified determinantal expression in terms of two-point partition functions. In this expression we can easily perform the differentiations in x and y which essentially acts on the prefactor in front of the determinant only, since it vanishes at $\widetilde{x}+\imath L\varepsilon=x$ and $\widetilde{y}+\imath L'\varepsilon=y$. Only for the diagonal elements do we have to differentiate the kernel which yields one point functions. Therefore the result is

$$\begin{aligned}&\widehat{R}_{k,l}^{(N,a,b,\mathrm{C})}(x;y)\\ =\ &\det\left[\begin{array}{c|c}K_{01}^{(N,a,b,\mathrm{C})}(x_i,x_j) & K_{11}^{(N,a,b,\mathrm{C})}(x_i;y_j)\\ \hline K_{00}^{(N,a,b,\mathrm{C})}(x_j;y_i) & K_{10}^{(N,a,b,\mathrm{C})}(y_i,y_j)\end{array}\right]+\text{lower order terms},\end{aligned}\tag{2.33}$$

where the indices take the same values as in Eq. (2.20). The kernels are

$$\begin{aligned}
K_{11}^{(N,a,b,\mathrm{C})}(x_i;y_j) &= -\frac{x_i^a y_j^b}{x_i+y_j} + x_i^a y_j^b \int_0^1 dt G_{2,3}^{2,1}\left(\begin{matrix} -a-N; N+b \\ 0, b; -a \end{matrix} \middle| t x_i \right) \\
&\quad \times G_{2,3}^{2,1}\left(\begin{matrix} -b-N; N+a \\ 0, a; -b \end{matrix} \middle| t y_j \right), \\
K_{01}^{(N,a,b,\mathrm{C})}(x_i,x_j) &= x_i^{a+b} \int_0^1 dt t^{a+b} G_{2,3}^{1,1}\left(\begin{matrix} -N-a-b; N \\ 0; -a, -a-b \end{matrix} \middle| t x_j \right) \\
&\quad \times G_{2,3}^{2,1}\left(\begin{matrix} -N-a-b; N \\ 0, -b; -a-b \end{matrix} \middle| t x_i \right), \\
K_{10}^{(N,a,b,\mathrm{C})}(y_i,y_j) &= y_j^{a+b} \int_0^1 dt t^{a+b} G_{2,3}^{2,1}\left(\begin{matrix} -N-a-b; N \\ 0, -a; -a-b \end{matrix} \middle| t y_j \right) \\
&\quad \times G_{2,3}^{1,1}\left(\begin{matrix} -N-a-b; N \\ 0; -b, -a-b \end{matrix} \middle| t y_i \right), \\
K_{00}^{(N,a,b,\mathrm{C})}(x_j;y_i) &= \int_0^1 dt t^{a+b} G_{2,3}^{1,1}\left(\begin{matrix} -N-a-b; N \\ 0; -a, -a-b \end{matrix} \middle| t x_j \right) \\
&\quad \times G_{2,3}^{1,1}\left(\begin{matrix} -N-a-b; N \\ 0; -b, -a-b \end{matrix} \middle| t y_i \right).
\end{aligned} \tag{2.34}$$

From this result one can read off the (k,l)-point correlation function without "self-energy" terms,

$$R_{k,l}^{(N,a,b,\mathrm{C})}(x;y) = \frac{(N-k)!(N-l)!N^{k+l}}{(N!)^2} \det \left[\begin{array}{c|c} K_{01}^{(N,a,b,\mathrm{C})}(x_i,x_j) & K_{11}^{(N,a,b,\mathrm{C})}(x_i;y_j) \\ \hline K_{00}^{(N,a,b,\mathrm{C})}(x_j;y_i) & K_{10}^{(N,a,b,\mathrm{C})}(y_i,y_j) \end{array} \right], \tag{2.35}$$

which is the result of Ref. [16]. Note that our k-point functions are normalized, i.e. $\int d[x]d[y] R_{k,l}^{(N,a,b,\mathrm{C})}(x;y) = 1$. In the particular case of the two kinds of level densities this result reads

$$\begin{aligned}
R_{1,0}^{(N,a,b,\mathrm{C})}(x) &= K_{01}^{(N,a,b,\mathrm{C})}(x,x) \qquad (2.36) \\
&= \int_0^1 dt G_{2,3}^{1,1}\left(\begin{matrix} -N; N+a+b \\ a+b; 0, b \end{matrix} \middle| tx \right) G_{2,3}^{2,1}\left(\begin{matrix} -N-a-b; N \\ 0, -b; -a-b \end{matrix} \middle| tx \right)
\end{aligned}$$

and

$$\begin{aligned}
R_{0,1}^{(N,a,b,\mathrm{C})}(y) &= K_{10}^{(N,a,b,\mathrm{C})}(y,y) \qquad (2.37) \\
&= \int_0^1 dt G_{2,3}^{1,1}\left(\begin{matrix} -N; N+a+b \\ a+b; 0, a \end{matrix} \middle| ty \right) G_{2,3}^{2,1}\left(\begin{matrix} -N-a-b; N \\ 0, -a; -a-b \end{matrix} \middle| ty \right),
\end{aligned}$$

which shall conclude this section. Due to the symmetry of the joint probability density (2.2) the spectral statistics is invariant by exchanging $\{x_i\} \leftrightarrow \{y_i\}$ and $a \leftrightarrow b$ which in particular is reflected in the level densities $R_{1,0}^{(N,a,b,\mathrm{C})}$ and $R_{0,1}^{(N,a,b,\mathrm{C})}$. Now we are well-prepared for calculating the eigenvalue statistics of the Bures ensemble.

3. Relationship between Bures and Cauchy two-matrix ensemble

We aim at two things in this section. First we want to work along the same ideas and calculations as we have done it for the Cauchy two-matrix ensemble in section 2. Hence we start with the partition function

$$Z_{k|l}^{(N,a,\mathrm{B})}(\kappa,\lambda)=\frac{1}{N!}\int_{\mathbb{R}_+^N}d[z]\frac{\Delta_N^2(z)}{\prod_{1\le i<j\le N}(z_i+z_j)}\prod_{j=1}^{N}z_j^a\mathrm{e}^{-z_j}\frac{\prod_{i=1}^{l}(z_j-\lambda_i)}{\prod_{i=1}^{k}(z_j-\kappa_i)} \tag{3.1}$$

and express every other quantity in terms of this including the normalization constant, the skew-orthogonal polynomials, and the k-point correlation function. Thereby for the latter we again choose the definition

$$\begin{aligned}\widehat{R}_k^{(N,a,\mathrm{B})}(z) &:= \frac{1}{Z_{0|0}^{(N,a,\mathrm{B})}}\frac{1}{N!}\int_{\mathbb{R}_+^N}d[z']\frac{\Delta_N^2(z')\prod_{j=1}^{N}{z'_j}^a\mathrm{e}^{-z'_j}}{\prod_{1\le i<j\le N}(z'_i+z'_j)}\prod_{j=1}^{k}\left(\frac{1}{N}\sum_{i=1}^{N}\delta(z_j-z'_i)\right)\\ &= \frac{1}{Z_{0|0}^{(N,a,\mathrm{B})}}\lim_{\varepsilon\to 0}\sum_{L_j=\pm}\prod_{j=1}^{k}\left(\frac{L_j}{2\pi\imath N}\frac{\partial}{\partial\widetilde{z}_j}\right)Z_{k|k}^{(N,a,\mathrm{B})}(\widetilde{z}+\imath L\varepsilon,z)\Big|_{\widetilde{z}=z},\end{aligned} \tag{3.2}$$

where $x=(x_1,\ldots,x_k)$, and $L=(L_1,\ldots,L_k)$, including the "self-energy" terms. From this quantity we can easily read off the correlation function without the "self-energy" terms

$$R_k^{(N,a,\mathrm{B})}(z) := \frac{1}{Z_{0|0}^{(N,a,\mathrm{B})}}\frac{1}{N!}\int_{\mathbb{R}_+^{N-k}}\prod_{j=k+1}^{N}dz_j\frac{\Delta_N^2(z')\prod_{j=1}^{N}{z'_j}^a\mathrm{e}^{-z'_j}}{\prod_{1\le i<j\le N}(z'_i+z'_j)}. \tag{3.3}$$

Also these two eigenvalue correlation functions are related via

$$\widehat{R}_k^{(N,a,\mathrm{B})}(z)=\frac{N!}{(N-k)!N^k}R_k^{(N,a,\mathrm{B})}(z)+\text{lower order terms}, \tag{3.4}$$

where the lower order terms comprise the correlation functions $R_{k-1}^{(N,a,\mathrm{B})}$, $R_{k-2}^{(N,a,\mathrm{B})},\ldots$ which are all proportional to some Dirac delta-functions like $\delta(z_i-z_j)$ with $i\neq j$.

Our second aim is to establish a relation between the Bures and the Cauchy two-matrix model. Thereby we show in subsection 3.1 that each square of the partition function (3.1) for the Bures measure can be expressed as a partition function (2.3) for the Cauchy two-matrix model. However to make sense of this relation it has to be inverted. This means we have to take the square root correctly such that we do not lose any algebraical structure which we had for the Cauchy two-matrix ensemble. For the Cauchy two-matrix ensemble we have recalled in subsection 2.3 that it corresponds to a determinantal point process. In subsection 3.2 we show that this determinantal point process carries over to a Pfaffian point process for the Bures ensemble. In this way we calculate the skew-orthogonal polynomials, the kernels for the partition function (3.1) (both in subsection 3.2), and the kernels for the k-point correlation function (3.3) without "self-energy" terms (in subsection 3.3).

3.1. **Going from Bures to Cauchy.**

Some statements we make in subsections 3.1 and 3.2 can be applied to more general weights than the one of the joint probability

density (1.5). Therefore let us define the partition functions

$$Z^{(N,\mathrm{C})}_{k_1|l_1;k_2|l_2}[\alpha](\kappa_1,\lambda_1;\kappa_2,\lambda_2) := \frac{1}{(N!)^2}\int_{\mathbb{R}_+^{2N}} d[x]d[y]\frac{\Delta_N^2(x)\Delta_N^2(y)}{\prod_{i,j=1}^N(x_i+y_j)} \times\prod_{j=1}^N\left(\alpha(x_j)\alpha(y_j)y_j\frac{\prod_{i=1}^{l_1}(x_j-\lambda_{1,i})\prod_{i=1}^{l_2}(y_j-\lambda_{2,i})}{\prod_{i=1}^{k_1}(x_j-\kappa_{1,i})\prod_{i=1}^{k_2}(y_j-\kappa_{2,i})}\right) \tag{3.5}$$

for a Cauchy-like two-matrix model and

$$Z^{(N,\mathrm{B})}_{k|l}[\alpha](\kappa,\lambda) := \frac{1}{N}\int_{\mathbb{R}_+^N} d[z]\frac{\Delta_N^2(z)}{\prod_{1\le i<j\le N}(z_i+z_j)}\prod_{j=1}^N\alpha(z_j)\frac{\prod_{i=1}^{l}(z_j-\lambda_i)}{\prod_{i=1}^{k}(z_j-\kappa_i)} \tag{3.6}$$

for a Bures-like ensemble. The weight $\alpha(z)$ is a one point weight and is in the case of the original Bures ensemble $\alpha(z)=z^a\mathrm{e}^{-z}$ such that we have

$$Z^{(N,\mathrm{C})}_{k_1|l_1;k_2|l_2}[z^a\mathrm{e}^{-z}] = Z^{(N,a,a+1\mathrm{C})}_{k_1|l_1;k_2|l_2} \text{ and } Z^{(N,\mathrm{B})}_{k|l}[z^a\mathrm{e}^{-z}] = Z^{(N,a,\mathrm{B})}_{k|l}. \tag{3.7}$$

With these definitions we prove the following proposition in appendix A.

Proposition 3.1. *For an arbitrary, suitable integrable scalar function $\alpha(x)$ and two sets of variables $\lambda_1,\dots,\lambda_l\in\mathbb{C}$ and $\kappa_1,\dots,\kappa_k\in\mathbb{C}\setminus\mathbb{R}_0^+$ pairwise different, and $N\in\mathbb{N}$ the partition functions of the Bures-like ensemble, $Z^{(N,\mathrm{B})}_{k|l}[\alpha](\kappa,\lambda)$, and of the Cauchy-like two-matrix ensemble, $Z^{(N,\mathrm{C})}_{k|l;k|l}[\alpha](\kappa,\lambda;\kappa,\lambda)$, are related as*

$$\left(Z^{(N,\mathrm{B})}_{k|l}[\alpha](\kappa,\lambda)\right)^2 = 2^N Z^{(N,\mathrm{C})}_{k|l;k|l}[\alpha](\kappa,\lambda;\kappa,\lambda). \tag{3.8}$$

We remark that the derivation of Eq. (3.8) makes use of a rewrite of the Cauchy-Vandermonde determinant (2.21) as well as the Schur Pfaffian identity [43]

$$\frac{\Delta_N(z)}{\prod_{1\le i<j\le N}(z_i+z_j)} = \prod_{1\le i<j\le N}\frac{z_j-z_i}{z_j+z_i} \tag{3.9}$$
$$= \begin{cases} \mathrm{Pf}\left[\dfrac{z_a-z_b}{z_a+z_b}\right]_{1\le a,b\le N}, & N \text{ even},\\ \mathrm{Pf}\left[\begin{array}{c|ccc} 0 & -1 & \cdots & -1\\ \hline 1 & & & \\ \vdots & & \left\{\dfrac{z_a-z_b}{z_a+z_b}\right\}_{1\le a,b\le N} & \\ 1 & & & \end{array}\right], & N \text{ odd}. \end{cases}$$

As a simple corollary the normalization constants of the Bures ensemble and the Cauchy two-matrix ensemble are directly related, see Ref. [14] where it was first proven.

Corollary 3.2. *The case $k=l=0$ of proposition 3.1 yields the normalization constant and explicitly reads*

$$\left(\frac{1}{N!}\int_{\mathbb{R}_+^N} d[z]\frac{\Delta_N^2(z)\prod_{j=1}^N\alpha(z_j)}{\prod_{1\le i<j\le N}(z_i+z_j)}\right)^2 = \frac{2^N}{(N!)^2}\int_{\mathbb{R}_+^{2N}} d[x]d[y]\frac{\prod_{j=1}^N\alpha(x_j)\alpha(y_j)y_j\,\Delta_N^2(x)\Delta_N^2(y)}{\prod_{i,j=1}^N(x_i+y_j)}. \tag{3.10}$$

Our proportionality constants in (3.10) are different to those in Ref. [14]. In fact a check can be made on this latter point, by making the choice $\alpha(x) = x^a \mathrm{e}^{-x}$. The LHS of (3.10) can then be evaluated using matrix integral methods [67],

$$\begin{aligned} Z_{0|0}^{(N,a,\mathrm{B})} &= \frac{1}{N!}\int_{\mathbb{R}_+^N} d[z] \prod_{j=1}^N z_j^a \mathrm{e}^{-z_j} \prod_{1\le i<j\le N} \frac{(z_j-z_i)^2}{z_j+z_i} \\ &= \pi^{N/2} 2^{-N^2-2Na} \prod_{j=0}^{N-1} \frac{\Gamma(1+j)\Gamma(2a+2+j)}{\Gamma(j+a+3/2)}. \end{aligned} \tag{3.11}$$

while from [14, Eq. (2-6)] (note that Eq. (2-7) of [14], obtained from Eq. (2-6) using the duplication formula for the Gamma function, contains a typo) we deduce that

$$\begin{aligned} Z_{0|0;0|0}^{(N,a,a+1,\mathrm{C})} &= \frac{1}{(N!)^2}\int_{\mathbb{R}_+^{2N}} d[x]d[y] \frac{\prod_{j=1}^N x_j^a \mathrm{e}^{-x_j} y_j^{a+1} \mathrm{e}^{-y_j} \, \Delta^2(x)\Delta^2(y)}{\prod_{i,j=1}^N (x_i+y_j)} \\ &= \prod_{j=0}^{N-1} \Big(\frac{\Gamma(1+j)\Gamma(2a+2+j)}{\Gamma(j+a+3/2)} \Big)^2 \frac{\pi}{2^{4j+4a+3}}, \end{aligned} \tag{3.12}$$

cf. Eq. (2.5). Using these in Eq. (3.10) the claimed proportionality of the identity is verified.

3.2. **Going from Cauchy to Bures.** Looking at the established relationship between the Bures ensemble and the Cauchy two matrix model one can ask if one can invert the result of Proposition 3.1 and derive all correlation functions for the Bures ensemble from the Cauchy two-matrix model. Indeed one can readily take the square root of the partition functions $Z_{k|l;k|l}^{(N,a,\mathrm{C})}$ to find $Z_{k|l}^{(N,a,\mathrm{B})}$. However when doing so we may loose the algebraic structure which is a determinantal point process for the Cauchy two-matrix ensemble, see Eq. (2.20) for $k_1 = k_2 = k$ and $l_1 = l_2 = l$. We would expect a Pfaffian expression for the Bures ensemble when taking the square root of Eq. (2.20). Indeed this is the case due to the Schur Pfaffian identity (3.9). Then the joint probability density has the form of the class of ensembles discussed in [46] with the two-point weight $g(z_1, z_2) = z_1^a z_2^a e^{-z_1-z_2}(z_1-z_2)/(z_1+z_2)$. Therefore the partition function (3.6) has the representation ($\widetilde{N} = N + l - k > 1$)

$$\begin{aligned} Z_{k|l}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda) &= (-1)^{k(k-1)/2+l(l-1)/2} \frac{Z_{0|0}^{(\widetilde{N},\mathrm{B})}[\alpha]}{B_{k|l}(\kappa;\lambda)} \\ &\times \mathrm{Pf} \left[\begin{array}{c|c} (\kappa_i-\kappa_j)\dfrac{Z_{2|0}^{(\widetilde{N}+2,\mathrm{B})}[\alpha](\kappa_i,\kappa_j)}{Z_{0|0}^{(\widetilde{N},\mathrm{B})}[\alpha]} & \dfrac{1}{\kappa_i-\lambda_j}\dfrac{Z_{1|1}^{(\widetilde{N},\mathrm{B})}[\alpha](\kappa_i,\lambda_j)}{Z_{0|0}^{(\widetilde{N},\mathrm{B})}[\alpha]} \\ \hline \dfrac{1}{\lambda_i-\kappa_j}\dfrac{Z_{1|1}^{(\widetilde{N},\mathrm{B})}[\alpha](\kappa_i,\lambda_j)}{Z_{0|0}^{(\widetilde{N},\mathrm{B})}[\alpha]} & (\lambda_i-\lambda_j)\dfrac{Z_{0|2}^{(\widetilde{N}-2,\mathrm{B})}[\alpha](\lambda_i,\lambda_j)}{Z_{0|0}^{(\widetilde{N},\mathrm{B})}[\alpha]} \end{array} \right] \end{aligned} \tag{3.13}$$

for $k+l$ even and

$$Z_{k|l}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda) = (-1)^{k(k-1)/2+l(l-1)/2}\frac{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]}{B_{k|l}(\kappa;\lambda)}$$
$$\times\mathrm{Pf}\left[\begin{array}{c|c|c} (\kappa_i-\kappa_j)\dfrac{Z_{2|0}^{(\widetilde{N}+3,\mathrm{B})}[\alpha](\kappa_i,\kappa_j)}{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]} & \dfrac{1}{\kappa_i-\lambda_j}\dfrac{Z_{1|1}^{(\widetilde{N}+1,\mathrm{B})}[\alpha](\kappa_i,\lambda_j)}{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]} & \dfrac{Z_{1|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha](\kappa_i)}{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]} \\ \hline \dfrac{1}{\lambda_i-\kappa_j}\dfrac{Z_{1|1}^{(\widetilde{N}+1,\mathrm{B})}[\alpha](\kappa_i,\lambda_j)}{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]} & (\lambda_i-\lambda_j)\dfrac{Z_{0|2}^{(\widetilde{N}-1,\mathrm{B})}[\alpha](\lambda_i,\lambda_j)}{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]} & \dfrac{Z_{0|1}^{(\widetilde{N}-1,\mathrm{B})}[\alpha](\lambda_i)}{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]} \\ \hline -\dfrac{Z_{1|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha](\kappa_j)}{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]} & -\dfrac{Z_{0|1}^{(\widetilde{N}-1,\mathrm{B})}[\alpha](\lambda_j)}{Z_{0|0}^{(\widetilde{N}+1,\mathrm{B})}[\alpha]} & 0 \end{array}\right] \tag{3.14}$$

for $k+l$ odd. The indices take the values $1 \le i,j \le k$ in the first few rows and columns and $1 \le i,j \le l$ in the second set of rows and columns in both equations. See Ref. [61], Appendix C of Ref. [45] or [30, §6.3.2&§6.3.3] for a general calculation of integrals over a product of a Pfaffian and a determinant. After having this Pfaffian the kernels can be identified by choosing particular values of k and l.

The results (3.13) and (3.14) cannot be obtained in a trivial way by only taking the square root of the partition functions of the Cauchy two-matrix model since the determinant (2.20) is not over an anti-symmetric matrix and the fact that it is an exact square is obscured. To uncover this fact one needs relations between the two-point partition functions (2.22-2.25) of the Cauchy two-matrix model. We underline that those relations as well as the determinantal structure with its kernels in terms of these two-point partition functions also hold for general weight α. We derive these relations in the proof given in appendix B, of the following proposition.

Proposition 3.3. *With the requirements of proposition 3.1 and $N+l-k>1$ the partition function of the Cauchy-like two-matrix model can be rewritten as*

$$\begin{aligned} & Z_{k|l;k|l}^{(N,\mathrm{C})}[\alpha](\kappa,\lambda;\kappa,\lambda) \\ = \; & \frac{Z_{0|0;0|0}^{(N+l-k,\mathrm{C})}[\alpha]}{\mathrm{B}_{k|l}^2(\kappa;\lambda)}\det\left[\begin{array}{c|c} \widehat{K}_{11}^{(N+l-k+1)}(\kappa_i,\kappa_j) & -\widehat{K}_{01}^{(N+l-k)}(\kappa_i,\lambda_j) \\ \hline \widehat{K}_{01}^{(N+l-k)}(\kappa_j,\lambda_i) & \widehat{K}_{00}^{(N+l-k-1)}(\lambda_i,\lambda_j) \end{array}\right] \end{aligned} \tag{3.15}$$

for $k+l$ even with $1 \le i,j \le k$ in the first rows and columns and $1 \le i,j \le l$ in the last ones and

$$Z_{k|l;k|l}^{(N,\mathrm{C})}[\alpha](\kappa,\lambda;\kappa,\lambda) = \frac{Z_{0|0;0|0}^{(N+l-k+1,\mathrm{C})}[\alpha]}{\mathrm{B}_{k|l}^2(\kappa;\lambda)}$$
$$\times\det\left[\begin{array}{c|c|c} \widehat{K}_{11}^{(N+l-k+2)}(\kappa_i,\kappa_j) & -\widehat{K}_{01}^{(N+l-k+1)}(\kappa_i,\lambda_j) & -\widehat{K}_{1}^{(N+l-k+1)}(\kappa_i) \\ \hline \widehat{K}_{01}^{(N+l-k+1)}(\kappa_j,\lambda_i) & \widehat{K}_{00}^{(N+l-k)}(\lambda_i,\lambda_j) & \widehat{K}_{0}^{(N+l-k)}(\lambda_i) \\ \hline \widehat{K}_{1}^{(N+l-k+1)}(\kappa_j) & -\widehat{K}_{0}^{(N+l-k)}(\lambda_j) & 0 \end{array}\right] \tag{3.16}$$

for $k+l$ odd with $1 \le i,j \le k$ in the first set of rows and columns and $1 \le i,j \le l$ in the second set of rows and columns, cf. eq. (3.15)*. The kernels of these determinants*

read

$$\begin{aligned}
\widehat{K}_{11}^{(L)}(\kappa_1,\kappa_2) &= \frac{Z_{1|0;1|0}^{(L,\mathrm{C})}[\alpha](\kappa_1;\kappa_2)-Z_{1|0;1|0}^{(L,\mathrm{C})}[\alpha](\kappa_2;\kappa_1)}{2Z_{0|0;0|0}^{(L-1,\mathrm{C})}[\alpha]},\\
\widehat{K}_{01}^{(L)}(\kappa,\lambda) &= \frac{Z_{0|0;1|1}^{(L,\mathrm{C})}[\alpha](\kappa,\lambda)+Z_{1|1;0|0}^{(L,\mathrm{C})}[\alpha](\kappa,\lambda)}{2Z_{0|0;0|0}^{(L,\mathrm{C})}[\alpha](\kappa-\lambda)},\\
\widehat{K}_{00}^{(L)}(\lambda_1,\lambda_2) &= \frac{Z_{0|1;0|1}^{(L,\mathrm{C})}[\alpha](\lambda_2;\lambda_1)-Z_{0|1;0|1}^{(L,\mathrm{C})}[\alpha](\lambda_1;\lambda_2)}{2Z_{0|0;0|0}^{(L+1,\mathrm{C})}[\alpha]},\\
\widehat{K}_{1}^{(L)}(\kappa) &= -\frac{Z_{0|0;1|0}^{(L,\mathrm{C})}[\alpha](\kappa)+Z_{1|0;0|0}^{(L,\mathrm{C})}[\alpha](\kappa)}{2Z_{0|0;0|0}^{(L,\mathrm{C})}[\alpha]},\\
\widehat{K}_{0}^{(L)}(\lambda) &= \frac{Z_{0|0;0|1}^{(L,\mathrm{C})}[\alpha](\lambda)-Z_{0|1;0|0}^{(L,\mathrm{C})}[\alpha](\lambda)}{2Z_{0|0;0|0}^{(L+1,\mathrm{C})}[\alpha]}
\end{aligned}\tag{3.17}$$

for $L\in\mathbb{N}$ and $L>1$.

Now we can take the square root and find the following corollary and one of our main results.

Corollary 3.4. *With the requirements of proposition 3.1 the partition function of the Bures-like ensemble can be expressed in terms of two- and one-point partition functions of the Cauchy-like two-matrix ensemble according to*

$$\begin{aligned}
Z_{k|l}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda) &= (-1)^{k(k-1)/2+l(l-1)/2}2^{N/2}\frac{\sqrt{Z_{0|0;0|0}^{(N+l-k,\mathrm{C})}[\alpha]}}{\mathrm{B}_{k|l}(\kappa;\lambda)}\\
&\quad\times\mathrm{Pf}\left[\begin{array}{c|c}-s\widehat{K}_{11}^{(N+l-k+1)}(\kappa_i,\kappa_j) & -\widehat{K}_{01}^{(N+l-k)}(\kappa_i,\lambda_j)\\ \hline \widehat{K}_{01}^{(N+l-k)}(\kappa_j,\lambda_i) & -s\widehat{K}_{00}^{(N+l-k-1)}(\lambda_i,\lambda_j)\end{array}\right]
\end{aligned}\tag{3.18}$$

for $k+l$ even and

$$\begin{aligned}
Z_{k|l}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda) &= (-1)^{k(k-1)/2+l(l-1)/2}2^{N/2}\frac{\sqrt{Z_{0|0;0|0}^{(N+l-k+1,\mathrm{C})}[\alpha]}}{\mathrm{B}_{k|l}(\kappa;\lambda)}\\
&\quad\times\mathrm{Pf}\left[\begin{array}{c|c|c}-s\widehat{K}_{11}^{(N+l-k+2)}(\kappa_i,\kappa_j) & -\widehat{K}_{01}^{(N+l-k+1)}(\kappa_i,\lambda_j) & -\widehat{K}_{1}^{(N+l-k+1)}(\kappa_i)\\ \hline \widehat{K}_{01}^{(N+l-k+1)}(\kappa_j,\lambda_i) & -s\widehat{K}_{00}^{(N+l-k)}(\lambda_i,\lambda_j) & -s\widehat{K}_{0}^{(N+l-k)}(\lambda_i)\\ \hline \widehat{K}_{1}^{(N+l-k+1)}(\kappa_j) & s\widehat{K}_{0}^{(N+l-k)}(\lambda_j) & 0\end{array}\right]
\end{aligned}\tag{3.19}$$

for $k+l$ odd. The indices i and j take the same values as in Eqs. (3.15) *and* (3.16)*, respectively. The variable s is the sign of the mean value of the difference of the two variable sets of the Cauchy-like two-matrix ensemble*

$$\widehat{Z}^{(L,\mathrm{C})}[\alpha]=\frac{1}{(L!)^2}\int_{\mathbb{R}_+^{2L}}d[x]d[y]\frac{\Delta_L^2(x)\Delta_L^2(y)}{\prod_{i,j=1}^{L}(x_i+y_j)}\left(\prod_{j=1}^{L}\alpha(x_j)y_j\alpha(y_j)\right)\sum_{j=1}^{L}(x_j-y_j)\tag{3.20}$$

and

$$s=\operatorname{sign}\widehat{Z}^{(L,\mathrm{C})}[\alpha].\tag{3.21}$$

Additionally, in Eqs. (3.18) *and* (3.19) *the kernels can be identified with partition functions of the Bures-like ensemble as well as with partition functions of the Cauchy-like two-matrix ensemble,*

$$\begin{aligned}
\frac{Z_{2|0}^{(N,\mathrm{B})}[\alpha](\kappa_1,\kappa_2)}{Z_{0|0}^{(N,\mathrm{B})}[\alpha]} &= \frac{1}{\kappa_2-\kappa_1}\frac{Z_{1|0;1|0}^{(N-1,\mathrm{C})}[\alpha](\kappa_1;\kappa_2)-Z_{1|0;1|0}^{(N-1,\mathrm{C})}[\alpha](\kappa_2;\kappa_1)}{\widehat{Z}^{(N-1,\mathrm{C})}[\alpha]},\\
\frac{Z_{1|1}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda)}{Z_{0|0}^{(N,\mathrm{B})}[\alpha]} &= \frac{Z_{0|0;1|1}^{(N,\mathrm{C})}[\alpha](\kappa,\lambda)+Z_{1|1;0|0}^{(N,\mathrm{C})}[\alpha](\kappa,\lambda)}{2Z_{0|0;0|0}^{(N,\mathrm{C})}[\alpha]},\\
\frac{Z_{0|2}^{(N,\mathrm{B})}[\alpha](\lambda_1,\lambda_2)}{Z_{0|0}^{(N,\mathrm{B})}[\alpha]} &= \frac{1}{\lambda_1-\lambda_2}\frac{Z_{0|1;0|1}^{(N+1,\mathrm{C})}[\alpha](\lambda_1;\lambda_2)-Z_{0|1;0|1}^{(N+1,\mathrm{C})}[\alpha](\lambda_2;\lambda_1)}{\widehat{Z}^{(N+1,\mathrm{C})}[\alpha]},\\
\frac{Z_{1|0}^{(N,\mathrm{B})}[\alpha](\kappa)}{Z_{0|0}^{(N,\mathrm{B})}[\alpha]} &= \frac{Z_{0|0;1|0}^{(N,\mathrm{C})}[\alpha](\kappa)+Z_{1|0;0|0}^{(N,\mathrm{C})}[\alpha](\kappa)}{2Z_{0|0;0|0}^{(N,\mathrm{C})}[\alpha]},\\
\frac{Z_{0|1}^{(N,\mathrm{B})}[\alpha](\lambda)}{Z_{0|0}^{(N,\mathrm{B})}[\alpha]} &= \frac{Z_{0|1;0|0}^{(N+1,\mathrm{C})}[\alpha](\lambda)-Z_{0|0;0|1}^{(N+1,\mathrm{C})}[\alpha](\lambda)}{\widehat{Z}^{(N+1,\mathrm{C})}[\alpha]}.
\end{aligned} \tag{3.22}$$

We emphasize that the normalization in the relations (3.22) agree with the formerly chosen one but in the way given in Eq. (3.22) they can be easily checked. The overall sign can be identified with the one in Eqs. (3.13) and (3.14). Moreover we underline that corollary 3.4 provides only a way to derive all spectral correlations of the Bures measure with the help of the Cauchy two-matrix model. In contrast, proposition 3.1 does not imply that the correlations of all Cauchy two-matrix models are determined by the Bures measure since the weights for the two sets $\{x_i\}$ and $\{y_j\}$ have to be the same up to a factor y_j.

Thus let us come back to the original problem where $\alpha(z) = z^a \mathrm{e}^{-z}$. For this measure we know already the normalization constants $Z_{0|0}^{(L,a,\mathrm{B})}$ and $Z_{0|0;0|0}^{(L,a,a+1,\mathrm{C})}$, see Eqs. (3.11) and (3.12), respectively. The third normalization constant appearing in Eq. (3.22) is

$$\begin{aligned}
\widehat{Z}^{(L,a,a+1,\mathrm{C})} &= \int_0^{2\pi}\frac{d\varphi}{2\pi}\mathrm{e}^{-\imath(2n-1)\varphi}(\widetilde{p}_{2n}^{(a,a+1)}(\mathrm{e}^{\imath\varphi})-p_{2n}^{(a,a+1)}(\mathrm{e}^{\imath\varphi}))\\
&= -L\frac{2a+L+1}{2a+2L+1}Z_{0|0;0|0}^{(L,a,a+1,\mathrm{C})},
\end{aligned} \tag{3.23}$$

which is essentially the $(2L-1)$st coefficient of the difference of the two bi-orthogonal polynomials of order $2n$ for the Cauchy two-matrix model, see
Eqs. (2.6) and (2.7).

What can be said about the skew-orthogonal polynomials $q_n^{(a)}$ of the Bures ensemble, in particular those polynomials which are skew orthogonal with respect to the two-point weight $g^{(a)}(z_1,z_2) = (z_1z_2)^a \mathrm{e}^{-z_1-z_2}(z_1-z_2)/(z_1+z_2)$? First of all

they have to satisfy the relations

$$\begin{aligned}\int_{\mathbb{R}_+^2} dz_1dz_2 g^{(a)}(z_1,z_2)q_{2n}^{(a)}(z_1)q_{2m}^{(a)}(z_1) &= \int_{\mathbb{R}_+^2} dz_1dz_2 g^{(a)}(z_1,z_2)q_{2n+1}^{(a)}(z_1)q_{2m+1}^{(a)}(z_1)=0,\\ \int_{\mathbb{R}_+^2} dz_1dz_2 g^{(a)}(z_1,z_2)q_{2n}^{(a)}(z_1)q_{2m+1}^{(a)}(z_1) &= \frac{Z_{0|0}^{(2n+2,a,\mathrm{B})}}{Z_{0|0}^{(2n,a,\mathrm{B})}}\delta_{mn}\\ &= \frac{\pi}{16^{2n+a+1}}\frac{(2n+1)!(2n)!\Gamma(2n+2a+3)\Gamma(2n+2a+2)}{\Gamma(2n+a+5/2)\Gamma(2n+a+3/2)}\delta_{mn}\end{aligned} \tag{3.24}$$

for all $m,n\in\mathbb{N}$. For the polynomials of even order it is well known [29, 7] that it is simply the average of one characteristic polynomial. It takes the following form shown in different representations, i.e. in terms of a partition function, the bi-orthogonal polynomials (2.6) and (2.7), a finite explicit sum, a generalized hypergeometric function, and a Meijer G-function, respectively,

$$\begin{aligned}q_{2n}^{(a)}(x) &= \frac{Z_{0|1}^{(2n,a,\mathrm{B})}(x)}{Z_{0|0}^{(2n,a,\mathrm{B})}}\\ &= \lim_{y\to\infty}\frac{y^{2n}}{p_{2n+1}^{(a,b)}(y)-\widetilde{p}_{2n+1}^{(a,b)}(y)}\left(p_{2n+1}^{(a,b)}(x)-\widetilde{p}_{2n+1}^{(a,b)}(x)\right)\\ &= \sum_{j=0}^{2n}(-1)^j\binom{2n}{j}\frac{\Gamma(2a+2n+j+3)\Gamma(2a+2n+2)\Gamma(a+2n+2)}{\Gamma(2a+4n+3)\Gamma(2a+j+2)\Gamma(a+j+2)}x^j\\ &= \frac{(2a+2)_{2n}(a+2)_{2n}}{(2a+2n+3)_{2n}}\,{}_2F_2\left(\begin{matrix}-2n,2a+2n+3\\ a+2,2a+2\end{matrix}\middle|\,x\right)\\ &= \frac{(2n)!\Gamma(2a+2n+2)\Gamma(a+2n+2)}{\Gamma(2a+4n+3)}G_{2,3}^{1,1}\left(\begin{matrix}-2a-2n-2;2n+1\\ 0;-2a-1,-a-1\end{matrix}\middle|\,x\right).\end{aligned} \tag{3.25}$$

It is also well-known [7] what the odd polynomials look like in terms of partition functions, namely

$$q_{2n+1}^{(a)}(x)=\frac{1}{Z_{0|0}^{(2n,a,\mathrm{B})}}\frac{1}{(2n)!}\int_{\mathbb{R}_+^{2n}}d[z]\frac{\Delta_{2n}^2(z)\prod_{j=1}^{2n}z_j^a(x-z_j)\mathrm{e}^{-z_j}}{\prod_{1\le i<j\le 2n}(z_i+z_j)}\left(x+\sum_{j=1}^{2n}z_j+c\right), \tag{3.26}$$

where c is an arbitrary constant and reflects the ambiguity of the solution of the skew-orthogonality relations (3.24). We make use of fixing this constant later on to simply our results.

The constant c as well as the variable x can be pulled out the integral (3.26) leaving the polynomial $q_{2n}^{(a)}$ as a factor. The trace can be written as a derivative of an auxiliary parameter t which is introduced in the exponent, in particular we replace the two point weight $g^{(a)}(z_1,z_2)\to g_t^{(a)}(z_1,z_2)=(z_1z_2)^a\mathrm{e}^{-t(z_1+z_2)}(z_1-z_2)/(z_1+z_2)$. Then the integration variables z can be rescaled, $z\to z/t$ such that

we find the identity

$$q^{(a)}_{2n+1}(x) = (x+c)q^{(a)}_{2n}(x) - \frac{\partial}{\partial t} t^{-n(2n+3+2a)} q^{(a)}_{2n}(tx)\Big|_{t=1}. \tag{3.27}$$

Fixing now the constant $c = -n(2n+3+2a)$ we find the simple result

$$\begin{aligned} q^{(a)}_{2n+1}(x) &= x\left(1 - \frac{\partial}{\partial x}\right) q^{(a)}_{2n}(x) \\ &= \frac{(2n)!\Gamma(2a+2n+2)\Gamma(a+2n+2)}{\Gamma(2a+4n+3)} \left(G^{1,1}_{2,3}\left(\begin{matrix} -2a-2n-1; 2n+2 \\ 1; -2a, -a \end{matrix} \Big| x \right) \right. \\ &\quad \left. - G^{1,2}_{3,4}\left(\begin{matrix} 0, -2a-2n-2; 2n+1 \\ 0; 1, -2a-1, -a-1 \end{matrix} \Big| x \right) \right). \end{aligned} \tag{3.28}$$

Note that we do not need the polynomials of odd order in our approach. We only show them for the sake of completeness.

Also the kernels of the Pfaffians (3.13) and (3.14) can be read off yielding the following corollary.

Corollary 3.5. *With the requirements of proposition 3.1 and $\widetilde{N} = N+l-k$ the kernels of the Pfaffian representations* (3.13) *and* (3.14) *for the partition function* (3.1) *of the original Bures ensemble ($\alpha(z) = z^a \mathrm{e}^{-z}$) are*

$$\begin{aligned} &(\lambda_1 - \lambda_2) \frac{Z^{(\widetilde{N}-2,a,\mathrm{B})}_{0|2}(\lambda_1,\lambda_2)}{Z^{(\widetilde{N},a,\mathrm{B})}_{0|0}} \\ &= -\frac{1}{4}\int_0^1 dt \left[G^{1,1}_{2,3}\left(\begin{matrix} -2a-\widetilde{N}-1; \widetilde{N} \\ 0; -a, -2a-1 \end{matrix} \Big| t\lambda_1 \right) G^{1,1}_{2,3}\left(\begin{matrix} -2a-\widetilde{N}-1; \widetilde{N} \\ 0; -a-1, -2a-1 \end{matrix} \Big| t\lambda_2 \right) \right. \\ &\quad \left. - G^{1,1}_{2,3}\left(\begin{matrix} -2a-\widetilde{N}-1; \widetilde{N} \\ 0; -a, -2a-1 \end{matrix} \Big| t\lambda_2 \right) G^{1,1}_{2,3}\left(\begin{matrix} -2a-\widetilde{N}-1; \widetilde{N} \\ 0; -a-1, -2a-1 \end{matrix} \Big| t\lambda_1 \right) \right], \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\frac{1}{\kappa-\lambda}\frac{Z^{(\widetilde{N},a,\mathrm{B})}_{1|1}(\kappa,\lambda)}{Z^{(\widetilde{N},a,\mathrm{B})}_{0|0}} = \frac{1}{\kappa-\lambda} \\ &+\frac{1}{2}\int_0^1 dt \left[G^{1,1}_{2,3}\left(\begin{matrix} -2a-\widetilde{N}-1; \widetilde{N} \\ 0; -a, -2a-1 \end{matrix} \Big| t\lambda \right) G^{3,1}_{2,3}\left(\begin{matrix} -\widetilde{N}; \widetilde{N}+2a+1 \\ 0, a, 2a+1 \end{matrix} \Big| -t\kappa \right) \right. \\ &\quad \left. + G^{1,1}_{2,3}\left(\begin{matrix} -2a-\widetilde{N}-1; \widetilde{N} \\ 0; -a-1, -2a-1 \end{matrix} \Big| t\lambda \right) G^{3,1}_{2,3}\left(\begin{matrix} -\widetilde{N}; \widetilde{N}+2a+1 \\ 0, a+1, 2a+1 \end{matrix} \Big| -t\kappa \right) \right], \end{aligned} \tag{3.30}$$

$$(\kappa_1-\kappa_2)\frac{Z_{2|0}^{(\widetilde{N}+2,a,\mathrm{B})}(\kappa_1,\kappa_2)}{Z_{0|0}^{(\widetilde{N},a,\mathrm{B})}}=\int_{\mathbb{R}_+^2}dxdy\frac{(xy)^a(y-x)\mathrm{e}^{-x-y}}{(\kappa_1-x)(\kappa_2-y)(x+y)}$$
$$+(\kappa_1\kappa_2)^a\int_0^1 dt\Bigg[\kappa_2 G_{2,3}^{3,1}\left(\begin{matrix}-\widetilde{N}-a;\widetilde{N}+a+1\\0,-a,a+1\end{matrix}\middle|-t\kappa_1\right)G_{2,3}^{3,1}\left(\begin{matrix}-\widetilde{N}-a-1;\widetilde{N}+a\\0,-a-1,a\end{matrix}\middle|-t\kappa_2\right)$$
$$-\kappa_1 G_{2,3}^{3,1}\left(\begin{matrix}-\widetilde{N}-a;\widetilde{N}+a+1\\0,-a,a+1\end{matrix}\middle|-t\kappa_2\right)G_{2,3}^{3,1}\left(\begin{matrix}-\widetilde{N}-a-1;\widetilde{N}+a\\0,-a-1,a\end{matrix}\middle|-t\kappa_1\right)$$
$$-\kappa_2 G_{2,3}^{3,1}\left(\begin{matrix}-a;a+1\\0,-a,a+1\end{matrix}\middle|-t\kappa_1\right)G_{2,3}^{3,1}\left(\begin{matrix}-a-1;a\\0,-a-1,a\end{matrix}\middle|-t\kappa_2\right)$$
$$+\kappa_1 G_{2,3}^{3,1}\left(\begin{matrix}-a;a+1\\0,-a,a+1\end{matrix}\middle|-t\kappa_2\right)G_{2,3}^{3,1}\left(\begin{matrix}-a-1;a\\0,-a-1,a\end{matrix}\middle|-t\kappa_1\right)\Bigg] \tag{3.31}$$

for the two-point kernels and

$$\frac{Z_{0|1}^{(\widetilde{N}-1,a,\mathrm{B})}(\lambda)}{Z_{0|0}^{(\widetilde{N}+1,a,\mathrm{B})}}=\frac{2^{2(\widetilde{N}+a)}\Gamma[\widetilde{N}+a+3/2]}{\widetilde{N}!\Gamma[\widetilde{N}+2a+2]}G_{2,3}^{1,1}\left(\begin{matrix}-\widetilde{N}-2a-1;\widetilde{N}\\0;-2a-1,-a-1\end{matrix}\middle|\lambda\right), \tag{3.32}$$

and

$$\frac{Z_{1|0}^{(\widetilde{N}+1,a,\mathrm{B})}(\kappa)}{Z_{0|0}^{(\widetilde{N}+1,a,\mathrm{B})}}=-\frac{1}{\widetilde{N}!\Gamma[\widetilde{N}+a+1]}G_{2,3}^{3,1}\left(\begin{matrix}-\widetilde{N}-1;\widetilde{N}+2a\\-1,a-1,2a\end{matrix}\middle|-\kappa\right) \tag{3.33}$$

for the partition functions with only one characteristic polynomial.

With the help of this corollary we are now ready to find all k-point correlation functions of the Bures ensemble.

3.3. **Correlation functions of the Bures ensemble.** Proposition 3.1 relates the partition functions of the Bures ensemble with those of the Cauchy two-matrix model. Thereby the individual level density of a random matrix has to be exchanged in the following way

$$\sum_{j=1}^N\delta(z-z_j)\to\sum_{j=1}^N\big[\delta(z-x_j)+\delta(z-y_j)\big] \tag{3.34}$$

for the three species $\{x_j\}$, $\{y_j\}$, and $\{z_j\}$ in Eqs. (2.3) and (3.1), respectively. On the other hand, Eq. (2.35), in particular Eqs. (2.36) and (2.37) for the densities, gives the correlations relating to the averaged level densities of the two species regarded as separate entities. Thus knowledge of the eigenvalue correlations of the Cauchy two-matrix model implies the correlations of the Bures ensemble. In

particular, we have for the level densities

$$\begin{aligned} R_1^{(N,a,\mathrm{B})}(z) &= \lim_{\varepsilon\to 0}\sum_{L=\pm 1}\frac{L}{2\pi\imath N}\frac{\partial}{\partial\widetilde{z}}\ln Z_{1|1}^{(N,a,\mathrm{B})}(\widetilde{z}+\imath L\varepsilon,z)\Big|_{\widetilde{z}=z} \\ &= \frac{1}{2}\lim_{\varepsilon\to 0}\sum_{L=\pm 1}\frac{L}{2\pi\imath N}\frac{\partial}{\partial\widetilde{z}}\ln Z_{1|1;1|1}^{(N,a,a+1\mathrm{C})}(\widetilde{z}+\imath L\varepsilon,z;\widetilde{z}+\imath L\varepsilon,z)\Big|_{\widetilde{z}=z} \\ &= \frac{1}{2}\left(R_{1,0}^{(N,a,a+1,\mathrm{C})}(z)+R_{0,1}^{(N,a,a+1,\mathrm{C})}(z)\right). \end{aligned} \tag{3.35}$$

The same calculation can be done for the two-point correlation function with "self-energy" terms,

$$\begin{aligned} \widehat{R}_2^{(N,a,\mathrm{B})}(z_1,z_2) &= \lim_{\varepsilon\to 0}\sum_{L_1,L_2=\pm 1}\frac{L_1L_2}{(2\pi\imath N)^2}\frac{\partial^2}{\partial\widetilde{z}_1\partial\widetilde{z}_2}\Big[\ln Z_{2|2}^{(N,a,\mathrm{B})}(\widetilde{z}+\imath L\varepsilon,z) \\ &\quad +\ln Z_{1|1}^{(N,a,\mathrm{B})}(\widetilde{z}_1+\imath L_1\varepsilon,z_1)\ln Z_{1|1}^{(N,a,\mathrm{B})}(\widetilde{z}_2+\imath L_1\varepsilon,z_2)\Big]\Big|_{\widetilde{z}=z}. \end{aligned} \tag{3.36}$$

The first term is the connected correlation function and is the analogue to the cumulant for ordinary random variables. Employing relation (2.35) we have

$$\begin{aligned} &\widehat{R}_2^{(N,a,\mathrm{B})}(z_1,z_2) \\ =\ &\frac{1}{2}\Big[\widehat{R}_{2,0}^{(N,a,a+1,\mathrm{C})}(z_1,z_2)+\widehat{R}_{1,1}^{(N,a,a+1,\mathrm{C})}(z_1,z_2)+\widehat{R}_{1,1}^{(N,a,a+1,\mathrm{C})}(z_2,z_1) \\ &+\widehat{R}_{0,2}^{(N,a,a+1,\mathrm{C})}(z_1,z_2) \\ &-\frac{1}{2}\left(R_{1,0}^{(N,a,a+1,\mathrm{C})}(z_1)+R_{0,1}^{(N,a,a+1,\mathrm{C})}(z_1)\right)\left(R_{1,0}^{(N,a,a+1,\mathrm{C})}(z_2)+R_{0,1}^{(N,a,a+1,\mathrm{C})}(z_2)\right)\Big] \\ =\ &\frac{1}{2}\Big[\frac{1}{2}\left(K_{01}^{(N,a,a+1,\mathrm{C})}(z_1,z_1)+K_{10}^{(N,a,a+1,\mathrm{C})}(z_1,z_1)\right) \\ &\times\left(K_{01}^{(N,a,a+1,\mathrm{C})}(z_2,z_2)+K_{10}^{(N,a,a+1,\mathrm{C})}(z_2,z_2)\right) \\ &-K_{01}^{(N,a,a+1,\mathrm{C})}(z_1,z_2)K_{01}^{(N,a,a+1,\mathrm{C})}(z_2,z_1)-K_{10}^{(N,a,a+1,\mathrm{C})}(z_1,z_2)K_{10}^{(N,a,a+1,\mathrm{C})}(z_2,z_1) \\ &-K_{11}^{(N,a,a+1,\mathrm{C})}(z_1,z_2)K_{00}^{(N,a,a+1,\mathrm{C})}(z_1,z_2)-K_{11}^{(N,a,a+1,\mathrm{C})}(z_2,z_1)K_{00}^{(N,a,a+1,\mathrm{C})}(z_2,z_1) \\ &+\frac{1}{N}\delta(z_1-z_2)\left(K_{01}^{(N,a,a+1,\mathrm{C})}(z_1,z_1)+K_{10}^{(N,a,a+1,\mathrm{C})}(z_1,z_1)\right)\Big]. \end{aligned} \tag{3.37}$$

The term proportional to the Dirac delta-function is the "self-energy" term and will be omitted in the following. The kernels satisfy the relations

$$\begin{aligned} K_{00}^{(N,a,a+1,\mathrm{C})}(z_1;z_2)+K_{00}^{(N,a,a+1,\mathrm{C})}(z_1;z_2) &= w(z_1)w(z_2), \\ K_{01}^{(N,a,a+1,\mathrm{C})}(z_1,z_2)-K_{10}^{(N,a,a+1,\mathrm{C})}(z_1,z_2) &= v(z_1)w(z_2), \\ K_{11}^{(N,a,a+1,\mathrm{C})}(z_1;z_2)+K_{11}^{(N,a,a+1,\mathrm{C})}(z_1;z_2) &= -v(z_1)v(z_2), \end{aligned} \tag{3.38}$$

where the functions w and v can be read off from the relations satisfied by the two-point partition functions derived in appendix B. With the help of these relations one

can show that the two-point correlation function without the "self-energy" terms is equal to the Pfaffian

$$R_2^{(N,a,\mathrm{B})}(z_1,z_2) = -\frac{N-1}{4N} \times \mathrm{Pf}\left[\begin{array}{c|c} \Delta K_{11}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) & \Sigma K_{01}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) \\ \hline -\Sigma K_{01}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) & \Delta K_{00}^{(N,a,a+1,\mathrm{C})}(z_j;z_i) \end{array}\right]_{1\le i,j\le 2} \tag{3.39}$$

with the abbreviations

$$\begin{aligned} \Delta K_{11}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) &= K_{11}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) - K_{11}^{(N,a,a+1,\mathrm{C})}(z_j;z_i), \\ \Sigma K_{01}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) &= K_{01}^{(N,a,a+1,\mathrm{C})}(z_i,z_j) + K_{10}^{(N,a,a+1,\mathrm{C})}(z_i,z_j), \\ \Delta K_{00}^{(N,a,a+1,\mathrm{C})}(z_j;z_i) &= K_{00}^{(N,a,a+1,\mathrm{C})}(z_j;z_i) - K_{00}^{(N,a,a+1,\mathrm{C})}(z_i;z_j). \end{aligned} \tag{3.40}$$

Indeed we can also find the results (3.35) and (3.39) via the relation of the partition functions derived in subsection 3.2.

Let us employ the definition (3.2) to the result (3.18) with $k = l$. Then the only important contribution is the action of the derivatives in $\widetilde{z}$ on the prefactor $1/B_{k|k}(\widetilde{z}+\imath L\varepsilon, z)$ in front of the Pfaffian. Almost all other terms vanish under the sum of the signs $L_j = \pm 1$ and in the limit $\widetilde{z} = z$ and $\varepsilon \to 0$. Indeed there are also contributions from the derivatives in the diagonal entries of the off-diagonal blocks in the Pfaffian. However they yield the same kernel as the other matrix entries in the off-diagonal blocks apart from the Dirac delta-functions $\delta(z_i - z_j)$ which result from the first term $1/(\kappa - \lambda)$ of the two-point partition function (3.30). Omitting these Dirac delta-functions we find the k-point correlation function summarized in the following corollary.

Corollary 3.6. *Let $z_1, \ldots, z_k \in \mathbb{R}_+$ be pairwise different. Then the k-point correlation function without the "self-energy" terms of the Bures ensemble is*

$$R_k^{(N,a,\mathrm{B})}(z) = (-1)^{k(k-1)/2}\frac{N!}{(2N)^k(N-k)!} \times \mathrm{Pf}\left[\begin{array}{c|c} \Delta K_{11}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) & \Sigma K_{01}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) \\ \hline -\Sigma K_{01}^{(N,a,a+1,\mathrm{C})}(z_i;z_j) & \Delta K_{00}^{(N,a,a+1,\mathrm{C})}(z_j;z_i) \end{array}\right]_{1\le i,j\le k}, \tag{3.41}$$

where we used the abbreviations (3.40) *and the kernels* (2.34) *of the Cauchy two-matrix model.*

Thus we have traced all k-point correlation functions for the Bures ensemble back to the kernels of the (k,l)-point correlation functions (2.35) for the Cauchy two-matrix model. In particular the case $k = N$ yields the joint probability density in terms of a Pfaffian. This shows that the process given by Eq. (1.5) is a Pfaffian point process.

4. Hard Edge Scaling Limit

Finally we consider the hard edge scaling limit which is the double scaling limit $N \to \infty$ with $N^2 x_j$ and a fixed. This limit is a direct corollary of our result in combination with [16, Theorem 2.2].

Corollary 4.1. *Let $z_1, \dots, z_k \in \mathbb{R}_+$ be pairwise different. Then the hard edge scaling limit of the k-point correlation function of the Bures ensemble is*

$$\begin{aligned} R_k^{(\infty,a,\mathrm{B})}(z) &= \lim_{N\to\infty} N^{-2k} R_k^{(N,a,\mathrm{B})}\left(\frac{z}{N^2}\right) \\ &= \frac{(-1)^{k(k-1)/2}}{2^k}\mathrm{Pf}\left[\begin{array}{c|c} \Delta K_{11}^{(\infty,a)}(z_i;z_j) & \Sigma K_{01}^{(\infty,a)}(z_i;z_j) \\ \hline -\Sigma K_{01}^{(\infty,a)}(z_i;z_j) & \Delta K_{00}^{(\infty,a)}(z_j;z_i) \end{array}\right]_{1\le i,j\le k}. \end{aligned} \tag{4.1}$$

The kernels are given by

$$\begin{aligned} \Delta K_{11}^{(\infty,a)}(z_i;z_j) &= \lim_{N\to\infty} N^{4a}\Delta K_{11}^{(N,a,a+1,\mathrm{C})}\left(\frac{z_i}{N^2};\frac{z_j}{N^2}\right) \\ &= z_i^a z_j^a\left(\frac{z_i - z_j}{z_i + z_j}\right. \\ &\quad \left. + \int_0^1 dt\left[z_j G_{0,3}^{2,0}\left(\begin{matrix}-\\ 0, a+1; -a\end{matrix}\middle| tz_i\right) G_{0,3}^{2,0}\left(\begin{matrix}-\\ 0, a; -a-1\end{matrix}\middle| tz_j\right)\right.\right. \\ &\quad \left.\left. - z_i G_{0,3}^{2,0}\left(\begin{matrix}-\\ 0, a; -a-1\end{matrix}\middle| tz_i\right) G_{0,3}^{2,0}\left(\begin{matrix}-\\ 0, a+1; -a\end{matrix}\middle| tz_j\right)\right]\right), \\ \Sigma K_{01}^{(\infty,a)}(z_i;z_j) &= \lim_{N\to\infty} N^{-2}\Sigma K_{01}^{(N,a,a+1,\mathrm{C})}\left(\frac{z_i}{N^2};\frac{z_j}{N^2}\right) \\ &= \int_0^1 dt\left[G_{0,3}^{1,0}\left(\begin{matrix}-\\ 0; -a, -2a-1\end{matrix}\middle| tz_j\right) G_{0,3}^{2,0}\left(\begin{matrix}-\\ a, 2a+1; 0\end{matrix}\middle| tz_i\right)\right. \\ &\quad \left. + G_{0,3}^{2,0}\left(\begin{matrix}-\\ a+1, 2a+1; 0\end{matrix}\middle| tz_j\right) G_{0,3}^{1,0}\left(\begin{matrix}-\\ 0; -a-1, -2a-1\end{matrix}\middle| tz_i\right)\right], \\ \Delta K_{00}^{(\infty,a)}(z_j;z_i) &= \lim_{N\to\infty} N^{-4a-4}\Delta K_{00}^{(N,a,a+1,\mathrm{C})}\left(\frac{z_j}{N^2};\frac{z_i}{N^2}\right) \\ &= \int_0^1 dt t^{2a+1}\left[G_{0,3}^{1,0}\left(\begin{matrix}-\\ 0; -a, -2a-1\end{matrix}\middle| tz_j\right) G_{0,3}^{1,0}\left(\begin{matrix}-\\ 0; -a-1, -2a-1\end{matrix}\middle| tz_i\right)\right. \\ &\quad \left. - G_{0,3}^{1,0}\left(\begin{matrix}-\\ 0; -a-1, -2a-1\end{matrix}\middle| tz_j\right) G_{0,3}^{1,0}\left(\begin{matrix}-\\ 0; -a, -2a-1\end{matrix}\middle| tz_i\right)\right]. \end{aligned} \tag{4.2}$$

The proof of this corollary is straightforward and will be skipped. One has simply to combine [16, Theorem 2.2] with Eq. (3.40). There is also no problem with the different scalings of the three kernels due to multilinearity of the Pfaffian. Then the factors N^{4a}, N^{-2}, and N^{-4a-4} can be pulled into the corresponding rows and columns and the total factor of N can be counted in the overall factor.

The Meijer G-kernel in the Pfaffian is up to now unique. It would be interesting if this kernel can be found for other random matrix ensembles as well. This spectral behavior would be quite natural as long as the pair interaction of the eigenvalues near the origin, $z_i, z_j \ll 1$, behave like $\exp[2\ln|z_i - z_j| - \ln|z_i + z_j|]$. Working in [34] tells us that this functional form gives the equilibrium problem for the Raney density indexed by $(3/2, 1/2)$, the significance of this being that the evidence is that it is the Raney parameters (p, r) which determine the hard edge universality class; see again [34]. It is to be expected that this hard edge scaling limit is independent of the confining potential $\exp[-V(z_j)]$ implying that the a condition like $\sum_j z_j = 1$ does not affect this behavior because it can be rewritten in term of such a potential via a Fourier-Laplace transform, see e.g. [50]. The reason why we expect this result

also for a general class of ensembles is the separation of scales which is the origin of universality of spectral statistics.

5. Discussion and Outlook

We established a relationship between the kernels of the Pfaffian point process of the Bures ensemble and the kernels of the determinantal point process of the Cauchy two-matrix model. Thereby we started from the partially known fact [14] that the square of any partition function of the Bures ensemble is equal to a partition function of a Cauchy two-matrix model, see proposition 3.1. Since the kernels of the Pfaffian point process can be also identified as partition functions we had to invert this relation. Surprisingly, the square root can be made exact such that we end up with precisely the same one-fold integrals over Meijer G-functions as were found for the Cauchy two-matrix model [16], see corollaries 3.5 and 3.6. In particular, each of the kernels of the Bures ensemble is only a linear combination of two kernels of the Cauchy two-matrix model. Additionally the skew-orthogonal polynomials corresponding to the Bures ensemble are expressed in the bi-orthogonal polynomials of the Cauchy two-matrix model. All these relations together represent a complete exact solution of the Bures ensemble.

Two problems can now be studied. First, as already begun we can study any large N asymptotics of the level statistics including the macroscopic and microscopic level densities, the hard edge and the soft edge correlation functions, and the correlation functions in the bulk. Here we expect the sine-kernel for the Dyson index $\beta = 2$ in the bulk of the spectrum since the level repulsion is $(\lambda_1 - \lambda_2)^2$ which should be the only relevant input on the scale of the mean level spacing. Also at the soft edge we expect the standard Airy kernel behaviour for the joint probability density (1.5) and for the fixed trace ensemble (1.3) as known for the Laguerre ensemble and its fixed trace counterpart [50]. The latter corresponds to the case of the Hilbert-Schmidt measure on the set of density matrices.

At the hard edge we derived that the Bures ensemble lies in a universality class which is described by Meijer G-functions and is reminiscent to those already found for product matrices, see Refs. [3, 4, 33, 48, 49, 55]. Thus the Bures ensemble is the first ensemble of such a class for which the singular values exhibit a Pfaffian point process. We expect that the Bures ensemble with the fixed trace condition (1.3) shares the same behaviour at the hard edge since the condition should only effect the upper bound on the local scale of the mean level spacing. The macroscopic level density for the Bures ensemble with an infinite rank $N - M$ fixed was already calculated in Ref. [68]. However, neither the level density at finite N nor when $N/(N - M)$ is fixed were considered. The first case is important when small quantum systems as qubits and qutrits are studied [64, 65, 66, 63]. The second case with $N/(N - M)$ fixed for $N \to \infty$ is the quite natural case for any experiment which considers a prepared quantum state which is almost pure.

In the second problem, one can go back to the original Bures measure where we have to include the fixed trace condition which results in a joint probability density (1.3). As previously remarked, the Pfaffian based correlations found for Eq. (1.5) now involve an auxiliary scaling parameter, and a Fourier-Laplace transform must be taken with respect to this parameter. Although there is good reason to think that this has no effect on scaling limits [50], the details of the calculation remain. The additional integral from the Fourier-Laplace transformation destroys

the algebraic structure of the partition functions and the k-point correlation functions in terms of Pfaffians. This additional integral will have a crucial influence on the macroscopic level density and the edge behaviour at the upper bound because the support of the eigenvalues is squeezed to the allowed interval $[0, 1]$.

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Appendix A. Proof of proposition 3.1

Let $k = l$ without loss of generality since we can take the limits $|\lambda_j| \to \infty$ or $|\kappa_j| \to \infty$ for some j which reduces the partition function for $k = l$ to the general case where k and l are different.

The proof starts with two identities, namely the Schur Pfaffian (3.9) and the extension

$$\Delta_N(z) \prod_{j=1}^{N} \frac{\prod_{i=1}^{k}(z_j - \lambda_i)}{\prod_{i=1}^{k}(z_j - \kappa_i)} = \frac{\mathrm{B}_{k|N+k}(\kappa; z, \lambda)}{\mathrm{B}_{k|k}(\kappa; \lambda)} \tag{A.1}$$

of the Cauchy-Vandermonde determinant [12, 45]

$$\begin{aligned} \mathrm{B}_{k|N+k}(\kappa; z, \lambda) &= \frac{\Delta_N(z)\Delta_k(\kappa)\Delta_k(\lambda)}{\prod_{i,j=1}^{k}(\kappa_i - \lambda_j)} \prod_{j=1}^{N} \frac{\prod_{i=1}^{k}(z_j - \lambda_i)}{\prod_{i=1}^{k}(z_j - \kappa_i)} \qquad (A.2) \\ &= (-1)^{k(k-1)/2} \det \begin{bmatrix} \left\{ z_a^{b-1} \right\}_{1\le a,b\le N} & \left\{ \dfrac{1}{\kappa_b - z_a} \right\}_{\substack{1\le a\le N \\ 1\le b\le k}} \\ \left\{ \lambda_a^{b-1} \right\}_{\substack{1\le a\le k \\ 1\le b\le N}} & \left\{ \dfrac{1}{\kappa_b - \lambda_a} \right\}_{1\le a,b\le k} \end{bmatrix}, \\ \mathrm{B}_{k|k}(\kappa; \lambda) &= \frac{\Delta_k(\kappa)\Delta_k(\lambda)}{\prod_{i,j=1}^{k}(\kappa_i - \lambda_j)} = (-1)^{k(k-1)/2} \det \left[\frac{1}{\kappa_a - \lambda_b} \right]_{1\le a,b\le k}. \end{aligned}$$

which is equivalent to (2.21). The two identities (3.9) and (A.1) can be plugged into the partition function $Z_{k|k}^{(N,\mathrm{B})}[\alpha](\kappa, \lambda)$ for the Bures-like measure. Applying a modified version of de Bruijn's integral identity [20, 45] we find

$$\begin{aligned} Z_{k|k}^{(N,\mathrm{B})}[\alpha](\kappa, \lambda) &= \pm \frac{1}{\mathrm{B}_{k|k}(\kappa; \lambda)} \mathrm{Pf}\left[M_{\mathrm{even}} \right] \qquad (A.3) \\ &= \pm \frac{1}{\mathrm{B}_{k|k}(\kappa; \lambda)} \mathrm{Pf} \begin{bmatrix} \{M_{ab}\}_{1\le a,b\le N} & \{F_a(\kappa_b)\}_{\substack{1\le a\le N \\ 1\le b\le k}} & \left\{ \lambda_b^{a-1} \right\}_{\substack{1\le a\le N \\ 1\le b\le k}} \\ \{-F_b(\kappa_a)\}_{\substack{1\le a\le k \\ 1\le b\le N}} & \{G(\kappa_a, \kappa_b)\}_{1\le a,b\le k} & \left\{ \dfrac{1}{\kappa_a - \lambda_b} \right\}_{1\le a,b\le k} \\ \left\{ -\lambda_a^{b-1} \right\}_{\substack{1\le a\le k \\ 1\le b\le N}} & \left\{ \dfrac{1}{\lambda_a - \kappa_b} \right\}_{1\le a,b\le k} & 0 \end{bmatrix} \end{aligned}$$

for even N and

$$Z_{k|k}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda) = \pm\frac{1}{\mathrm{B}_{k|k}(\kappa;\lambda)}\mathrm{Pf}\left[M_{\mathrm{odd}}\right] \tag{A.4}$$

$$= \pm\frac{1}{\mathrm{B}_{k|k}(\kappa;\lambda)}\mathrm{Pf}\left[\begin{array}{c|cc} 0 & \{m_b\}_{1\le b\le N} & \{f(\kappa_b)\}_{1\le b\le k} \quad 0 \\ \hline \begin{array}{c}\{-m_a\}_{1\le a\le N}\\ \{-f(\kappa_a)\}_{1\le a\le k}\\ 0\end{array} & \multicolumn{2}{c}{M_{\mathrm{even}}} \end{array}\right]$$

for odd N with the abbreviations

$$\begin{aligned} M_{ab} &= \int_0^\infty dz_1dz_2\alpha(z_1)\alpha(z_2)z_1^{a-1}z_2^{b-1}\frac{z_1-z_2}{z_1+z_2}, \\ F_a(\kappa_b) &= \int_0^\infty dz_1dz_2\frac{\alpha(z_1)\alpha(z_2)z_1^{a-1}}{\kappa_b-z_2}\frac{z_1-z_2}{z_1+z_2}, \\ G(\kappa_a,\kappa_b) &= \int_0^\infty dz_1dz_2\frac{\alpha(z_1)\alpha(z_2)}{(\kappa_a-z_1)(\kappa_b-z_2)}\frac{z_1-z_2}{z_1+z_2}, \\ m_b &= \int_0^\infty dz\alpha(z)z^{b-1}, \\ f(\kappa_b) &= \int_0^\infty \frac{dz\alpha(z)}{\kappa_b-z}. \end{aligned} \tag{A.5}$$

Note that the global sign is not important for the proof since we square the Pfaffian which yields determinants, i.e.

$$\left(Z_{k|k}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda)\right)^2 = \left(\frac{1}{\mathrm{B}_{k|k}(\kappa;\lambda)}\right)^2\begin{cases}\det M_{\mathrm{even}}, & N \text{ even},\\ \det M_{\mathrm{odd}}, & N \text{ odd}.\end{cases} \tag{A.6}$$

In the next step we apply the simple relation

$$\frac{z_1-z_2}{z_1+z_2} = \frac{2z_1}{z_1+z_2} - 1 \tag{A.7}$$

and define the vectors

$$\vec{x}_{\mathrm{even}}^T = \left[\begin{array}{cccc} \{m_b\}_{1\le b\le N} & \{f(\kappa_b)\}_{1\le b\le k} & 0\ldots0 \end{array}\right] \tag{A.8}$$

of dimension $N+2k$ and

$$\vec{x}_{\mathrm{odd}}^T = \left[\begin{array}{c|ccc} -1 & \{m_b\}_{1\le b\le N} & \{f(\kappa_b)\}_{1\le b\le k} & 0\ldots0 \end{array}\right] \tag{A.9}$$

of dimension $N+2k+1$. Then we can use the following algebraic manipulation

$$\begin{aligned} \det M_{\mathrm{even}} &= \det M_{even}(1+\vec{x}_{\mathrm{even}}^T M_{\mathrm{even}}^{-1}\vec{x}_{\mathrm{even}}) \\ &= \det M_{\mathrm{even}}\det(\mathbb{I}_{N+2k}+M_{\mathrm{even}}^{-1}\vec{x}_{\mathrm{even}}\vec{x}_{\mathrm{even}}^T) \\ &= \det(M_{\mathrm{even}}+\vec{x}_{\mathrm{even}}\vec{x}_{\mathrm{even}}^T) \end{aligned} \tag{A.10}$$

for even N and similarly for odd N,

$$\det M_{\mathrm{odd}} = \det(M_{\mathrm{odd}}+\vec{x}_{\mathrm{odd}}\vec{x}_{\mathrm{odd}}^T). \tag{A.11}$$

Both relations are based on the fact that M_{even} as well as M_{odd} are antisymmetric, namely for any antisymmetric matrix A and any vector $\vec{v}$ the expectation value

$\vec{v}^T A \vec{v}$ vanishes. Defining the abbreviations

$$\begin{aligned}
\widehat{M}_{ab} &= \int_0^\infty dz_1 dz_2 \alpha(z_1)\alpha(z_2) z_1^a z_2^{b-1} \frac{1}{z_1+z_2}, \qquad (A.12)\\
\widehat{F}_a^{(1)}(\kappa_b) &= \int_0^\infty dz_1 dz_2 \frac{\alpha(z_1)\alpha(z_2) z_1^a}{\kappa_b - z_2} \frac{1}{z_1+z_2},\\
\widehat{F}_b^{(2)}(\kappa_a) &= \int_0^\infty dz_1 dz_2 \frac{\alpha(z_1)\alpha(z_2) z_1 z_2^{b-1}}{\kappa_a - z_1} \frac{1}{z_1+z_2},\\
\widehat{G}(\kappa_a,\kappa_b) &= \int_0^\infty dz_1 dz_2 \frac{\alpha(z_1)\alpha(z_2) z_1}{(\kappa_a - z_1)(\kappa_b - z_2)} \frac{1}{z_1+z_2}.
\end{aligned}$$

we use the identities (A.10) and (A.11) such that

$$\left(Z_{k|k}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda)\right)^2 = 2^N \left(\frac{1}{\mathrm{B}_{k|k}(\kappa;\lambda)}\right)^2 \det \widehat{M}_{\mathrm{even}} \qquad (A.13)$$

$$= 2^N \left(\frac{1}{\mathrm{B}_{k|k}(\kappa;\lambda)}\right)^2 \times \det \begin{bmatrix} \left\{\widehat{M}_{ab}\right\}_{1\le a,b\le N} & \left\{\widehat{F}_a^{(1)}(\kappa_b)\right\}_{\substack{1\le a\le N\\ 1\le b\le k}} & \left\{\lambda_b^{a-1}\right\}_{\substack{1\le a\le N\\ 1\le b\le k}} \\ \left\{\widehat{F}_b^{(2)}(\kappa_a)\right\}_{\substack{1\le a\le k\\ 1\le b\le N}} & \left\{\widehat{G}(\kappa_a,\kappa_b)\right\}_{1\le a,b\le k} & \left\{\frac{1}{\kappa_a-\lambda_b}\right\}_{1\le a,b\le k} \\ \left\{-\lambda_a^{b-1}\right\}_{\substack{1\le a\le k\\ 1\le b\le N}} & \left\{\frac{1}{\lambda_a-\kappa_b}\right\}_{1\le a,b\le k} & 0 \end{bmatrix}$$

for even N and

$$\begin{aligned}
\left(Z_{k|k}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda)\right)^2 &= 2^N \left(\frac{1}{\mathrm{B}_{k|k}(\kappa;\lambda)}\right)^2 \det \widehat{M}_{\mathrm{odd}} \qquad (A.14)\\
&= 2^N \left(\frac{1}{\mathrm{B}_{k|k}(\kappa;\lambda)}\right)^2 \det \left[\begin{array}{c|c} 1 & 0 \\ \hline \{-2m_a\}_{1\le a\le N} & \\ \{-2f(\kappa_a)\}_{1\le a\le k} & \widehat{M}_{\mathrm{even}} \\ 0 & \end{array}\right]\\
&= 2^N \left(\frac{1}{\mathrm{B}_{k|k}(\kappa;\lambda)}\right)^2 \det \widehat{M}_{\mathrm{even}}
\end{aligned}$$

for odd N.

Now we can identify the integration variable z_1 in the kernels of the determinant (A.13) with the integration variables y_j in the proposition and the variable z_2 with the variables x_j. Then we can apply a generalized version of Andréief's integration theorem [10, 45] backwards and find

$$\begin{aligned}
\left(Z_{k|k}^{(N,\mathrm{B})}[\alpha](\kappa,\lambda)\right)^2 &= (-1)^{N(N-1)/2} 2^N \int_{\mathbb{R}_+^{2N}} \prod_{j=1}^N \alpha(x_j)\alpha(y_j) y_j \, \mathrm{B}_{N|N}(x;-y)\\
&\quad \times \frac{\mathrm{B}_{k|N+k}(\kappa;x,\lambda)}{\mathrm{B}_{k|k}(\kappa;\lambda)} \frac{\mathrm{B}_{k|N+k}(\kappa;y,\lambda)}{\mathrm{B}_{k|k}(\kappa;\lambda)} (dx)(dy) \qquad (A.15)
\end{aligned}$$

independent of the fact whether N is even or odd. In the last step we employ the definitions of the Cauchy-Vandermonde determinants (A.2) and find the proposition for the case $l = k$. We can lift this restriction by extending the original integral for $k \neq l$ to the case $k = l$ and then taking the limit $\kappa_j \to \infty$ or $\lambda_j \to \infty$ for $j = \min\{k, l\} + 1, \ldots, \max\{k, l\}$.

Appendix B. Proof of proposition 3.3

Choosing $L \in \mathbb{N}$ we define the matrix

$$\Delta M_L \quad = \quad \left[\Delta M_{ij} = \frac{1}{2}(\widehat{M}_{ij} - \widehat{M}_{ji})\right]_{1 \leq i,j \leq L}, \tag{B.1}$$

the vectors

$$\begin{aligned}
\vec{m}_L^T &= (m_1, \ldots, m_L), \\
\Delta\vec{F}_L(\kappa) &= \left(\frac{\widehat{F}_1^{(1)}(\kappa) - \widehat{F}_1^{(2)}(\kappa)}{2}, \ldots, \frac{\widehat{F}_L^{(1)}(\kappa) - \widehat{F}_L^{(2)}(\kappa)}{2}\right), \\
\vec{\lambda}_L^T &= \left(1, \lambda, \ldots, \lambda^{L-1}\right), \\
\vec{e}_{L+1}^T &= (\overbrace{0, \ldots, 0}^{L}, 1),
\end{aligned} \tag{B.2}$$

and the scalar function

$$\Delta G(\kappa_1, \kappa_2) \quad = \quad \frac{1}{2}\left(\widehat{G}(\kappa_1, \kappa_2) - \widehat{G}(\kappa_2, \kappa_1)\right). \tag{B.3}$$

Then we can split the functions (A.12) in symmetric and anti-symmetric parts,

$$\begin{aligned}
\widehat{M}_{ij} &= \Delta M_{ij} + \frac{m_i m_j}{2}, \\
\widehat{F}_j^{(1)}(\kappa) &= \Delta F_j(\kappa) + \frac{f(\kappa) m_j}{2}, \\
\widehat{F}_j^{(2)}(\kappa) &= -\Delta F_j(\kappa) + \frac{f(\kappa) m_j}{2}, \\
\widehat{G}(\kappa_1, \kappa_2) &= \Delta\widehat{G}(\kappa_1, \kappa_2) + \frac{f(\kappa_1) f(\kappa_2)}{2}.
\end{aligned} \tag{B.4}$$

These relations follow from the particular form of the integrands, see Eqs. (A.5) and (A.12). We employ this splitting after we apply a generalized version of Andréief's integration theorem [10, 45] to the four partition functions in the kernels of the

determinant (2.20),

$$\begin{aligned}
Z^{(L+1,\mathrm{C})}_{1|0;1|0}[\alpha](\kappa_1;\kappa_2) &= \det\begin{bmatrix} \left\{\widehat{M}_{ij}\right\}_{1\le i,j\le L} & \left\{\widehat{F}^{(1)}_i(\kappa_1)\right\}_{1\le i\le L} \\ \left\{\widehat{F}^{(2)}_j(\kappa_2)\right\}_{1\le j\le L} & \widehat{G}(\kappa_1,\kappa_2) \end{bmatrix} \\
&= \det\left[\begin{array}{c|c} \Delta M_L + \dfrac{\vec{m}_L\vec{m}_L^T}{2} & \Delta\vec{F}_L(\kappa_1) + \dfrac{f(\kappa_1)\vec{m}_L}{2} \\ \hline -\Delta\vec{F}_L^T(\kappa_2) + \dfrac{f(\kappa_2)\vec{m}_L^T}{2} & \Delta\widehat{G}(\kappa_1,\kappa_2) + \dfrac{f(\kappa_1)f(\kappa_2)}{2} \end{array}\right], \\
\frac{Z^{(L,\mathrm{C})}_{0|0;1|1}[\alpha](\kappa,\lambda)}{\kappa-\lambda} &= \det\begin{bmatrix} \left\{\widehat{M}_{ij}\right\}_{1\le i,j\le L} & \left\{\lambda^{i-1}\right\}_{1\le i\le L} \\ \left\{\widehat{F}^{(2)}_j(\kappa)\right\}_{1\le j\le L} & \dfrac{1}{\kappa-\lambda} \end{bmatrix} \\
&= \det\left[\begin{array}{c|c} \Delta M_L + \dfrac{\vec{m}_L\vec{m}_L^T}{2} & \vec{\lambda}_L \\ \hline -\Delta\vec{F}_L^T(\kappa) + \dfrac{f(\kappa)\vec{m}_L^T}{2} & \dfrac{1}{\kappa-\lambda} \end{array}\right], \\
\frac{Z^{(L,\mathrm{C})}_{1|1;0|0}[\alpha](\kappa,\lambda)}{\kappa-\lambda} &= \det\begin{bmatrix} \left\{\widehat{M}_{ij}\right\}_{1\le i,j\le L} & \left\{\widehat{F}^{(1)}_i(\kappa)\right\}_{1\le i\le L} \\ \left\{\lambda^{j-1}\right\}_{1\le j\le L} & \dfrac{1}{\kappa-\lambda} \end{bmatrix} \\
&= \det\left[\begin{array}{c|c} \Delta M_L + \dfrac{\vec{m}_L\vec{m}_L^T}{2} & \Delta\vec{F}_L(\kappa) + \dfrac{f(\kappa)\vec{m}_L}{2} \\ \hline \vec{\lambda}_L^T & \dfrac{1}{\kappa-\lambda} \end{array}\right], \\
Z^{(L-1,\mathrm{C})}_{0|1;0|1}[\alpha](\lambda_1;\lambda_2) &= \det\begin{bmatrix} \left\{\widehat{M}_{ij}\right\}_{1\le i,j\le L} & \left\{\lambda_b^{i-1}\right\}_{1\le i\le L} \\ \left\{\lambda_a^{j-1}\right\}_{1\le j\le L} & 0 \end{bmatrix} \\
&= \det\left[\begin{array}{c|c} \Delta M_L + \dfrac{\vec{m}_L\vec{m}_L^T}{2} & \vec{\lambda}_{L,2} \\ \hline \vec{\lambda}_{L,1}^T & 0 \end{array}\right]. \qquad \text{(B.5)}
\end{aligned}$$

The next steps only apply for the case $L \in 2\mathbb{N}$ even since the antisymmetric matrix ΔM_L is only invertible in this case. For the case L odd we have to modify this procedure. Exactly for this case we need the constant vector $\vec{e}_{L+1}$.

Let L be even. Then the inverse of the matrix consisting of the moments of the Cauchy two-matrix ensemble reads

$$\left(\Delta M_L + \frac{\vec{m}_L\vec{m}_L^T}{2}\right)^{-1} = \Delta M_L^{-1} - \frac{1}{2}\Delta M_L^{-1}\vec{m}_L\vec{m}_L^T\Delta M_L^{-1} \qquad \text{(B.6)}$$

since we can perform a Taylor expansion in the dyadic matrix $\vec{m}_L\vec{m}_L^T$. This Taylor expansion is finite because $\vec{m}_L^T\Delta M_L^{-1}\vec{m}_L = 0$ resulting from the asymmetry of

ΔM_L which carries over to its inverse. Therefore we find

$$\begin{aligned}
\frac{Z^{(L+1,\mathrm{C})}_{1|0;1|0}[\alpha](\kappa_1;\kappa_2)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]} &= \Delta\widehat{G}(\kappa_1,\kappa_2)+\frac{f(\kappa_1)f(\kappa_2)}{2}+\left(\Delta\vec{F}_L^T(\kappa_2)-\frac{f(\kappa_2)\vec{m}_L^T}{2}\right)\\
&\quad\times\left(\Delta M_L^{-1}-\frac{1}{2}\Delta M_L^{-1}\vec{m}_L\vec{m}_L^T\Delta M_L^{-1}\right)\left(\Delta\vec{F}_L(\kappa_1)+\frac{f(\kappa_1)\vec{m}_L}{2}\right)\\
&= -\frac{Z^{(L+1,\mathrm{C})}_{1|0;1|0}[\alpha](\kappa_2;\kappa_1)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]}\\
&\quad+(f(\kappa_1)-\vec{m}_L^T\Delta M_L^{-1}\Delta\vec{F}_L(\kappa_1))(f(\kappa_2)-\vec{m}_L^T\Delta M_L^{-1}\Delta\vec{F}_L(\kappa_2)),\\
\frac{Z^{(L,\mathrm{C})}_{0|0;1|1}[\alpha](\kappa,\lambda)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha](\kappa-\lambda)} &= \frac{1}{\kappa-\lambda}+\left(\Delta\vec{F}_L^T(\kappa)-\frac{f(\kappa)\vec{m}_L^T}{2}\right)\\
&\quad\times\left(\Delta M_L^{-1}-\frac{1}{2}\Delta M_L^{-1}\vec{m}_L\vec{m}_L^T\Delta M_L^{-1}\right)\vec{\lambda}_L\\
&= \frac{Z^{(L,\mathrm{C})}_{1|1;0|0}[\alpha](\kappa,\lambda)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha](\kappa-\lambda)}\\
&\quad+(f(\kappa)-\vec{m}_L^T\Delta M_L^{-1}\Delta\vec{F}_L(\kappa))\vec{\lambda}_L^T\Delta M_L^{-1}\vec{m},\\
\frac{Z^{(L-1,\mathrm{C})}_{0|1;0|1}[\alpha](\lambda_1;\lambda_2)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]} &= -\vec{\lambda}_{L,1}^T\Delta M^{-1}\vec{\lambda}_{L,2}+\frac{1}{2}\vec{m}_L^T\Delta M_L^{-1}\vec{\lambda}_{L,1}\vec{m}_L^T\Delta M_L^{-1}\vec{\lambda}_{L,2}\\
&= -\frac{Z^{(L-1,\mathrm{C})}_{0|1;0|1}[\alpha](\lambda_2;\lambda_1)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]}+\vec{m}_L^T\Delta M_L^{-1}\vec{\lambda}_{L,1}\vec{m}_L^T\Delta M_L^{-1}\vec{\lambda}_{L,2}.
\end{aligned}\tag{B.7}$$

These relations prove the proposition for the case L even and $k+l$ even. The reason for this is that we can define the vector

$$\vec{v}=\begin{bmatrix}\left\{f(\kappa_a)-\vec{m}_L^T\Delta M_L^{-1}\Delta\vec{F}_L(\kappa_j)\right\}_{1\le j\le k}\\ \left\{\vec{m}_L^T\Delta M_L^{-1}\vec{\lambda}_{L,j}\right\}_{1\le j\le l}\end{bmatrix}\tag{B.8}$$

such that the partition function is

$$Z^{(L-l+k,\mathrm{C})}_{k|l;k|l}[\alpha](\kappa,\lambda;\kappa,\lambda)=\frac{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]}{\mathrm{B}^2_{k|l}(\kappa;\lambda)}\det\left(K+\frac{1}{2}\vec{v}\vec{v}^T\right)=\frac{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]}{\mathrm{B}^2_{k|l}(\kappa;\lambda)}\det K,\tag{B.9}$$

where K is the remaining antisymmetric matrix in the determinant (3.15). We extent this result to L even and $k+l$ odd by introducing an auxiliary variable λ_{l+1} in the partition function

$$Z^{(L-l+k,\mathrm{C})}_{k|l;k|l}[\alpha](\kappa,\lambda;\kappa,\lambda)=\lim_{\lambda_{l+1}\to\infty}\frac{Z^{(L-l+k,\mathrm{C})}_{k|l+1;k|l+1}[\alpha](\kappa,\lambda;\kappa,\lambda)}{\lambda_{l+1}^{2(L-l+k)}}.\tag{B.10}$$

Let us come to the other case where $L\in 2\mathbb{N}_0+1$ odd. Then the antisymmetric matrix ΔM_L is not invertible anymore. Therefore we extend the determinants (B.5)

by the vector $\vec{e}_{L+1}$,

$$
\begin{aligned}
& Z^{(L+1,\mathrm{C})}_{1|0;1|0}[\alpha](\kappa_1;\kappa_2) \\
= \; & -\det\left[\begin{array}{c|c|c} \Delta M_{L+1} + \dfrac{\vec{m}_{L+1}\vec{m}_{L+1}^T}{2} & \Delta\vec{F}_{L+1}(\kappa_1) + \dfrac{f(\kappa_1)\vec{m}_{L+1}}{2} & \vec{e}_{L+1} \\ \hline -\Delta\vec{F}_{L+1}^T(\kappa_2) + \dfrac{f(\kappa_2)\vec{m}_{L+1}^T}{2} & \Delta\widehat{G}(\kappa_1,\kappa_2) + \dfrac{f(\kappa_1)f(\kappa_2)}{2} & 0 \\ \hline \vec{e}_{L+1}^T & 0 & 0 \end{array}\right], \\
& \frac{Z^{(L,\mathrm{C})}_{0|0;1|1}[\alpha](\kappa,\lambda)}{\kappa-\lambda} \\
= \; & -\det\left[\begin{array}{c|c|c} \Delta M_{L+1} + \dfrac{\vec{m}_{L+1}\vec{m}_{L+1}^T}{2} & \vec{\lambda}_{L+1} & \vec{e}_{L+1} \\ \hline -\Delta\vec{F}_{L+1}^T(\kappa) + \dfrac{f(\kappa)\vec{m}_{L+1}^T}{2} & \dfrac{1}{\kappa-\lambda} & 0 \\ \hline \vec{e}_{L+1}^T & 0 & 0 \end{array}\right], \\
& \frac{Z^{(L,\mathrm{C})}_{1|1;0|0}[\alpha](\kappa,\lambda)}{\kappa-\lambda} \\
= \; & -\det\left[\begin{array}{c|c|c} \Delta M_{L+1} + \dfrac{\vec{m}_{L+1}\vec{m}_{L+1}^T}{2} & \Delta\vec{F}_{L+1}(\kappa) + \dfrac{f(\kappa)\vec{m}_{L+1}}{2} & \vec{e}_{L+1} \\ \hline \vec{\lambda}_{L+1}^T & \dfrac{1}{\kappa-\lambda} & 0 \\ \hline \vec{e}_{L+1}^T & 0 & 0 \end{array}\right], \\
& Z^{(L-1,\mathrm{C})}_{0|1;0|1}[\alpha](\lambda_1;\lambda_2) \\
= \; & -\det\left[\begin{array}{c|c|c} \Delta M_{L+1} + \dfrac{\vec{m}_{L+1}\vec{m}_{L}^T}{2} & \vec{\lambda}_{L+1,2} & \vec{e}_{L+1} \\ \hline \vec{\lambda}_{L+1,1}^T & 0 & 0 \\ \hline \vec{e}_{L+1}^T & 0 & 0 \end{array}\right]. \qquad \text{(B.11)}
\end{aligned}
$$

Now the antisymmetric matrix ΔM_{L+1} is invertible and we can derive relations analogous to those in Eq. (B.7),

$$
\begin{aligned}
\frac{Z^{(L+1,\mathrm{C})}_{1|0;1|0}[\alpha](\kappa_1;\kappa_2)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]} &= -\frac{Z^{(L+1,\mathrm{C})}_{1|0;1|0}[\alpha](\kappa_2;\kappa_1)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]} \\
&\quad +2\vec{e}_{L+1}^T\Delta M_{L+1}^{-1}\Delta\vec{F}_{L+1}(\kappa_1)\vec{e}_{L+1}^T\Delta M_{L+1}^{-1}\Delta\vec{F}_{L+1}(\kappa_2) \\
\frac{Z^{(L,\mathrm{C})}_{0|0;1|1}[\alpha](\kappa,\lambda)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha](\kappa-\lambda)} &= \frac{Z^{(L,\mathrm{C})}_{1|1;0|0}[\alpha](\kappa,\lambda)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha](\kappa-\lambda)} \\
&\quad +2\vec{e}_{L+1}^T\Delta M_{L+1}^{-1}\Delta\vec{F}_{L+1}(\kappa)\vec{e}_{L+1}\Delta M_{L+1}^{-1}\vec{\lambda}_{L+1}, \\
\frac{Z^{(L-1,\mathrm{C})}_{0|1;0|1}[\alpha](\lambda_1;\lambda_2)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]} &= -\frac{Z^{(L-1,\mathrm{C})}_{0|1;0|1}[\alpha](\lambda_2;\lambda_1)}{Z^{(L,\mathrm{C})}_{0|0;0|0}[\alpha]} \\
&\quad +2\vec{e}_{L+1}\Delta M_{L+1}^{-1}\vec{\lambda}_{L+1,1}\vec{e}_{L+1}\Delta M_{L+1}^{-1}\vec{\lambda}_{L+1,2}. \qquad \text{(B.12)}
\end{aligned}
$$

Hence we have to define a new vector

$$\vec{v} = \begin{bmatrix} \left\{ \vec{e}_{L+1}^{T} \Delta M_{L+1}^{-1} \Delta \vec{F}_{L+1}(\kappa_j) \right\}_{1 \le j \le k} \\ \left\{ \vec{e}_{L+1} \Delta M_{L+1}^{-1} \vec{\lambda}_{L+1,j} \right\}_{1 \le j \le l} \end{bmatrix} \tag{B.13}$$

with which one can do a calculation similar to Eq. (B.9) and so proving the proposition for the case L odd and $k+l$ even. The case for $k+l$ odd can be performed exactly in the same way as for L even which finishes the proof.

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Supersymmetry for Products of Random Matrices *

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We consider the singular value statistics of products of independent random matrices. In particular we compute the corresponding averages of products of characteristic polynomials. To this aim we apply the projection formula recently introduced for chiral random matrix ensembles which serves as a short cut of the supersymmetry method. The projection formula enables us to study the local statistics where free probability currently fails. To illustrate the projection formula and underline the power of our approach we calculate the hard edge scaling limit of the Meijer G-ensembles comprising the Wishart-Laguerre (chiral Gaussian), the Jacobi (truncated orthogonal, unitary or unitary symplectic) and the Cauchy-Lorentz (heavy tail) random matrix ensembles. All calculations are done for real, complex, and quaternion matrices in a unifying way. In the case of real and quaternion matrices the results are completely new and point to the universality of the hard edge scaling limit for a product of these matrices, too. Moreover we identify the non-linear σ-models corresponding to product matrices.

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1. Introduction

Sums and products of random matrices are the simplest generalization of random matrix theory (RMT) to introduce some kind of dimension. Sums of random matrices can be understood as a convolution and regularly appear in the field of Dyson's Brownian motion [1]. Product matrices are versatile as well. Applications of them can be found in mesoscopic physics [2, 3], QCD [4], and wireless telecommunication [5, 6]. In the past years a lot of progress was made on products of random matrices, see the new review [7] reporting on this progress. For example, free probability has proven as an efficient tool for calculating the macroscopic level density [9]. With the help of orthogonal polynomials one could calculate algebraic structures like determinants and Pfaffians, their kernels, and certain universal statistics on the local scale of the spectrum [6, 10, 11, 12, 13]. In particular products of random matrices drawn from Meijer G-ensembles

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(the weight is essentially given by Meijer G-functions, see [14] for a definition of these functions) exhibit a new kind of universal kernel in the hard scaling limit (microscopic limit around the origin). This limit is called Meijer G-kernel. Its name is reminiscent to the fact that the kernel essentially depends on Meijer G-functions. The "standard candles" of RMT, the Wishart-Laguerre ensemble [15] (χGβE), the Cauchy-Lorentz ensemble [16] ($\mathcal{L}\beta$E), and the Jacobi ensemble [17] (JβE) are particular cases of Meijer G-ensembles. Also products of matrices drawn from these three ensembles are Meijer G-ensembles since this class of ensembles is expected to be closed under matrix products.

Most results on the singular value statistics about product matrices are known for complex matrices ($\beta = 2$), only. The only exception, the macroscopic level density, can be computed for real ($\beta = 1$) and quaternion ($\beta = 4$) matrices with free probability [9] because they share the level density with $\beta = 2$. However the local statistics of the singular values is still highly involved for $\beta = 1, 4$ due to unknown group integrals like the Itzykson-Zuber integral [18] and its polynomial counterpart [19, 13]. The projection formula recently proposed [20] circumvents such problems. This formula is a short cut of the supersymmetry method [21, 22] and directly relates the original probability density with the weight in the dual superspace.

After introducing the required notation in Sec. 2 we briefly review the projection formula in Sec. 3. Thereby we only consider the average of a product of characteristic polynomials to keep the calculation simple. We emphasize that the projection formula holds for all three Dyson indices $\beta = 1, 2, 4$ which is the strength of this approach.

In Sec. 4, we demonstrate via the three ensembles, χGβE, $\mathcal{L}\beta$E, and JβE, how the projection formula works. Thereby we explicitly compute the well-known orthogonal polynomials for $\beta = 2$ and show that the average of one characteristic polynomial for $\beta = 1$ and the square root of a characteristic polynomial for $\beta = 4$ is apart from some shifts in the parameters the same as in the case $\beta = 2$. Another example is presented in Sec. 5 where we generalize the approach to a product of independently distributed matrices. Also for product matrices we explicitly calculate the orthogonal polynomials in the case $\beta = 2$. However the completely new results are the ones for $\beta = 1, 4$ which are expressed in terms of integrals over Dyson's circular ensembles (CβE) [23]. In this way we show in Sec. 6 that the universality in the hard edge scaling limit holds for real and quaternion product matrices, too. We are also able to identify the non-linear σ-models which are necessary when comparing the universal results with physical field theories.

2. Preliminaries

We consider rectangular random matrices which are either real ($\beta = 1$), complex ($\beta = 2$), or quaternion ($\beta = 4$). We are particularly interested in

the singular value statistics of a random matrix

$$W \in \mathrm{gl}^{(\beta)}(n; n+\nu) = \begin{cases} \mathbb{R}^{n\times(n+\nu)}, & \beta = 1, \\ \mathbb{C}^{n\times(n+\nu)}, & \beta = 2, \\ \mathbb{H}^{n\times(n+\nu)}, & \beta = 4 \end{cases} \tag{1}$$

distributed by $P(WW^\dagger)$. We assume $\nu = 0$ in the following to keep the computations simple such that we choose the abbreviation $\mathrm{gl}^{(\beta)}(n) = \mathrm{gl}^{(\beta)}(n; n)$. Nonetheless this restriction is not that strong since a product of rectangular matrices can be always rephrased to a product of square matrices [12]. Examples of such induced measures resulting from rectangular matrices are given in Sec. 5.

Since we choose the complex representation of the quaternion numbers $\mathbb{H}$ in terms of Pauli matrices we introduce the convenient parameters

$$\widetilde{\beta} = \frac{4}{\beta},\ \gamma = \begin{cases} 1, & \beta = 1, 2, \\ 2, & \beta = 4, \end{cases} \quad \widetilde{\gamma} = \begin{cases} 2, & \beta = 1, \\ 1, & \beta = 2, 4. \end{cases} \tag{2}$$

For the sake of readability we restrict ourselves to partition functions of the form

$$Z(M) = \int d[W] P(WW^\dagger) \det{}^{1/(\gamma\widetilde{\gamma})}(WW^\dagger \otimes \mathbf{1}_{\widetilde{\gamma}k} - M). \tag{3}$$

The fixed matrix $M = \{M_{ab,ij}\}$ has the dimension $(\gamma n \times \gamma n) \otimes (\widetilde{\gamma}k \times \widetilde{\gamma}k) = \gamma\widetilde{\gamma}nk \times \gamma\widetilde{\gamma}nk$. It has to satisfy the symmetry

$$M^T = \begin{cases} \mathbf{1}_n \otimes [\tau_2 \otimes \mathbf{1}_k]\, M\, \mathbf{1}_n \otimes [\tau_2 \otimes \mathbf{1}_k], & \beta = 1, \\ [\tau_2 \otimes \mathbf{1}_n] \otimes \mathbf{1}_k\, M\, [\tau_2 \otimes \mathbf{1}_n] \otimes \mathbf{1}_k, & \beta = 4, \end{cases} \tag{4}$$

where τ_2 is the second Pauli matrix. Other properties of M are not required.

The partition function (3) needs an explanation. The determinant acts on the tensor space of $(\gamma n \times \gamma n)$ matrices containing the matrix $WW^\dagger$ and a space of dimension $(\widetilde{\gamma}k \times \widetilde{\gamma}k)$. In the case that $M = \mathbf{1}_{\gamma n} \otimes \mathrm{diag}\,(m_1, \ldots, m_{\widetilde{\gamma}k})$ the determinant is a short hand notation for a product of characteristic polynomials of $WW^\dagger$ which is a well-known partition function in random matrix theory [8]. The reason why we wrote this product in such an uncommon, compact form is the application we aim at, namely the singular value statistics of matrix products. Then the matrix M does not take such a simple form.

Another particularity of Eq. (3) which needs an explanation is the exponent of the determinant, $-1/(\gamma\widetilde{\gamma})$ and the matrix dimensions. In the case of complex matrices ($\beta = 2$), the exponent and the dimensions become self-explanatory since they become trivial, e.g. $-1/(\gamma\widetilde{\gamma})|_{\beta=2} = -1$. When W is real ($\beta = 1$) then $WW^\dagger$ is real symmetric and $n \times n$ dimensional. The space dual to the polynomials consists of self-dual matrices. The resulting Kramers degeneracy cancels the exponent $1/2$ and doubles the dimension, $k \to 2k$. Exactly the opposite happens in the case of a

quaternion matrix W ($\beta = 4$). Due to its quaternion structure the dimension is doubled, $n \to 2n$. However the dual space consists of symmetric matrices. Since symmetric matrices may have also odd dimensions we do not need a doubling of the dimension k. The corresponding square roots of the characteristic polynomials are exact and, thus, a polynomial because the spectrum of $WW^\dagger$ is Kramers degenerate. Such a square root is known as quaternion determinant and is equivalent to a Pfaffian determinant [24].

An important ingredient needed for the supersymmetry method is the invariance of the probability density P under the transformation $P(WW^\dagger) = P(UWW^\dagger U^\dagger)$ for all $U \in \mathrm{U}^{(\beta)}(n)$ where

$$\mathrm{U}^{(\beta)}(n) = \begin{cases} \mathrm{O}(n), & \beta = 1, \\ \mathrm{U}(n), & \beta = 2, \\ \mathrm{USp}(2n), & \beta = 4. \end{cases} \tag{5}$$

Only due to this invariance it is possible to find an integral over a supermatrix whose dimension is independent of the ordinary dimension n and which yields exactly the same partition function as Eq. (3). This can be achieved in four steps which we briefly sketch in section 3.

For this purpose we have to introduce two supermatrix spaces and one ordinary matrix space. Let $p, q, N \in \mathbb{N}$, and $\mathrm{U}(p|q)$ and $\mathrm{UOSp}(p|2q)$ be the unitary and the unitary ortho-symplectic supergroup, respectively, see [25, 26, 27]. The space of rectangular supermatrices is defined by

$$\mathrm{gl}^{(\beta)}(p|q;p'|q') = \mathrm{u}^{(\beta)}(p+p'|q+q')/[\mathrm{u}^{(\beta)}(p|q) \times \mathrm{u}^{(\beta)}(p'|q')], \tag{6}$$

where $\mathrm{u}^{(\beta)}(p|q)$ is the Lie superalgebra of the supergroup

$$\mathrm{U}^{(\beta)}(p|q) = \begin{cases} \mathrm{UOSp}^{(+)}(p|2q), & \beta = 1, \\ \mathrm{U}(p|q), & \beta = 2, \\ \mathrm{UOSp}^{(-)}(2p|q), & \beta = 4. \end{cases} \tag{7}$$

The coset is taken via the addition as a group action on the Lie superalgebra. Therefore a matrix $\rho \in \mathrm{gl}^{(\beta)}(p|q;N)$ is $(\gamma p|\widetilde{\gamma} q) \times (\gamma p|\widetilde{\gamma} q)$ dimensional and has the following form

$$\rho = \begin{bmatrix} \rho_{\mathrm{BB}} & \rho_{\mathrm{BF}} \\ \rho_{\mathrm{FB}} & \rho_{\mathrm{FF}} \end{bmatrix}. \tag{8}$$

The $\gamma p \times \gamma p$ dimensional boson-boson block ρ_{BB} and the $\widetilde{\gamma} q \times \widetilde{\gamma} q$ dimensional fermion-fermion block ρ_{FF} comprise commuting variables while the other two block contain anti-commuting ones.

We employ the same notation for the two inequivalent fundamental representations of the supergroup $\mathrm{UOSp}(p|2q)$ as in [26, 27] where the superscripts indicate the transformation property under the complex conjugation, i.e.

$$\rho^* = \begin{cases} \mathrm{diag}\,(\mathbf{1}_p, -\imath\tau_2 \otimes \mathbf{1}_q)\, \rho\, \mathrm{diag}\,(\mathbf{1}_{p'}, \imath\tau_2 \otimes \mathbf{1}_{q'}), & \beta = 1, \\ \mathrm{diag}\,(-\imath\tau_2 \otimes \mathbf{1}_p, \mathbf{1}_q)\, \rho\, \mathrm{diag}\,(\imath\tau_2 \otimes \mathbf{1}_{p'}, \mathbf{1}_{q'}), & \beta = 4 \end{cases} \tag{9}$$

for $\rho \in \mathrm{gl}^{(\beta)}(p|q;p'|q')$ and

$$U^* = \begin{cases} \mathrm{diag}\,(\mathbf{1}_p, -\imath\tau_2 \otimes \mathbf{1}_q)\, U\, \mathrm{diag}\,(\mathbf{1}_p, \imath\tau_2 \otimes \mathbf{1}_q), & \beta = 1, \\ \mathrm{diag}\,(-\imath\tau_2 \otimes \mathbf{1}_p, \mathbf{1}_q)\, U\, \mathrm{diag}\,(\imath\tau_2 \otimes \mathbf{1}_p, \mathbf{1}_q), & \beta = 4 \end{cases} \tag{10}$$

for $U \in \mathrm{U}^{(\beta)}(p|q) \subset \mathrm{U}(\gamma p|\widetilde{\gamma} q)$. The two relations (9) and (10) are generalization of the definitions of real and quaternion matrices to superspace.

The ordinary matrix space announced is the coset

$$\mathrm{C}\beta\mathrm{E}(\gamma k) = \begin{cases} \mathrm{U}(k)/\mathrm{O}(k), & \beta = 1, \\ [\mathrm{U}(k) \times \mathrm{U}(k)]/\mathrm{U}(k) \simeq \mathrm{U}(k), & \beta = 2, \\ \mathrm{U}(2k)/\mathrm{USp}(2k), & \beta = 4 \end{cases} \tag{11}$$

equipped with a normalized Haar measure $d\mu(U)$ induced by the Haar measures on the defining groups. These three sets are the circular ensembles first studied by Dyson [23]. These co-sets are also the fermionic part of the supermatrices involved in the superbosonization formula [22]. Since we only discuss the average of products of determinants and not ratios superbosonization reduces to bosonization only involving the circular ensembles (11). Let us recall the properties of a matrix $U \in \mathrm{C}\beta\mathrm{E}(\gamma k)$. The matrix U is unitary and satisfies the symmetries $U^T = U$ for $\beta = 1$ and $U^T = (\tau_2 \otimes \mathbf{1}_k) U (\tau_2 \otimes \mathbf{1}_k)$ for $\beta = 4$.

Also the superdeterminant and the supertrace play an important role in the ensuing calculations. They are defined via the ordinary determinant and trace and explicitly read

$$\mathrm{Sdet}\,\rho = \frac{\det(\rho_{\mathrm{BB}} - \rho_{\mathrm{BF}}\rho_{\mathrm{FF}}^{-1}\rho_{\mathrm{FB}})}{\det \rho_{\mathrm{FF}}}, \ \mathrm{Str}\,\rho = \mathrm{tr}\,\rho_{\mathrm{BB}} - \mathrm{tr}\,\rho_{\mathrm{FF}} \tag{12}$$

for an arbitrary square supermatrix $\rho \in \mathrm{gl}^{(\beta)}(p|q;p|q)$ whose fermion-fermion block ρ_{FF} is invertible. The definitions are chosen in such a way that many properties of the trace and the determinant carry over to superspace. For example the circularity $\mathrm{Str}\,AB = \mathrm{Str}\,BA$, the factorization $\mathrm{Sdet}\,AB = \mathrm{Sdet}\,A\,\mathrm{Sdet}\,B$, and the relation $\ln \mathrm{Sdet}\,A = \mathrm{Str}\,\ln A$ still hold for two arbitrary invertible square supermatrices A and B. The circularity property of the supertrace works for rectangular supermatrices, as well. A more profound introduction in supersymmetric analysis and algebra can be found in [28].

3. What is the Projection Formula?

The projection formula in its general form projects functions living on a very large superspace to functions on a much smaller superspace [20]. In this way it directly relates the original weight P to a weight Q in the smaller superspace. Hence the projection formula is a short cut of the supersymmetry method [20]. For our particular purposes the large superspace is $\mathrm{gl}^{(\beta)}(n + \widetilde{\gamma} l|\gamma l; n|0)$ with l being an integer larger than or equal to k/γ. The enlargement of the dimensions $k \to 2l$ in the case k odd and $\beta = 4$

is crucial. The reason is a Cauchy-like integration theorem [29, 27] first derived in a general framework by Wegner [30] which only applies to an even dimensional reduction of a matrix space in the case of $\beta = 1, 4$.

In the first step of deriving the projection formula we need the following version of this Cauchy-like theorem [20]

$$P(WW^\dagger) = \frac{\int d[\widehat{\Omega}] P(\Omega\Omega^\dagger)}{\int d[\widehat{\Omega}] \exp[-\mathrm{Str}\, \widehat{\Omega}\widehat{\Omega}^\dagger]} \tag{13}$$

with $W \in \mathrm{gl}^{(\beta)}(n) = \mathrm{gl}^{(\beta)}(n|0; n|0)$ and $\widehat{\Omega} \in \mathrm{gl}^{(\beta)}(\widetilde{\gamma}l|\gamma l; n|0)$. The matrices are embedded as follows

$$\Omega = \begin{bmatrix} W \\ \widehat{\Omega} \end{bmatrix} = \begin{bmatrix} W' \\ \Omega' \end{bmatrix} \in \mathrm{gl}^{(\beta)}(n + \widetilde{\gamma}l|\gamma l; n). \tag{14}$$

The second splitting in $W' \in \mathrm{gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l-k; n|0)$ and $\Omega' \in \mathrm{gl}^{(\beta)}(0|k; n|0)$ becomes relevant in the third step of the derivation of the projection formula. The measure $d[\widehat{\Omega}]$ is the product of all differentials of independent matrix entries of $\widehat{\Omega}$. The normalization with a Gaussian is true because the proportionality constant is independent of P and thus can be fixed by any weight.

In Eq. (13) we have chosen a supersymmetric extension of P to the superspace $\mathrm{gl}^{(\beta)}(n + \widetilde{\gamma}l|\gamma l; n|0)$ which is by far not unique. However the final result is independent of this choice as already discussed in [12]. Such an extension indeed exists for a smooth distribution P. Since P is invariant under the group $\mathrm{U}^{(\beta)}(n)$ we can apply the Cayley-Hamilton theorem implying that P can be expressed in matrix invariants like traces and determinants of $WW^\dagger$. Those invariants have invariant extensions, namely the supertrace and the superdeterminant, cf. Eq. (12).

In the next step we rewrite the determinant in Eq. (3) as a Gaussian integral over a matrix $V = \{V_{aj}\} \in \mathrm{gl}^{(\beta)}(0|k; n|0)$ which only consists of Grassmann (anti-commuting) variables [28],

$$\begin{aligned} &\det{}^{1/(\gamma\widetilde{\gamma})}(WW^\dagger \otimes \mathbf{1}_{\widetilde{\gamma}k} - M) \\ &= \frac{\int d[V] \exp[\mathrm{tr}\, VWW^\dagger V^\dagger - \sum_{a,b=1}^{\widetilde{\gamma}k} \sum_{i,j=1}^{\gamma n} M_{ab,ij} V_{ai} V_{bj}^*]}{\int d[V] \exp[\mathrm{tr}\, VV^\dagger]}. \end{aligned} \tag{15}$$

Then the partition function is up to a constant

$$Z(M) \propto \int d[\Omega] d[V] P(\Omega\Omega^\dagger) \exp\left[-\mathrm{Str}\, \Omega\Omega^\dagger \widehat{V}^\dagger \widehat{V} - \sum_{a,b=1}^{\widetilde{\gamma}k} \sum_{i,j=1}^{\gamma n} M_{ab,ij} V_{ai} V_{bj}^*\right] \tag{16}$$

with

$$\widehat{V} = \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}, \ \widehat{V}^\dagger = \begin{bmatrix} 0 & V^\dagger \\ 0 & 0 \end{bmatrix} \in \mathrm{gl}^{(\beta)}(n + \widetilde{\gamma}l|\gamma l; n + \widetilde{\gamma}l|\gamma l). \tag{17}$$

The first $(\gamma n + 2\gamma\widetilde{\gamma}l - \widetilde{\gamma}k)$ rows and the last $2\gamma\widetilde{\gamma}l$ columns of $\widehat{V}$ are equal to 0. The change of the sign in front of the first term in the exponential function relates to the fact that Grassmann variables are anti-commuting.

The integrals over V and Ω can be interchanged such that we find the function

$$\widehat{P}(\widehat{V}^\dagger\widehat{V}) = \int d[\Omega]P(\Omega\Omega^\dagger)\exp[-\mathrm{Str}\,\Omega\Omega^\dagger\widehat{V}^\dagger\widehat{V}]. \tag{18}$$

The invariance of $P(\Omega\Omega^\dagger) = P(U\Omega\Omega^\dagger U^\dagger)$ for all $U \in \mathrm{U}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l)$ carries over to a symmetry for $\widehat{P}(\widehat{V}^\dagger\widehat{V}) = \widehat{P}(U\widehat{V}^\dagger\widehat{V}U^\dagger)$ for all $U \in \mathrm{U}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l)$. Therefore the following duality holds

$$\widehat{P}(\widehat{V}^\dagger\widehat{V}) = \widehat{P}(\widehat{V}\widehat{V}^\dagger) \tag{19}$$

which is the third important step of the derivation. Employing the definition (18) backwards the partition function is

$$\begin{aligned} Z(M) \;&\propto\; \int d[\Omega]d[V]P(\Omega\Omega^\dagger)\exp\left[\mathrm{tr}\,V^\dagger\Omega'\Omega'^\dagger V - \sum_{a,b=1}^{\widetilde{\gamma}k}\sum_{i,j=1}^{\gamma n} M_{ab,ij}V_{ai}V_{bj}^*\right] \\ &\propto\; \int d[\Omega']Q(\Omega'\Omega'^\dagger)\mathrm{det}^{1/(\gamma\widetilde{\gamma})}(\mathbf{1}_{\gamma n}\otimes\Omega'\Omega'^\dagger - M). \end{aligned} \tag{20}$$

In the last step we integrated over the remaining degrees of freedom W', cf. the splitting (14), which do not show up in the determinant. This integration yields the function

$$Q(\Omega'\Omega'^\dagger) \propto \int d[W']P\left(\begin{bmatrix} W'W'^\dagger & W'\Omega'^\dagger \\ \Omega'W'^\dagger & \Omega'\Omega'^\dagger \end{bmatrix}\right). \tag{21}$$

This equation is the essence of the projection formula. The remaining things to do is cosmetics.

We want to express the dyadic matrix $\Omega'\Omega'^\dagger$ as a single square matrix U which is an element in $\mathrm{C}\widetilde{\beta}\mathrm{E}(\gamma k)$. Note that the circular ensemble really relates to the Dyson index $\widetilde{\beta} = 4/\beta$ and not β which originates from the symmetries fulfilled by V.

Exactly this is done in the last step. We apply the superbosonization formula [22] which reduces to pure bosonization in our case. This yields the partition function

$$Z(M) = \int d\mu(U)Q(U)\mathrm{det}^{1/(\gamma\widetilde{\gamma})}(\mathbf{1}_{\gamma n}\otimes U - M)\mathrm{det}^{-n/\widetilde{\gamma}}U. \tag{22}$$

with the normalized distribution

$$Q(U) = \frac{\int d[W_1]d[W_2]P\left(\begin{bmatrix} W_1W_1^\dagger + W_2W_2^\dagger & W_2U^{1/2} \\ U^{1/2}W_2^\dagger & U \end{bmatrix}\right)}{\int d\mu(U)d[W_1]d[W_2]\,\mathrm{det}^{-n/\widetilde{\gamma}}U\exp[-\mathrm{Str}\,(W_1W_1^\dagger + W_2W_2^\dagger) + \mathrm{tr}\,U]}. \tag{23}$$

The reduction of the integral (21) to the final expression (23) as an integral over the two matrices $W_1 \in \mathrm{gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l-k; n+\widetilde{\gamma}l|\gamma l-k)$ and $W_2 \in \mathrm{gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l-k; 0|k)$ was done in [27] and is skipped here due to the lack of space.

We remark that apart from the case k odd and $\beta = 4$ the auxiliary parameter l can be chosen $l = k/\gamma$. Then the matrix W_1 is an ordinary square matrix and W_2 is a rectangular matrix only consisting of Grassmann variables.

4. Application to Standard Random Matrix Ensembles

Three particular cases of Meijer G-ensembles are the Gaussian χGβE, the heavy-tailed $\mathcal{L}\beta$E, and the compactly supported JβE. We discuss them in subsections 4.1, 4.2, and 4.3, respectively. These ensembles play important roles in a vast of applications and cover a broad range of systems [15, 10, 11, 8, 17, 16].

4.1. Wishart-Laguerre (Gaussian) Ensemble

The first ensemble we consider is the χGβE,

$$P_{\mathrm{WL}}(WW^\dagger) \propto \det{}^{\nu/\widetilde{\gamma}} WW^\dagger \exp[-\mathrm{tr}\, WW^\dagger/\Gamma^2] \tag{24}$$

with $\nu \in \mathbb{N}_0$ and $\Gamma > 0$. It is the oldest random matrix ensemble first studied by Wishart [15]. The determinant in front of the Gaussian originates from a transformation of a rectangular matrix $W' \in \mathrm{gl}^{(\beta)}(n, n+\nu)$ to the square matrix $W \in \mathrm{gl}^{(\beta)}(n)$. Therefore one can understand Eq. (24) as an induced measure [12]. The corresponding weight Q_{WL} is given by Eq. (23),

$$\begin{aligned} Q_{\mathrm{WL}}(U) \;&\propto\; \int d[W_1]d[W_2]\mathrm{Sdet}^{\,\nu/\widetilde{\gamma}} \begin{bmatrix} W_1W_1^\dagger + W_2W_2^\dagger & W_2U^{1/2} \\ U^{1/2}W_2^\dagger & U \end{bmatrix} \\ &\quad \times \exp[-\mathrm{Str}\,(W_1W_1^\dagger + W_2W_2^\dagger) + \mathrm{tr}\, U/\Gamma^2] \\ &\propto\; \det{}^{-\nu/\widetilde{\gamma}} U e^{\mathrm{tr}\, U/\Gamma^2}. \end{aligned} \tag{25}$$

Therefore the partition function (3) for $P_{\mathrm{WL}}(WW^\dagger)$ reads

$$Z_{\mathrm{WL}}(M) = \frac{\int d\mu(U) \det{}^{-(n+\nu)/\widetilde{\gamma}} U \det{}^{1/(\gamma\widetilde{\gamma})}(\mathbf{1}_{\gamma n} \otimes U - M) e^{\mathrm{tr}\, U/\Gamma^2}}{\int d\mu(U) \det{}^{-(n+\nu)/\widetilde{\gamma}} U e^{\mathrm{tr}\, U/\Gamma^2}}. \tag{26}$$

This result agrees with the one derived in [27]. The normalization can be fixed by considering the expansion of the partition function for large M.

The result (26) exhibits nice implications. For example the case $k = \gamma$ and $M = m\mathbf{1}_{\gamma^2\widetilde{\gamma}n}$ is equal to the orthogonal polynomials for $\beta = 2$ and to the skew-orthogonal polynomials of even order for $\beta = 1, 4$, see [24]. Hence the contour for $\beta = 2$ is a representation of the modified Laguerre

polynomials $L_n^{(\nu)}$, see [14], i.e.

$$Z_{\mathrm{WL}}^{(\beta=2,k=1)}(m\mathbf{1}_n) \propto \oint dz z^{-(n+\nu+1)}(z-m)^n e^{z/\Gamma^2} \propto \sum_{j=0}^{n} \frac{1}{j!(n-j)!(\nu+j)!}\left(-\frac{m}{\Gamma^2}\right)^j \propto L_n^{(\nu)}\left(\frac{m}{\Gamma^2}\right). \quad (27)$$

These polynomials also appear for $\beta = 1, 4$ if we set $k = 1$. Only the argument m is modified to $\widetilde{\gamma}m$. Interestingly, the case $\beta = 4$ is an average over a square root of a determinant which is equivalent to a Pfaffian.

For the case $k = 2\gamma$ and $M = \mathbf{1}_{\gamma^2\widetilde{\gamma}n} \otimes \mathrm{diag}\,(m_1, m_2)$ we find one of the kernels corresponding to the χGβE [24]. When computing the contour integral (26) we immediately find the corresponding Christoffel-Darboux formulas.

4.2. Cauchy-Lorentz Ensemble

The $\mathcal{L}\beta$E is the next case we want to study. It is defined by the probability density [16, 20]

$$P_{\mathrm{CL}}(WW^\dagger) \propto \det{}^{\nu/\widetilde{\gamma}} WW^\dagger \det{}^{-\mu}(\Gamma^2\mathbf{1}_{\gamma n} + WW^\dagger) \quad (28)$$

with $\Gamma > 1$, $\nu \in \mathbb{N}_0$ and $\mu > k/\gamma + (2n+\nu)/\widetilde{\gamma} - (\gamma\widetilde{\gamma}-1)/2$ for guaranteeing the convergence of the integral (3)[1]. It is a heavy-tailed distribution and was employed for modelling financial correlations [16].

The choice $\Gamma > 1$ is convenient for the projection formula but is not a restriction at all because it only rescales the ensemble. The term $\det{}^{\nu/\widetilde{\gamma}} WW^\dagger$ can be again understood as a remnant of a rectangular matrix $W' \in \mathrm{gl}^{(\beta)}(n, n+\nu)$. However we underline that such a transformation from W' to W also changes the exponent μ.

The weight for the dual space is calculated by Eq. (23),

$$\begin{aligned} Q_{\mathrm{CL}}(U) &\propto \int d[W_1]d[W_2]\mathrm{Sdet}^{\,\nu/\widetilde{\gamma}} \begin{bmatrix} W_1W_1^\dagger + W_2W_2^\dagger & W_2U^{1/2} \\ U^{1/2}W_2^\dagger & U \end{bmatrix} \\ &\quad \times \mathrm{Sdet}^{\,-\mu} \begin{bmatrix} \Gamma^2\mathbf{1}_{\gamma n+\gamma\widetilde{\gamma}l|\gamma\widetilde{\gamma}l-\widetilde{\gamma}k} + W_1W_1^\dagger + W_2W_2^\dagger & W_2U^{1/2} \\ U^{1/2}W_2^\dagger & \Gamma^2\mathbf{1}_{\widetilde{\gamma}k} + U \end{bmatrix} \\ &\propto \det{}^{-\nu/\widetilde{\gamma}} U \det{}^{\mu}(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k} + U) \int d[W_1]d[W_2]\mathrm{Sdet}^{\,\nu/\widetilde{\gamma}} W_1W_1^\dagger \\ &\quad \times \mathrm{Sdet}^{\,-\mu}\left[\Gamma^2\mathbf{1}_{\gamma n+\gamma\widetilde{\gamma}l|\gamma\widetilde{\gamma}l-\widetilde{\gamma}k} + W_1W_1^\dagger + \Gamma^2W_2(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k} + U)^{-1}W_2^\dagger\right] \\ &\propto \det{}^{-\nu/\widetilde{\gamma}} U \det{}^{\mu-k/\gamma-n/\widetilde{\gamma}}(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k} + U) \end{aligned} \quad (29)$$

[1] Note that the inequality satisfied by μ in [20] contains a mistake which we have corrected here. The inequality can be found by performing a singular value decomposition of W and then reading off the algebraic behaviour at infinity.

In the last step we have rescaled $W_2 \to W_2(\Gamma^2 \mathbf{1}_{\widetilde{\gamma}k} + U)^{1/2}$ such that the remaining integrals are independent of U. Thereby we recall that the Berezinian (Jacobian in superspace) is $\mathrm{det}^{-k/\gamma - n/\widetilde{\gamma}}(\Gamma^2 \mathbf{1}_{\widetilde{\gamma}k} + U)$. Hence we end up with the partition function

$$Z_{\mathrm{CL}}(M) = \frac{\int d\mu(U) \mathrm{det}^{-\frac{n+\nu}{\widetilde{\gamma}}} U \mathrm{det}^{\mu - \frac{k}{\gamma} - \frac{n}{\widetilde{\gamma}}}(\Gamma^2 \mathbf{1}_{\widetilde{\gamma}k} + U) \mathrm{det}^{\frac{1}{\gamma\widetilde{\gamma}}}(\mathbf{1}_{\gamma n} \otimes U - M)}{\int d\mu(U) \mathrm{det}^{-(n+\nu)/\widetilde{\gamma}} U \mathrm{det}^{\mu - k/\gamma - n/\widetilde{\gamma}}(\Gamma^2 \mathbf{1}_{\widetilde{\gamma}k} + U)}. \tag{30}$$

Starting from this formula one can again easily deduce the orthogonal or skew-orthogonal polynomials, the kernel involving two characteristic polynomials, and the Christoffel-Darboux formula associated to this kernel. For example the orthogonal polynomials corresponding to the complex $\mathcal{L}\beta\mathrm{E} = \mathrm{LUE}$ is

$$\begin{aligned} Z_{\mathrm{CL}}^{(\beta=2,k=1)}(m\mathbf{1}_n) &\propto \oint dz z^{-(n+\nu+1)}(\Gamma^2 + z)^{\mu-n-1}(z-m)^n \\ &\propto \sum_{j=0}^{n} \frac{1}{j!(n-j)!(\nu+j)!\Gamma[\mu-n-\nu-j]} \left(-\frac{m}{\Gamma^2}\right)^j. \end{aligned} \tag{31}$$

This polynomial can be understood as a Jacobi polynomial when analytically continuing the parameters to negative values, cf. Eq. (37). The same polynomials pop up for $\beta = 1, 4$ when setting $k = 1$. This time we have only to change the exponent $\mu \to \widetilde{\gamma}\mu - \widetilde{\gamma}/\gamma + 1$.

4.3. Jacobi (Truncated Unitary) Ensemble

The JβE is defined by [17]

$$P_{\mathrm{J}}(WW^\dagger) \propto \mathrm{det}^{\nu/\widetilde{\gamma}} WW^\dagger \mathrm{det}^{\kappa}(\Gamma^2 \mathbf{1}_{\gamma n} - WW^\dagger)\Theta(\Gamma^2 \mathbf{1}_{\gamma n} - WW^\dagger), \tag{32}$$

where $\nu \in \mathbb{N}_0$, $\kappa > -1/(2\gamma)$. The Heaviside step function for matrices Θ is unity if the matrix is positive definite and otherwise vanishes. Again the scaling $\Gamma > 1$ is only introduced to avoid problems with the contour integrals in the dual space. In the case $\gamma\widetilde{\gamma}\mu \in \mathbb{N}_0$ the random matrix W distributed by Eq. (32) can be understood as a truncation of an orthogonal ($\beta = 1$), a unitary ($\beta = 2$), or a unitary symplectic ($\beta = 4$) matrix, respectively, see [17, 12].

To apply the projection formula we have first to find the supersymmetric generalization of the Heaviside step function. For this reason we write this function as $\Theta(\Gamma^2 \mathbf{1}_{\gamma n} - W^\dagger W)$. Then it is clear that this function reads in terms of the supermatrix Ω as $\Theta(\Gamma^2 \mathbf{1}_{\gamma n} - \Omega^\dagger \Omega)$ because the dyadic matrix $\Omega^\dagger\Omega$ has still an ordinary dimension and can be embedded in the space of $\gamma n \times \gamma n$ matrices by a Taylor expansion in the Grassmann valued matrix entries. Such a Taylor expansion is always finite since Grassmann variables are nilpotent. Hence we do not have to fear any problems of convergence.

Let $n, p, q \in \mathbb{N}$ and $V \in \mathrm{gl}^{(\beta)}(p|q; n|0)$. Then the extension of the Heaviside step function is done by a limit,

$$\begin{aligned}\Theta(\mathbf{1}_{\gamma n} - V^\dagger V) &= \lim_{\epsilon\to\infty} \mathrm{det}^{-1}(\mathbf{1}_{\gamma n} + e^{-\epsilon} e^{\epsilon V^\dagger V}) \\ &= \lim_{\epsilon\to\infty} \exp\left[\sum_{j=1}^{\infty} \frac{(-1)^j e^{-j\epsilon}}{j} \mathrm{tr}\, e^{j\epsilon V^\dagger V}\right]. \end{aligned} \tag{33}$$

This limit vanishes if one or more eigenvalues of the numerical part of the dyadic matrix $V^\dagger V$ is larger than 1. We emphasize that indeed only the numerical part matters and not the nilpotent terms because of the Taylor expansion in the latter. In the next step we employ the duality $\mathrm{tr}\, e^{j\epsilon V^\dagger V} = \gamma(n-p) + \widetilde{\gamma} q + \mathrm{Str}\, e^{j\epsilon V V^\dagger}$. We have

$$\begin{aligned}\Theta(\mathbf{1}_{\gamma n} - V^\dagger V) &= \lim_{\epsilon\to\infty} (1+e^{-\epsilon})^{\gamma(p-n)-\widetilde{\gamma} q} \mathrm{Sdet}^{-1}(\mathbf{1}_{\gamma p|\widetilde{\gamma} q} + e^{-\epsilon} e^{\epsilon V V^\dagger}) \\ &= \lim_{\epsilon\to\infty} \mathrm{det}^{-1}(\mathbf{1}_{\gamma p|\widetilde{\gamma} q} + e^{-\epsilon} \{e^{\epsilon V V^\dagger}\}_{\mathrm{BB}}) \\ &= \Theta(\mathbf{1}_{\gamma p} - \{V V^\dagger\}_{\mathrm{BB}}^{\mathrm{num}}). \end{aligned} \tag{34}$$

The Heaviside step function is only taken for the numerical part $\{VV^\dagger\}_{\mathrm{BB}}^{\mathrm{num}}$ of the boson-boson block of the dyadic matrix $VV^\dagger$. Any expansion in the nilpotent terms yields a polynomial in ϵ which are suppressed by the exponential $e^{-\epsilon}$. This implies that the other three blocks of the supermatrix $e^{\epsilon VV^\dagger}$ cannot contribute because they are polynomials in ϵ. The boson-boson block is $\{e^{\epsilon VV^\dagger}\}_{\mathrm{BB}} = e^{\epsilon\{VV^\dagger\}_{\mathrm{BB}}^{\mathrm{num}}}(1+f(\epsilon))$ with f a polynomial with $f(0)=0$. Therefore Eq. (34) is the correct generalization of the Heaviside step function to the superspace. Interestingly the Taylor expansion in the nilpotent terms have no influence on the Heaviside step function. But this behaviour has to be expected because the Taylor expansion can only have an effect on the boundary. Only there one or more eigenvalues of the numerical part $\{VV^\dagger\}_{\mathrm{BB}}^{\mathrm{num}}$ are equal to 1 where the value of the function may change. However, the supersymmetric Heaviside step function vanishes at the boundary, too, due to the expansion in the nilpotent terms yielding an inverted polynomial in ϵ, e.g. $\mathrm{det}^{-1}(\mathbf{1}_{\gamma p|\widetilde{\gamma} q} + e^{-\epsilon}\{e^{\epsilon VV^\dagger}\}_{\mathrm{BB}}) \overset{\{VV^\dagger\}_{\mathrm{BB}}\to\mathbf{1}_{\gamma p}}{\longrightarrow} 1/f(\epsilon) \overset{\epsilon\to\infty}{\to} 0$ with f a polynomial.

We employ Eq. (34) in our setting and recognize that the matrix U is not a part of the boson-boson block of the matrix argument of P_{J} in Eq. (23). Hence, the function in the dual space is

$$\begin{aligned} Q_{\mathrm{J}}(U) \;\propto\; & \int d[W_1] d[W_2] \mathrm{Sdet}^{\,\nu/\widetilde{\gamma}} \begin{bmatrix} W_1 W_1^\dagger + W_2 W_2^\dagger & W_2 U^{1/2} \\ U^{1/2} W_2^\dagger & U \end{bmatrix} \\ & \times \mathrm{Sdet}^{\,\kappa} \begin{bmatrix} \Gamma^2 \mathbf{1}_{\gamma n + \gamma\widetilde{\gamma} l | \gamma\widetilde{\gamma} l - \widetilde{\gamma} k} - W_1 W_1^\dagger - W_2 W_2^\dagger & W_2 U^{1/2} \\ U^{1/2} W_2^\dagger & \Gamma^2 \mathbf{1}_{\widetilde{\gamma} k} - U \end{bmatrix} \\ & \times \Theta(\Gamma^2 \mathbf{1}_{\gamma n + \gamma\widetilde{\gamma} l} - \{W_1 W_1^\dagger + W_2 W_2^\dagger\}_{\mathrm{BB}}^{\mathrm{num}}) \\ \propto\; & \mathrm{det}^{-\nu/\widetilde{\gamma}} U \mathrm{det}^{-\kappa - k/\gamma - n/\widetilde{\gamma}} (\Gamma^2 \mathbf{1}_{\widetilde{\gamma} k} - U). \end{aligned} \tag{35}$$

We underline that the boson-boson block of $W_2W_2^\dagger$ only consists of nilpotent parts such that it does not contribute to the Heaviside step function. The corresponding partition function is

$$Z_{\rm J}(M)=\frac{\int d\mu(U)\det^{-\frac{n+\nu}{\widetilde{\gamma}}}U\det^{-\kappa-\frac{k}{\gamma}-\frac{n}{\widetilde{\gamma}}}(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k}-U)\det^{\frac{1}{\gamma\widetilde{\gamma}}}(\mathbf{1}_{\gamma n}\otimes U-M)}{\int d\mu(U)\det^{-(n+\nu)/\widetilde{\gamma}}U\det^{-\kappa-k/\gamma-n/\widetilde{\gamma}}(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k}-U)}. \tag{36}$$

One can readily check the correctness of this result by calculating the orthogonal or skew-orthogonal polynomials and the kernel involving two characteristic polynomials. For example, with the help of the residue theorem we generate the polynomials,

$$\begin{aligned} Z_{\rm J}^{(\beta=2,k=1)}(m\mathbf{1}_n) &\propto \oint dz z^{-(n+\nu+1)}(\Gamma^2-z)^{-(n+\kappa+1)}(z-m)^n \\ &\propto \sum_{j=0}^{n}\frac{\Gamma[n+\kappa+\nu+j+1]}{j!(n-j)!(\nu+j)!}\left(-\frac{m}{\Gamma^2}\right)^j \propto P_n^{(\kappa,\nu)}\left(\frac{2m}{\Gamma^2}-1\right), \end{aligned} \tag{37}$$

where $P_n^{(\kappa,\nu)}$ are the Jacobi polynomials with respect to the weight $(1-x)^\kappa(1+x)^\nu\Theta(1-x^2)$, see [14]. As in the case of the $\mathcal{L}\beta$E we find the same polynomials for $\beta=1,4$ and $k=1$ when replacing the exponent $\kappa\to\widetilde{\gamma}\kappa+\widetilde{\gamma}/\gamma-1$.

We also obtain the well-known Christoffel-Darboux formula of the Jacobi polynomials by setting $k=2$, $\beta=2$, and $M=\mathbf{1}_n\otimes\mathrm{diag}\,(m_1,m_2)$. Then the integral reduces to a double contour integral after diagonalizing U.

5. Application to Product Matrices

The computation of the partition function for a product of L matrices $W\to W^{(L)}=\prod_{j=1}^L W_j=W_1\cdots W_L$ independently distributed by $P(WW^\dagger)\to\prod_{j=1}^L P_j(W_jW_j^\dagger)$ works in a similar way as for a single matrix. Starting from the partition function

$$\begin{aligned} Z_\Pi(M) &= \int\left(\prod_{j=1}^L d[W_j]P_j(W_jW_j^\dagger)\right)\det^{1/(\gamma\widetilde{\gamma})}\left[W^{(L)}\left(W^{(L)}\right)^\dagger\otimes\mathbf{1}_{\widetilde{\gamma}k}-M\right] \\ &= \int\left(\prod_{j=1}^L d[W_j]P_j(W_jW_j^\dagger)\right)\det{}^{k/\gamma}W^{(L-1)}\left(W^{(L-1)}\right)^\dagger \\ &\quad\times\det{}^{1/(\gamma\widetilde{\gamma})}\left[W_LW_L^\dagger\otimes\mathbf{1}_{\widetilde{\gamma}k}-X_{L-1}^{-1}MY_{L-1}^{-1}\right] \end{aligned} \tag{38}$$

with $X_{L-1}=W^{(L-1)}\otimes\mathbf{1}_{\widetilde{\gamma}k}$ and $Y_{L-1}=(W^{(L-1)})^\dagger\otimes\mathbf{1}_{\widetilde{\gamma}k}$, we apply the projection formula for W_L after replacing the matrix $M\to X_{L-1}^{-1}MY_{L-1}^{-1}$.

Then we obtain

$$
\begin{aligned}
Z_\Pi(M) \;=\; & \int \left(\prod_{j=1}^{L-1} d[W_j] P_j(W_j W_j^\dagger) \right) d\mu(U_L) Q_L(U_L) \det{}^{-n/\widetilde{\gamma}} U_L \\
& \times \det{}^{1/(\gamma\widetilde{\gamma})} \left[W^{(L-1)} \left(W^{(L-1)} \right)^\dagger \otimes U_L - M \right] \\
= & \int \left(\prod_{j=1}^{L} d[W_j] P_j(W_j W_j^\dagger) \right) d\mu(U_L) Q_L(U_L) \det{}^{k/\gamma} W^{(L-2)} \left(W^{(L-2)} \right)^\dagger \\
& \times \det{}^{\frac{1}{\gamma\widetilde{\gamma}}} \left[W_{L-1} W_{L-1}^\dagger \otimes \mathbf{1}_{\widetilde{\gamma}k} - X_{L-2}^{-1} M Y_{L-2}^{-1} \right] \qquad (39)
\end{aligned}
$$

where $X_{L-2} = W^{(L-2)} \otimes \sqrt{U_L}$, $Y_{L-2} = (W^{(L-2)})^\dagger \otimes \sqrt{U_L}$, and Q_L is computed as in the projection formula (23). This procedure yields a recursion resulting in the following expression for the partition function,

$$
\begin{aligned}
Z_\Pi(M) \;=\; & \int \left(\prod_{j=1}^{L} d\mu(U_j) Q_j(U_j) \right) \det{}^{-n/\widetilde{\gamma}} U_L \cdots U_1 \\
& \times \det{}^{1/(\gamma\widetilde{\gamma})} \left[\mathbf{1}_{\gamma n} \otimes \sqrt{U_L} \cdots \sqrt{U_2} U_1 \sqrt{U_2} \cdots \sqrt{U_L} - M \right], \qquad (40)
\end{aligned}
$$

where each matrix U_j is an element in the circular ensemble $\mathrm{C}\widetilde{\beta}\mathrm{E}(\widetilde{\gamma}k)$.

In the final step we replace $U_j' = \sqrt{U_L} \cdots \sqrt{U_{j+1}} U_j \sqrt{U_{j+1}} \cdots \sqrt{U_L}$ which preserves the symmetries such that $U_j' \in \mathrm{C}\widetilde{\beta}\mathrm{E}(\widetilde{\gamma}k)$. For this purpose we use two facts. First, the Haar measure is invariant under $d\mu(U) = d\mu(VUV^T)$ for all $V \in \mathrm{U}(\widetilde{\gamma}k)$ resulting from the fact that the explicit form of the Haar measure of $\mathrm{C}\widetilde{\beta}\mathrm{E}(\widetilde{\gamma}k)$ is $d\mu(U) \propto \det{}^{-k/\gamma-(\gamma-\widetilde{\gamma})/2} U d[U]$ with $d[U]$ the product of the differentials of all independent matrix entries [23, 22]. Second, the weights Q_j are also invariant under $Q_j(U) = Q_j(VUV^\dagger)$ for all $V \in \mathrm{U}^{(\beta)}(k)$. Hence these weights have an expression in terms of functions of matrix invariants. With the help of a slight abuse of notation one can say that the weights Q_j satisfy a cyclic permutation symmetry, $Q_j(AB) = Q_j(BA)$ for any two matrices $A, B \in \mathrm{U}(\widetilde{\gamma}k)$.

Finally, we find the result

$$
Z_\Pi(M) = \int \left(\prod_{j=1}^{L} d\mu(U_j') Q_j(U_j' U_{j+1}'^{-1}) \right) \det{}^{-n/\widetilde{\gamma}} U_1' \det{}^{1/(\gamma\widetilde{\gamma})} \left[\mathbf{1}_{\gamma n} \otimes U_1' - M \right] \qquad (41)
$$

with $U_{L+1}' = \mathbf{1}_{\widetilde{\gamma}k}$. This result is surprisingly compact. It also reflects the nature of the original product of matrices which is equivalent to a Mellin-like convolution in a matrix space. Also the dual space exhibits this structure of a Mellin-like convolution.

As an example we calculate the orthogonal polynomials ($k = 1$) of a product of L_{WL} complex $\chi\mathrm{G}\beta\mathrm{E} = \chi\mathrm{GUE}$, Eq. (24), L_{CL} complex $\mathcal{L}\beta\mathrm{E} =$

LUE, Eq. (24), and $L_{\rm J}$ complex JβE = JUE, (24). We assume this product to be ordered, i.e. first the Wishart-Laguerre, then the Cauchy-Lorentz, and finally the Jacobi matrices. The result does not depend on this ordering, see the discussion in [12]. Then the orthogonal polynomials are

$$\begin{aligned} Z^{(\beta=2,k=1)}_{L_{\rm WL}L_{\rm CL}L_{\rm J}}(m\mathbf{1}_n) \;\propto\; & \oint (1-mz_1^{-1})^n \left(\prod_{j=1}^{L_{\rm WL}} \frac{dz_j}{z_j}\left[\frac{z_{j+1}}{z_j}\right]^{\nu_j} e^{\frac{1}{\Gamma_j^2}\frac{z_j}{z_{j+1}}}\right) \\ & \times \left(\prod_{j=L_{\rm WL}+1}^{L_{\rm WL}+L_{\rm CL}} \frac{dz_j}{z_j}\left[\frac{z_{j+1}}{z_j}\right]^{\nu_j}\left[\Gamma_j^2+\frac{z_j}{z_{j+1}}\right]^{\mu_j'-1}\right) \\ & \times \left(\prod_{j=L_{\rm WL}+L_{\rm CL}+1}^{L_{\rm WL}+L_{\rm CL}+L_{\rm J}} \frac{dz_j}{z_j}\left[\frac{z_{j+1}}{z_j}\right]^{\nu_j}\left[\Gamma_j^2-\frac{z_j}{z_{j+1}}\right]^{-\kappa_j'-1}\right) \\ \propto & \sum_{j=0}^{n} \frac{\prod_a \Gamma[n+\kappa_a+\nu+j+1]}{j!(n-j)!(\prod_a(\nu_a+j)!)(\prod_a \Gamma[\mu_a-n-\nu-j])}\left(-\frac{m}{\Gamma^2}\right)^j. \end{aligned} \tag{42}$$

with $z_{L_{\rm WL}+L_{\rm CL}+L_{\rm J}+1}=1$, $\mu_j'=\mu_j-n$, $\kappa_j'=\kappa_j+n$, and $\Gamma^2=\prod_j\Gamma_j^2$. The product of the Gamma functions runs over the possible values for ν_a, κ_a, and μ_a. The polynomial (42) is a hypergeometric function and, thus, a Meijer G-function [14]. It agrees for certain values of the parameters $L_{\rm WL}$, $L_{\rm CL}$, and $L_{\rm J}$ with known results [11, 13]. What is completely new are the results for $\beta=1,4$ and $k=1$ which are essentially the same polynomials. Here the other approaches failed because of unknown group integrals.

6. Hard Edge Scaling limit of Product Matrices

Up to now every calculation was done for finite p such that we made no approximation and the projection formula was exact. However to make contact to physical systems and universality we have to zoom onto the local scale somewhere of the spectrum. A very prominent scaling is the one to a vicinity around the origin also known as the hard edge scaling limit.

As a simple but non-trivial example, we choose the matrix product of the previous section with the source $M=\widetilde{\gamma}(\prod_j\Gamma_j^2)\mathbf{1}_{\gamma n}\otimes\widehat{m}/[n(\prod_a(\mu_a-n/\widetilde{\gamma}))(\prod_a(\kappa_a+n/\widetilde{\gamma}))]$. In particular we consider the scaling limit $n\to\infty$ and ν_j, $\widehat{\mu}_j=(\mu_j/n-1/\widetilde{\gamma})$, $\widehat{\kappa}_j=(\kappa_j/n+1/\widetilde{\gamma})$, and $\widehat{m}$ fixed. Then one can easily show that the asymptotics of each weight, regardless what kind of random matrix we consider, is

$$Q_j(\alpha U) \overset{n\gg 1}{\propto} \det{}^{\nu_j/\widetilde{\gamma}} U e^{\operatorname{tr} U} \tag{43}$$

with $\alpha=\Gamma_j^2$ for χGβE, $\alpha=\Gamma_j^2/(n\widehat{\mu}_j)$ for $\mathcal{L}\beta$E, and $\alpha=\Gamma_j^2/(n\widehat{\kappa}_j)$ for $\mathcal{L}\beta$E. After a proper rescaling of the matrices U_j the partition function (41) takes the asymptotic form

$$Z_\Pi(M) \overset{n\gg 1}{\propto} \int \left(\prod_{j=1}^{L} d\mu(U_j')\det{}^{\nu_j/\widetilde{\gamma}}U_j\right) e^{\operatorname{tr}U_L+\sum_{j=1}^{L-1}\operatorname{tr}U_jU_{j+1}^{-1}-\operatorname{tr}\widehat{m}U_1^{-1}} \tag{44}$$

with $L = L_{\rm WL}+L_{\rm CL}+L_{\rm J}$. We underline that no saddlepoint approximation is needed for this limit. Hence the matrices U_j are still elements of the circular ensemble $\mathrm{C}\widetilde{\beta}\mathrm{E}(\widetilde{\gamma}k)$.

For $\beta = 2$ the partition function (44) yields the Meijer G-kernel of a product of matrices drawn from $\chi\mathrm{G}\beta\mathrm{Es}$, cf. [11]. This can be seen by diagonalizing the unitary matrices, applying the Itzykson-Zuber integral [18] and finally integrating over a determinantal point process. The entries of the resulting determinant are Meijer G-functions. Our result emphasizes the conjecture that also this kernel is universal. Indeed we could also have chosen another scaling which still leads to a hard edge scaling limit. Then we would get finite rank deformations of the result (44) which was recently discovered for a product of truncated unitary matrices in [13]. Nevertheless the limiting kernel is still a Meijer G-kernel but with other parameters.

From a physical point of view can one ask for the non-linear σ-model corresponding to the partition function (44). In this framework the function in the exponential function is identified as the potential. The integration domain $\mathrm{C}\widetilde{\beta}\mathrm{E}^L(\widetilde{\gamma}k)$ is the coset of the "flavour" group which keeps the "massless Lagrangian" ($\widehat{m} = 0$) in the full theory at finite "volume" n invariant divided by the group which keeps the ground state invariant. As in the case $L = 1$ the theory is spontaneously broken. For a product matrix the "flavour" symmetry at finite "volume" n is $\mathrm{U}^L(\widetilde{\gamma}k)$ for $\beta = 1, 4$ and $[\mathrm{U}(k) \times \mathrm{U}(k)]^L$ for $\beta = 2$ which can be readily checked by linearising the product $W^{(L)}$ in the matrices W_j. This group is spontaneously broken to $[\mathrm{U}^{(\widetilde{\beta})}(k)]^L$ and the source term for its condensate is the "mass" $\widehat{m}$. This non-linear σ model generalizes the one for the Wishart-Laguerre ensemble which were found in QCD [31] and mesoscopic systems [25]. Especially the coupling between different U_j is reminiscent but not completely the same as the recently proposed chiral Lagrangian for high density QCD [32].

7. Conclusions

We briefly presented the projection formula [20] for averages over products of characteristic polynomials which is a short cut of the supersymmetry method [21, 22]. The general results found by this approach were demonstrated in the case of Wishart-Laguerre ($\chi\mathrm{G}\beta\mathrm{E}$), Cauchy-Lorentz ($\mathcal{L}\beta\mathrm{E}$), and Jacobi ($\mathrm{J}\beta\mathrm{E}$) ensembles, in particular we rederived the corresponding orthogonal polynomials for $\beta = 2$. These polynomials are essentially the same when averaging over one characteristic polynomial for $\beta = 1$ and over a square root of a characteristic polynomial for $\beta = 4$.

Moreover we generalized the projection formula to products of matrices. Since the projection formula works in a unifying way for all three Dyson indices $\beta = 1, 2, 4$ this approach is an ideal alternative compared to other methods like orthogonal polynomials and free probability when studying real or quaternion matrices. Note that up to now free probability only applies to global spectral properties and to use orthogonal polynomials we need to know group integrals like the Itzykson-Zuber integral [18] or its polynomial counterpart [19, 13]. The projection formula circumvents this

problem. In particular we were able to show that the spectral statistics at the hard edge are the same for products of completely different random matrices only depending on the number of matrices defined and their indices $\nu_1, \ldots, \nu_l$ encoding the rectangularity of the matrices. This was done for all three cases $\beta = 1, 2, 4$ and underlines the strength of the projection formula where other methods fail. In the complex case ($\beta = 2$) we easily deduce from our results those for the Meijer G-ensembles studied in [11, 13].

The projection formula also enabled us to identify the non-linear σ-models and the symmetry breaking pattern for product matrices and derived the potential of the Goldstone manifold. This result is completely new and shows what the effective theory associated to such a product matrix would look like. In particular one can understand a product matrix by itself as a discrete one-dimensional system. Therefore our results show one way to generalize the zero-dimensional RMT to a one-dimensional theory. Indeed via the DMPK equation such a link to a one-dimensional system was established [3], though there a different limit is considered.

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Singular value statistics of matrix products with truncated unitary matrices

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Abstract

We prove that the squared singular values of a fixed matrix multiplied with a truncation of a Haar distributed unitary matrix are distributed by a polynomial ensemble. This result is applied to a multiplication of a truncated unitary matrix with a random matrix. We show that the structure of polynomial ensembles and of certain Pfaffian ensembles is preserved. Furthermore we derive the joint singular value density of a product of truncated unitary matrices and its corresponding correlation kernel which can be written as a double contour integral. This leads to hard edge scaling limits that also include new finite rank perturbations of the Meijer G-kernels found for products of complex Ginibre random matrices.

1 Introduction

The study of random matrices benefits greatly from explicit formulas of joint eigenvalue densities that are known for large classes of random matrix ensembles. Quite a lot of these densities have the structure of a determinantal or Pfaffian point process. Such structures are incredibly helpful to extract the spectral statistics in the limit of large matrix dimension of such ensembles. Various techniques as free probability [54], orthogonal polynomials [15, 52, 21, 45], and supersymmetry [34] have been developed to derive these limits.

In an important recent development it was found that explicit formulas also exist for the eigenvalue and singular value statistics of products of random matrices. This was first established for the eigenvalues [3, 40, 41, 26] and singular values [5, 6] of products of Ginibre matrices. Shortly after this development, results were also derived for the eigenvalues of products of truncated unitary matrices [1, 4, 41]. A common feature is that the joint probability densities are expressed in terms of Meijer G-functions which were also found in other recent works on random matrices [13, 27, 28, 29, 55].

Given the results of [1, 4, 41] on eigenvalues, it seems to be natural to expect that also the singular values of products of truncated unitary matrices have

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an explicit joint probability density. We aim at proving this statement. The squared singular values $x_1, \dots, x_n$ of such a product have the joint probability density

$$\frac{1}{Z_n} \prod_{j<k} (x_k - x_j) \det \left[w_k(x_j) \right]_{j,k=1}^n, \qquad \text{all } x_j > 0, \tag{1.1}$$

for certain functions w_k, see Corollary 2.6 below.

A joint probability density function of the form (1.1) is called a polynomial ensemble in [47]. It is an example of a biorthogonal ensemble [15] whose correlation kernel is built out of polynomials and certain dual functions. It reduces to an orthogonal polynomial ensemble [45] in the case $w_k(x) = x^{k-1} w(x)$ for a certain weight function w. The results of [5, 6] for the singular values of products of complex Ginibre matrices were interpreted in [47] in the sense of a transformation of polynomial ensembles. Recall that a complex Ginibre matrix is a random matrix whose entries are independent standard complex Gaussians. The main result of [47] is the following.

Theorem 1.1 (Theorem 2.1 in [47]). *Let n, l, ν be non-negative integers with $1 \leq n \leq l$. Let G be an $(n+\nu) \times l$ complex Ginibre matrix, and let X be a random matrix of size $l \times n$, independent of G, such that the squared singular values $x_1, \dots, x_n$ are a polynomial ensemble*

$$\propto \prod_{j<k} (x_k - x_j) \, \det \left[f_k(x_j) \right]_{j,k=1}^n, \qquad \textit{all } x_j > 0, \tag{1.2}$$

for certain functions $f_1, \dots, f_n$ defined on $[0, \infty)$. Then the squared singular values $y_1, \dots, y_n$ of $Y = GX$ are a polynomial ensemble

$$\propto \prod_{j<k} (y_k - y_j) \, \det \left[g_k(y_j) \right]_{j,k=1}^n, \qquad \textit{all } x_j > 0, \tag{1.3}$$

where

$$g_k(y) = \int_0^\infty x^\nu e^{-x} f_k\left(\frac{y}{x}\right) \frac{dx}{x}, \qquad y > 0. \tag{1.4}$$

Note that g_k is the Mellin convolution of f_k with the "Gamma density" $x^\nu e^{-x}$.

We aim at an analogue of Theorem 1.1 for the product of X with a truncated unitary matrix and find that the structure of a polynomial ensemble is preserved. Instead of a Mellin convolution with a "Gamma density" we find a Mellin convolution with a "Beta density" $x^\nu (1-x)^\mu$ with certain variables $\nu, \mu \in \mathbb{N}_0$ and defined on the interval $[0, 1]$, see Corollary 2.4. This result is an immediate consequence of a theorem on the transformation of squared singular values of a fixed matrix X when multiplied by a truncated unitary matrix that we present as our main theorem, see Theorem 2.1.

The spectral statistics of a generic truncation of a fixed matrix X is an old question. Especially in random matrix theory such truncations quite often occur because of its natural relation to the Jacobi ensemble [25] and a modification of the supersymmetry method [17] to calculate the average of an arbitrary product of characteristic polynomials where generic projections to lower dimensional matrices are needed. Applications of truncated unitary matrices in physics can be found in quantum mechanical evolution [33], chaotic scattering [30], mesoscopic

physics [12] and quantum information [58]. We claim that even in telecommunications one can certainly apply truncations of unitary matrices. Usually Ginibre matrices model the information channel between the transmitter and the receiver [56, 57, 37, 6]. However if the number of channels to the environment is on the scale of the number of transmitting channels or even smaller then deviations to the Gaussian assumption should be measurable. We guess that those deviations result from the fact that no signal is lost. It can only be absorbed by the environment. Therefore the evolution in all channels should be unitary and, thus, in the channels between the receiver and the transmitter a truncated unitary matrix.

Very recently it was shown by Ipsen and one of the authors [41] that truncated unitary matrices are also generally involved in the spectral statistics of products of rectangular random matrices, including products of matrices which are not Ginibre matrices or truncated unitary matrices. Moreover the truncation of matrices are also important in representation theory. For example, let H be an $m \times m$ Hermitian matrix and T an $n \times m$ complex matrix $(m > n)$ which is a truncation of a unitary matrix $U \in \mathcal{U}(m)$. Then the question for the generic eigenvalues of a truncation THT^* (T^* is the Hermitian conjugate of T) is deeply related to the fact which representations of $\mathcal{U}(n)$ are contained in a certain representation of $\mathcal{U}(m)$. In particular group integrals and coset integrals like integrals over Stiefel manifolds are deeply related to representation theory, see [35, 9, 10, 38, 19]. Though our results are more general they can be partially interpreted in this framework. The relation of truncated unitary matrices and representation theory goes back to a parametrization of the unitary group by Gelfand and Zeitlin[1] [31]. They found that the eigenvalues of THT^* with $n = m - 1$ satisfies the so-called interlacing property, i.e. $x_1 \leq x'_1 \leq x_2 \leq x'_2 \leq \cdots \leq x_{m-1} \leq x'_{m-1} \leq x_m$ where $x_1 \leq \cdots \leq x_m$ are the ordered eigenvalues of H and $x'_1 \leq \cdots \leq x'_{m-1}$ are the ordered eigenvalues of THT^*. This interlacing is reflected in the "quantum numbers" labelling the representations of $\mathcal{U}(n)$ and $\mathcal{U}(m)$.

As a consequence of our generalization of Theorem 1.1 to truncated unitary matrices we derive the joint probability density function for the squared singular values of an arbitrary product $Y = T_r \cdots T_1$, where each T_j, $j = 1, \ldots, r$ is a truncation of a unitary matrix. We find a polynomial ensemble with Meijer G-functions which is similar to the case of products of complex Ginibre matrices [5, 48]. The polynomial ensemble is a determinantal point process with a correlation kernel which is a double contour integral in products and ratios of Gamma functions and can be equivalently rewritten as a onefold integral over a product of two Meijer G-functions. Based on the double integral formula we are able to obtain hard edge scaling limits as was done in [48] for the product of complex Ginibre matrices. In addition to the Meijer G-kernels that are already in [48] we also find certain finite rank perturbations.

All results are summarized in section 2. The proofs of these results are contained in sections 3, 4, 5, and 6. In section 7 we briefly discuss open questions to this topic.

[1] We employ the transcription of Zeitlin used in [19].

2 Statement of results

Let us start with some preliminaries. A $k \times l$ truncation T of a matrix $U \in \mathcal{U}(m)$ ($m > \max(k,l)$) is a submatrix of U. Note that in the case that $k+l=m$ the matrix T is equivalent with an element $[T]$ in the coset $\mathcal{U}(m)/[\mathcal{U}(k) \times \mathcal{U}(m-k)]$ due to the embedding

$$T \hookrightarrow [T] = \left\{ U \begin{pmatrix} \sqrt{I_k - TT^*} & T \\ T^* & \sqrt{I_{m-k} - T^*T} \end{pmatrix} \in \mathcal{U}(m) \middle| U \in \mathcal{U}(k) \times \mathcal{U}(m-k) \right\}, \tag{2.1}$$

where I_k is the $k \times k$ identity matrix. If $k+l \neq m$ such an interpretation is not possible such that our studies are more general.

A natural measure for the truncated unitary matrix T is the induced Haar measure of the unitary matrix U. Let dU be the normalized Haar measure. Then the induced measure is

$$\int_{\mathcal{U}(n)} \prod_{a=1}^{k} \prod_{b=1}^{l} \delta^{(2)}(T_{ab} - U_{ab}) dU \tag{2.2}$$

with $\delta^{(2)}(x+iy) = \delta(x)\delta(y)$ the Dirac delta function in the complex plane.

We are interested in the singular value statistics of the matrix $Y = TX$ where $X \in \mathbb{C}^{l \times n}$. Recall that the squared singular values $y_1, \ldots, y_n$ of the matrix Y are the eigenvalues of Y^*Y and analogously for the matrix X. The joint probability density of these squared singular values involves the Vandermonde determinant which is given by the product $\Delta(y) = \prod_{j<k}^{n} (y_k - y_j)$.

We present in the following subsections five results. These results comprise the joint probability distribution of the squared singular values of a product TX of a fixed matrix X (subsection 2.1), of a random matrix X whose squared singular values are taken from a polynomial ensemble (subsection 2.3) or from a certain Pfaffian point process (subsection 2.4), and of a product of truncated unitary matrices (subsection 2.5). Moreover we present a remarkable group integral involved in one of our proofs which is the analogue to the Harish Chandra/Itzykson-Zuber (HCIZ) integral [35, 42] for the Ginibre ensembles, see subsection 2.2.

2.1 Main result

All corollaries we present in our work are based on the following theorem.

Theorem 2.1. *Let n, m, l, ν be non-negative integers with $1 \leq n \leq l \leq m$ and $m \geq n + \nu + 1$. Let T be an $(n+\nu) \times l$ truncation of a Haar distributed unitary matrix U of size $m \times m$. Let X be a non-random matrix of size $l \times n$ with non-zero squared singular values $x_1, \ldots, x_n$, and we assume them to be all pairwise distinct. Then the squared singular values $y_1, \ldots, y_n$ of $Y = TX$ have a joint probability density function on $[0,\infty)^n$*

$$\propto \left(\prod_{j=1}^{n} x_j^{-m+n} \right) \left(\prod_{j=1}^{n} y_j^{\nu} \right) \det \left[(x_k - y_j)_+^{m-n-\nu-1} \right]_{j,k=1}^{n} \frac{\Delta(y)}{\Delta(x)}, \tag{2.3}$$

where $(x-y)_+ = \max(0, x-y)$. The missing overall constant only depends on n, m and ν, but is independent of X.

We emphasize that this theorem can be easily generalized to a matrix X where one or more of its squared singular values x_j coincide. Then the result remains valid if we replace (2.3) by an appropriate limit using L'Hôpital's rule.

Remark 2.2. We can readily set $l = n$ without loss of generality. The reason for this is exactly the same one already discussed in [41]. We can perform a singular value decomposition of $X = U_{\mathrm{L}} X' U_{\mathrm{R}}$ ($U_{\mathrm{L}} \in \mathcal{U}(l)$, $U_{\mathrm{R}} \in \mathcal{U}(n)$, and X' a rectangular matrix with only one of its main-diagonals non-zero) and absorb the unitary matrix U_{L} in T. Since $l \geq n$ the matrix X' has $l - n$ rows equal to zero. This structure projects the matrix T to an even smaller matrix T' of size $(n + \nu) \times n$. Let $\widetilde{X}$ be the matrix $X' U_{\mathrm{R}}$ without these zero rows, in particular $\widetilde{X}$ is $n \times n$ dimensional. Then we can consider the product $Y = TX = T'\widetilde{X}$ and apply the Theorem 2.1 for the fixed matrix $\widetilde{X}$ and the truncated unitary matrix T'. Note that $\widetilde{X}$ and X have the same singular values.

2.2 An integral over the unitary group

We give two proofs of Theorem 2.1, the first one in section 4 and the second one in section 5. The first proof only works in the case where $m \geq 2n + \nu$. In this case the truncation T does not have generic squared singular values which are equal to 1. Then T is sufficiently small compared to the dimension of the underlying larger unitary matrix U. In this case the set of all matrices T which have a squared singular value equal to 1 is a set of measure zero.

The second proof works in all cases and is based on a calculation with test functions. Nonetheless, we decide to keep the first proof, too, since it is based on a remarkable integral over the unitary group that is of interest in its own right. It replaces the HCIZ integral that was used in the proof of Theorem 1.1. The HCIZ integral formula [35, 42] is the following well-known integral over the unitary group $\mathcal{U}(n)$,

$$\int_{\mathcal{U}(n)} \exp[t \operatorname{Tr} AUBU^*] dU = \left(\prod_{j=1}^{n-1} j! \right) \frac{\det \left[\exp(t a_j b_k) \right]_{j,k=1}^{n}}{t^{(n^2-n)/2} \Delta(a) \Delta(b)}, \tag{2.4}$$

where A and B are Hermitian matrices with pairwise distinct eigenvalues $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$, respectively, and $t \in \mathbb{C} \setminus \{0\}$.

The new integral over the unitary group involves the Heaviside step function of a matrix argument, defined on Hermitian matrices X as

$$\theta(X) = \begin{cases} 1, & \text{if } X \text{ is positive definite,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Then the generalization of (2.4) for our purposes is the following theorem which is proven in section 3.

Theorem 2.3. *Let A and B be $n \times n$ Hermitian matrices with respective eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ and pairwise distinct. Let dU be the normalized Haar measure on the unitary group $\mathcal{U}(n)$. Then for every $p \geq 0$,*

$$\int_{\mathcal{U}(n)} \det \left(A - UBU^*\right)^p \theta(A - UBU^*) \, dU = c_{n,p} \frac{\det \left[(a_j - b_k)_+^{p+n-1} \right]_{j,k=1}^{n}}{\Delta(a) \Delta(b)}. \tag{2.6}$$

The constant $c_{n,p}$ in (2.6) *is*

$$c_{n,p} = \prod_{j=0}^{n-1} \binom{p+n-1}{j}^{-1} \tag{2.7}$$

and $(a_j - b_k)_+$ is $(a_j - b_k)$ if the difference is positive and otherwise vanishes. If some of the a_j and/or b_j coincide we have to take the formula (2.6) *in the limiting sense using l'Hôpital's rule.*

Theorem 2.3 is known to hold when $\min a_k \geq \max b_j$, see [32, formula (3.21)] and also [36]. Then the Heaviside step function drops out. However if this condition is not met it is even more surprising that the result still looks that simple, especially that the result can be expressed in exactly the same determinantal form as if the condition has been met. We emphasise that in the general case we usually do not effectively integrate over the whole group $\mathcal{U}(n)$ but a smaller subset. The contribution of the subset of $U \in \mathcal{U}(n)$ for which $A - UBU^*$ is not positive definite vanishes due to the Heaviside step function.

What does really happen in the integral (2.6)? To understand this we may choose A and B diagonal since their diagonalizing matrices can be absorbed in the unitary group integral. Recall that Harish Chandra [35] traced the integral (2.4) back to a sum over the Weyl group acting on B. The Weyl group of $\mathcal{U}(n)$ is the permutation group of n elements, $\mathcal{S}_n \subset \mathcal{U}(n)$. This sum results into the determinant in (2.4). Exactly such a thing also happens here. The difference is that the action of $\omega \in \mathcal{S}_n \subset \mathcal{U}(n)$ to the matrix $M = A - \omega B \omega^*$ sometimes yields no contribution to the integral because of the Heaviside step function. Because M is also diagonal the Heaviside step function of M factorizes into Heaviside step functions of $a_k - b_{\omega(k)}$ for $k = 1, \ldots, n$ telling us that $a_k - b_{\omega(k)}$ has to be positive definite. Despite the fact that some of the terms in the sum over the Weyl group vanish we can extend the sum over the whole group because they are zero without changing the result. This is the reason why inside the determinant of (2.6) we have $(a_j - b_k)_+^{p+n-1}$ and not $(a_j - b_k)^{p+n-1}$. Hence, one can indeed understand Theorem 2.3 by this intuition.

2.3 Transformation of polynomial ensemble

Our main application of Theorem 2.1 is to the situation where X is random and statistically independent of T, in such a way that its squared singular values are a polynomial ensemble on $[0, \infty)$. The proof relies on the well-known Andréief formula [8, 22],

$$\int_{X^n} \det\left[\varphi_k(x_j)\right]_{j,k=1}^n \det\left[\psi_j(x_k)\right]_{j,k=1}^n d\mu(x_1) \cdots d\mu(x_n) \\ = n! \det\left[\int_X \varphi_k(x)\psi_j(x) d\mu(x)\right]_{j,k=1}^n, \tag{2.8}$$

that is valid for arbitrary functions φ_j and ψ_k on a measure space (X, μ) such that the integrals exist. The integral (2.8) is used several times in our proofs.

With the help of Andréief's formula one can readily deduce from Theorem 2.1 the following Corollary.

Corollary 2.4. *Let n, m, l, ν and T be as in Theorem 2.1. Let X be a random matrix of size $l \times n$, independent of T, such that the squared singular values $x_1, \ldots, x_n$ of X have the joint probability density function*

$$\propto \Delta(x) \det\left[f_k(x_j)\right]_{j,k=1}^{n}, \qquad \text{all } x_j > 0, \tag{2.9}$$

for certain distributions $f_1, \ldots, f_n$ such that the moments $\int_0^\infty f(x)x^a dx$ with $a = 0, \ldots, n-1$ exist. Then the squared singular values $y_1, \ldots, y_n$ of $Y = TX$ have the joint probability density

$$\propto \Delta(y) \det\left[g_k(y_j)\right]_{j,k=1}^{n}, \qquad \text{all } y_j > 0, \tag{2.10}$$

where

$$g_k(y) = \int_0^1 x^\nu (1-x)^{m-n-\nu-1} f_k\left(\frac{y}{x}\right) \frac{dx}{x}, \qquad y > 0, \tag{2.11}$$

is the Mellin convolution of f_k with the "Beta distribution" $x^\nu(1-x)^{m-n-\nu-1}$ on $[0,1]$.

We underline again that one can set $l = n$ without loss of generality.

Proof. We average (2.3) over $x_1, \ldots, x_n$ with the joint probability density function (2.9). The n-fold integral is evaluated using (2.8) with $d\mu(x) = x^{-m+n}dx$ on $X = [0, \infty)$ and identifying the functions $\varphi_k(x) = f_k(x)$ and $\psi_j(x) = (x - y_j)_+^{m-n-\nu-1}$. Then we find that the squared singular values of Y have a joint probability density

$$\propto n!\Delta(y) \left(\prod_{j=1}^{n} y_j^\nu\right) \det\left[\int_0^\infty x^{-m+n} f_k(x)(x-y_j)_+^{m-n-\nu-1} dx\right]_{j,k=1}^{n}, \tag{2.12}$$

which is of the form (2.10) with functions

$$\begin{aligned} g_k(y) &= y^\nu \int_y^\infty x^{-m+n} f_k(x)(x-y)^{m-n-\nu-1} dx \\ &= y^\nu \int_0^1 \left(\frac{y}{x}\right)^{-m+n} \left(\frac{y}{x} - y\right)^{m-n-\nu-1} f_k\left(\frac{y}{x}\right) \frac{y dx}{x^2}, \end{aligned} \tag{2.13}$$

Here, we applied the change of variables $x \mapsto y/x$. This is easily seen to reduce to (2.11). □

Corollary 2.4 was obtained in the recent preprint [46] in a different way.

Corollary 2.4 is the analogue of Theorem 1.1 for the case of a multiplication with truncated unitary matrix. It is interesting to note that Theorem 1.1 is obtained from Corollary 2.4 in the limit when $m \to \infty$ while keeping n, l, and ν fixed, since $\sqrt{m}T \to G$ where G is a complex Ginibre matrix. Recall that m is the dimension of the unitary matrix that T is a truncation of.

2.4 Transformation of Pfaffian ensembles

Theorem 2.1 can also be applied in a Pfaffian context. Instead of (2.8) we now use the de Bruijn formula [16], see [25, Proposition 6.3.5]

$$\int_{X^n} \det\left[\varphi_k(x_j)\right]_{j,k=1}^n \operatorname{Pf}\left[f(x_j,x_k)\right]_{j,k=1}^n d\mu(x_1)\cdots d\mu(x_n) \\ = n!\operatorname{Pf}\left[\int_X\int_X \varphi_j(x)\varphi_k(y)f(x,y)d\mu(x)d\mu(y)\right]_{j,k=1}^n, \quad (2.14)$$

which is valid for n even and f an anti-symmetric function on $X \times X$, i.e., $f(y,x) = -f(x,y)$ for all $x, y \in X$, such that all integrals exist. Then the following result is a consequence of Theorem 2.1 which can be deduced in a similar way as we obtained Corollary 2.4. Therefore we skip the proof and only state the Corollary.

Corollary 2.5. *Let n, m, l, ν and T be as in Theorem 2.1 with n even. Let X be independent of T such that the squared singular values of X have the joint probability density function*

$$\propto \Delta(x)\operatorname{Pf}\left[f(x_j,x_k)\right]_{j,k=1}^n, \qquad \text{all } x_j > 0, \quad (2.15)$$

for a certain anti-symmetric distribution f on $[0,\infty)\times[0,\infty)$ such that the mixed moments $\int_0^\infty\int_0^\infty x_1^j x_2^k f(x_1,x_2)dx_1dx_2$ exist for all $j,k = 0,\ldots,n-1$. Then the squared singular values $y_1,\ldots,y_n$ of $Y = TX$ have joint probability density

$$\propto \Delta(y)\operatorname{Pf}\left[g(y_j,y_k)\right]_{j,k=1}^n, \qquad \text{all } y_j > 0, \quad (2.16)$$

where

$$g(y_1,y_2) = \int_0^1\int_0^1 x_1^\nu(1-x_1)^{m-n-\nu-1}x_2^\nu(1-x_2)^{m-n-\nu-1} f\left(\frac{y_1}{x_1},\frac{y_2}{x_2}\right)\frac{dx_1}{x_1}\frac{dx_2}{x_2}. \quad (2.17)$$

This result can be combined with Corollary 2.4 for ensembles where we have a mixture of orthogonal and skew-orthogonal polynomials, i.e., the joint probability density of the squared singular values of X is given by

$$\propto \Delta(x)\operatorname{Pf}\begin{bmatrix} f(x_j,x_k) & f_i'(x_j) \\ -f_i'(x_j) & 0 \end{bmatrix}_{\substack{j,k=1,\ldots,n \\ i=1,\ldots,n'}}, \qquad \text{all } x_j > 0, \quad (2.18)$$

where $n + n'$ is even and f_i' is an additional set of distributions. Also this structure carries over to the product TX as can be easily checked.

The structure (2.18) is not only academically. It appears if X is real and n is odd (in this case we have $n' = 1$), see for example the real Laguerre ensemble [52] or the real Jacobi ensemble [25]. Also the case $n' > 1$ appears naturally in random matrix theory. For applications in QCD a random matrix model was proposed which breaks the Gaussian unitary ensemble by the chiral Gaussian unitary ensemble [20]. The joint probability density of this random matrix ensemble has the form (2.18), see [7, 44].

2.5 Products of truncated unitary matrices

Let us come to the product of a finite number of truncated unitary matrices. The joint probability density of the squared singular values readily follows from Corollary 2.4. As in the case of multiplying Ginibre matrices the Meijer G-functions play a crucial role in the spectral statistics. Meijer G-functions are defined via a contour integral

$$G_{p,q}^{m,n}\left(\left.\begin{matrix} a_1,\ldots,a_n;a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m;b_{m+1},\ldots,b_q \end{matrix}\right| z\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)} z^{-s}ds, \quad (2.19)$$

where the contour C separates the poles of $\prod_{j=1}^{m}\Gamma(b_j+s)$ from the poles of $\prod_{j=1}^{n}\Gamma(1-a_j-s)$, see [11, 50] for more details.

We apply Corollary 2.4 to the product of r truncated unitary matrices. For $j=1,\ldots,r$, let T_j be a matrix of size $(n+\nu_j)\times(n+\nu_{j-1})$ where $\nu_0=0$ and $\nu_1,\ldots,\nu_r$ are non-negative integers. Suppose T_j is the truncation of a Haar distributed unitary matrix of size $m_j\times m_j$ where $m_1\geq 2n+\nu_1$ and $m_j\geq n+\nu_j+1$ for $j=2,\ldots,r$. The squared singular values of T_1 have the joint probability density (4.2) with parameters n_1, m_1 and ν_1. This is a polynomial ensemble [25]

$$\frac{1}{Z_n}\Delta(x)\det\left[w_k^{(1)}(x_j)\right]_{j,k=1}^{n} \quad (2.20)$$

with

$$w_k^{(1)}(x)=\begin{cases} x^{\nu_1+k-1}(1-x)^{m_1-2n-\nu_1}, & \text{if } 0<x<1,\\ 0, & \text{otherwise.}\end{cases} \quad (2.21)$$

The constant Z_n normalizes the joint probability density (2.20). Then we find the following Corollary.

Corollary 2.6. *Let $Y=T_r\cdots T_1$ with truncated unitary matrices T_j as described above. Then the squared singular values of Y have the joint probability density*

$$\frac{1}{Z_n^{(r)}}\prod_{j<k}(y_k-y_j)\,\det\left[w_k^{(r)}(y_j)\right]_{j,k=1}^{n}, \quad (2.22)$$

where $w_k^{(r)}$ is given by (2.21) *in the case $r=1$ and by*

$$w_k^{(j)}(y)=\int_0^1 x^{\nu_j}(1-x)^{m_j-n-\nu_j-1}w_k^{(j-1)}\left(\frac{y}{x}\right)\frac{dx}{x} \quad (2.23)$$

for $j=2,\ldots,r$ when $r\geq 2$.

The weight functions $w_k^{(r)}$ are obtained as an $(r-1)$-fold Mellin convolution of the "Beta distribution". The function $w_k^{(1)}$ can be written as a Meijer G-function,

$$w_k^{(1)}(x)=c_1\,G_{1,1}^{1,0}\left(\left.\begin{matrix} m_1-2n+k \\ \nu_1+k-1\end{matrix}\right| x\right) \quad (2.24)$$

with $c_1 = \Gamma(m_1 - 2n - \nu_1 + 1)$. Since the class of Meijer G-functions is closed under the Mellin convolution, see [11, formula (5)], we find from (2.23) and (2.24),

$$\begin{aligned} w_k^{(r)}(y) &= c_r G^{r,0}_{r,r}\left(\begin{matrix} m_r - n, \dots, m_2 - n, \dots, m_1 - 2n + k \\ \nu_r, \nu_{r-1}, \dots, \nu_2, \nu_1 + k - 1 \end{matrix} \middle| y \right) \\ &= \frac{c_r}{2\pi i} \int_C \frac{\Gamma(\nu_1 + k - 1 + s) \prod_{j=2}^r \Gamma(\nu_j + s)}{\Gamma(m_1 - 2n + k + s) \prod_{j=2}^r \Gamma(m_j - n + s)} y^{-s} ds, \quad 0 < y < 1, \end{aligned} \tag{2.25}$$

Here the contour C is a positively oriented curve in the complex s-plane that starts and ends at $-\infty$ and encircles the negative real axis. The constant c_r in (2.25) is

$$c_r = \Gamma(m_1 - 2n - \nu_1 + 1) \prod_{j=2}^r \Gamma(m_j - n - \nu_j). \tag{2.26}$$

It can be checked from (2.25) that the linear span of $w_1^{(r)}, \dots, w_n^{(r)}$ consists of all functions of the form

$$\frac{1}{2\pi i} \int_C \frac{q(s) \prod_{j=1}^r \Gamma(s + \nu_j)}{\prod_{j=1}^r \Gamma(s + m_j - n)} y^{-s} ds, \qquad 0 < y < 1, \tag{2.27}$$

where $q(s)$ is a polynomial of degree smaller than n. Remarkably enough, this space does not depend on the ordering of the parameters $m_1, \dots, m_r$ and neither on the ordering of the parameters $\nu_1, \dots, \nu_r$.

The k-point correlation function of the joint probability density (2.22) satisfies a determinantal point process on $[0, 1]$. In the center of such a determinantal point process stands a correlation kernel that can always be written as

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x) Q_k(y). \tag{2.28}$$

For a polynomial ensemble, the function P_k is a polynomial of degree k and Q_k is in the linear span of $w_1^{(r)}, \dots, w_n^{(r)}$ satisfying the biorthogonality

$$\int_0^1 P_j(x) Q_k(x) dx = \delta_{j,k}, \tag{2.29}$$

see e.g. [15]. As in [48] we find integral representations for the biorthogonal functions P_k and Q_k and a double integral formula for K_n. In what follows we use the Pochhammer symbol

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{2.30}$$

Proposition 2.7. *For every $k = 0, \dots, n-1$, we have*

$$\begin{aligned} P_k(x) &= \frac{1}{2\pi i} \oint_{\Sigma_k} \frac{1}{(t-k)_{k+1}} \prod_{j=1}^r \frac{\Gamma(t + 1 + m_j - n)}{\Gamma(t + 1 + \nu_j)} \, x^t \, dt \\ &= G^{0,r+1}_{r+1,r+1}\left(\begin{matrix} k+1, n - m_1, \dots, n - m_r \\ 0, -\nu_1, \dots, -\nu_r \end{matrix} \middle| x \right), \end{aligned} \tag{2.31}$$

where Σ_k is a closed contour encircling the interval $[0,k]$ once in positive direction and is not encircling any pole of the integrand in (2.31) *that is outside $[0,k]$. Moreover we have*

$$Q_k(y) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_C (s-k)_k \prod_{j=1}^{r} \frac{\Gamma(s+\nu_j)}{\Gamma(s+m_j-n)}\, y^{-s} ds, & 0<y<1, \\ 0, & \text{elsewhere} \end{cases}$$

$$= G^{r+1,0}_{r+1,r+1}\left(\begin{matrix} -k, m_1-n, \ldots, m_r-n \\ 0, \nu_1, \ldots, \nu_r \end{matrix} \middle| y \right) \tag{2.32}$$

with the same contour C as in (2.25).

The kernel K_n from (2.28) *is*

$$K_n(x,y) = \frac{1}{(2\pi i)^2}\int_C ds \oint_{\Sigma_n} dt \prod_{j=0}^{r} \frac{\Gamma(s+1+\nu_j)\Gamma(t+1+m_j-n)}{\Gamma(t+1+\nu_j)\Gamma(s+1+m_j-n)} \frac{x^t y^{-s-1}}{s-t}$$

$$= -\int_0^1 G^{0,r+1}_{r+1,r+1}\left(\begin{matrix} n-m_0, \ldots, n-m_r \\ -\nu_0, \ldots, -\nu_r \end{matrix} \middle| ux \right)$$

$$\times G^{r+1,0}_{r+1,r+1}\left(\begin{matrix} m_0-n, \ldots, m_r-n \\ \nu_0, \ldots, \nu_r \end{matrix} \middle| uy \right) du \tag{2.33}$$

which is valid if Σ_n and C do not intersect. In (2.33) *it is understood that $m_0 = \nu_0 = 0$.*

The proposition is proved in section 6.

2.6 Hard edge scaling limits

Based on the double integral representation (2.33) we analyze some scaling limits of the correlation kernel as $n \to \infty$. In a forthcoming publication we show that the usual sine kernel limit can be found in the bulk and the Airy kernel limit at the soft edges. These results will be reported elsewhere, see also [49].

Here we want to look at a more straightforward scaling limit, namely the hard edge scaling at the origin in the following situation. Taking $n \to \infty$, we simultaneously have to let $m_j \to \infty$ for $j = 1, \ldots, r$, since $m_1 \geq 2n + \nu_1$ and $m_j \geq n + \nu_j + 1$ for $j \geq 2$. We keep ν_j fixed for every $j = 1, \ldots, r$. We choose a subset J of indices

$$J = \{j_1, \ldots, j_q\} \subset \{2, \ldots, r\}, \qquad \text{with } 0 \leq q = |J| < r \tag{2.34}$$

and integers $\mu_1, \ldots, \mu_q$ with $\mu_k \geq \nu_{j_k} + 1$, and we assume

$$\begin{aligned} &m_j - n \to \infty && \text{for } j \in \{1, \ldots, r\} \setminus J, \\ &m_{j_k} - n = \mu_k \text{ is constant} && \text{for } j = j_k \in J. \end{aligned}$$

This leads to our final result, which we also prove in section 6.

Theorem 2.8. *In the above setting, we put*

$$c_n = n \prod_{j \notin J} (m_j - n). \tag{2.35}$$

Then the kernels K_n from (2.33) *have the following hard edge scaling limit,*

$$\begin{aligned}\lim_{n\to\infty}&\frac{1}{c_n}K_n\left(\frac{x}{c_n},\frac{y}{c_n}\right)\\ &=\frac{1}{(2\pi i)^2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}ds\int_\Sigma dt\prod_{j=0}^{r}\frac{\Gamma(s+1+\nu_j)}{\Gamma(t+1+\nu_j)}\frac{\sin\pi s}{\sin\pi t}\prod_{k=1}^{q}\frac{\Gamma(t+1+\mu_k)}{\Gamma(s+1+\mu_k)}\frac{x^ty^{-s-1}}{s-t}\\ &=-\int_0^1 G^{1,q}_{q,r+1}\left(\begin{matrix}-\mu_1,\ldots,-\mu_q\\ 0;-\nu_1,\ldots,-\nu_r\end{matrix}\middle|ux\right)G^{r,0}_{q,r+1}\left(\begin{matrix}\mu_1,\ldots,\mu_q\\ \nu_1,\ldots,\nu_r;0\end{matrix}\middle|uy\right)du,\end{aligned}\tag{2.36}$$

where Σ is a contour around the positive real axis in the half-plane $\operatorname{Re} t>-\frac{1}{2}$.

The kernel (2.36) reduces to the Meijer G-kernel described in [48] in case $q=0$. These kernels appeared before for limits of products of Ginibre matrices [48], products with inverses of Ginibre matrices [26], for biorthogonal ensembles [47], and also as limits for Cauchy two matrix models [13, 14, 28]. According to Theorem 2.8 we obtain the same limits for products of truncated unitary matrices provided that the dimensions m_j of the underlying unitary matrices become large compared to n, in the sense that $m_j-n\to+\infty$ for every j.

The kernels (2.36) are new for $q\geq 1$, and these are finite rank perturbation of the Meijer G-kernels from [48]. To see this, we recall that $\mu_k\geq\nu_{j_k}+1$. Let us assume, for notational simplicity, that $j_k=r-q+k$ in (2.34). Then

$$R(t):=\prod_{k=1}^{q}\frac{\Gamma(t+1+\mu_k)}{\Gamma(t+1+\nu_{r-q+k})}\tag{2.37}$$

is a polynomial of degree $\deg R=\sum_{k=1}^q(\mu_k-\nu_{r-q+k})$, and (2.36) can be written as

$$\frac{1}{(2\pi i)^2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}ds\int_\Sigma dt\prod_{j=0}^{r-q}\frac{\Gamma(s+1+\nu_j)}{\Gamma(t+1+\nu_j)}\frac{\sin\pi s}{\sin\pi t}\frac{R(t)}{R(s)}\frac{x^ty^{-s-1}}{s-t}.\tag{2.38}$$

This is indeed a finite rank perturbation of the Meijer G-kernel with parameters $\nu_1,\ldots,\nu_{r-q}$, since R is a polynomial. In particular for $q=r-1$, it is a finite rank modification of the hard edge Bessel kernel. Such finite rank modifications were also obtained in [23] in a somewhat different context.

In the ensuing sections we prove our statements. We start in section 3 with the proof of Theorem 2.3 since it is used in the first proof of Theorem 2.1 that we present in section 4. The second proof is in section 5. This proof is a rather lenghty sequence of matrix integral evaluations and we have broken it up into six steps. The proofs of Proposition 2.7 and Theorem 2.8 are shown in section 6.

3 Proof of Theorem 2.3

For the proof of Theorem 2.3, we need the Ingham-Siegel formula [39, 53]

$$\int_{\mathcal{H}(n)}\exp[i\operatorname{Tr}HX]\det(H-zI_n)^{-n-p}dH=c\exp[iz\operatorname{Tr}X]\det X^p\,\theta(X)\tag{3.1}$$

with a normalization constant c depending on p and n, only, which can be fixed by the choice $X = I_n$ and $z = i$. This integral is valid for Hermitian matrices $X \in \mathcal{H}(n)$ and $\operatorname{Im} z > 0$. The integral is over the space $\mathcal{H}(n) = \mathfrak{gl}(n)/\mathfrak{u}(n)$ of $n \times n$ Hermitian matrices H with the flat Lebesgue measure,

$$dH = \prod_{j=1}^{n} dH_{jj} \prod_{j<k} d\operatorname{Re} H_{jk}\, d\operatorname{Im} H_{jk}. \tag{3.2}$$

Here $\mathfrak{gl}(n)$ and $\mathfrak{u}(n)$ are the Lie algebras of the general linear and the unitary group, respectively. If p is not an integer then we define via the spectral representation,

$$\det(H - zI_n)^{-n-p} = \prod_{j=1}^{n} (h_j - z)^{-n-p}, \tag{3.3}$$

where $h_1, \ldots, h_n$ are the real eigenvalues of the Hermitian matrix H and $(h - z)^{-n-p}$ is defined in the complex h-plane with a branch cut along $\{z+iy \mid y \geq 0\}$, and with $h^{n+p}(h - z)^{-n-p} \to 1$ as $h \to +\infty$.

For $n = 1$ the Ingham-Siegel formula (3.1) reduces to

$$\int_{-\infty}^{\infty} \frac{e^{ixs}}{(s - z)^{1+p}} ds = \begin{cases} \dfrac{2\pi i e^{\pi i p/2}}{\Gamma(p+1)} e^{izx} x^p, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{3.4}$$

which is valid for $\operatorname{Im} z > 0$ and $p \geq 0$. It can be verified by contour integration.

Proof of Theorem 2.3 We consider the integral

$$J(A, B) = \int_{\mathcal{U}(n)} \det(A - UBU^*)^p\, \theta(A - UBU^*)\, dU. \tag{3.5}$$

The determinant can be rewritten via the Ingham-Siegel formula (3.1) identifying $X = A - UBU^*$. We obtain

$$J(A, B) \propto e^{-iz \operatorname{Tr}(A-B)} \int_{\mathcal{U}(n)} \int_{\mathcal{H}(n)} e^{i \operatorname{Tr} H(A-UBU^*)} \det(H - zI_n)^{-n-p} dH dU \tag{3.6}$$

Both integrals are absolutely integrable because the integral over U is over a compact set with a continuous integrand and the integral over H is bounded by $|\Delta(h) \prod_{j=1}^{n} (h_j - z)^{-n-p}/\Delta(a)|$ for $\sum_{j=1}^{n} h_j^2 \to \infty$. Recall that $a_1, \ldots, a_n$ are the real, pairwise distinct eigenvalues of A. Hence we can interchange the order of integration.

The integral over the unitary group is evaluated with the HCIZ formula (2.4). Then (3.6) is up to a constant

$$J(A, B) \propto e^{-iz \operatorname{Tr}(A-B)} \int_{\mathcal{H}(n)} e^{i \operatorname{Tr} HA} \det(H - zI_n)^{-n-p} \frac{\det\left[e^{-ih_j b_k}\right]}{\Delta(h)\Delta(b)} dH. \tag{3.7}$$

We write $H = VhV^*$ for the eigenvalue decomposition of H where $V \in \mathcal{U}(n)$ and $h = \operatorname{diag}(h_1, \ldots, h_n)$. Moreover we use that $dH \propto \Delta(h)^2\, dV dh_1 \ldots dh_n$, see

e.g. [21]. Using this in (3.7) leads to

$$J(A,B) \propto \frac{e^{-iz\,\mathrm{Tr}(A-B)}}{\Delta(b)} \int_{\mathbb{R}^n} \int_{\mathcal{U}(n)} e^{i\,\mathrm{Tr}\,VhV^*A} \prod_{j=1}^n (h_j - z)^{-n-p} \\ \times \Delta(h) \det\left[e^{-ih_j b_k}\right]_{j,k=1}^n dV\, dh_1 \cdots dh_n \tag{3.8}$$

The integral over $V \in \mathcal{U}(n)$ is again a HCIZ integral (2.4),

$$J(A,B) \propto \frac{e^{-iz\,\mathrm{Tr}(A-B)}}{\Delta(a)\Delta(b)} \int_{\mathbb{R}^n} \det\left[e^{ih_j a_k}\right]_{j,k=1}^n \prod_{j=1}^n (\lambda_j - z)^{-n-p} \\ \times \det\left[e^{-ih_j b_k}\right]_{j,k=1}^n dh_1 \cdots dh_n. \tag{3.9}$$

The factors in the product $\prod_{j=1}^n (\lambda_j - z)^{-n-p}$ can be pulled into either one of the two determinants. Then, Andréief's identity (2.8) can be applied to find

$$J(A,B) \propto \frac{e^{-iz\,\mathrm{Tr}(A-B)}}{\Delta(a)\Delta(b)} \det\left[\int_{-\infty}^{\infty} \frac{e^{is(a_j-b_k)}}{(s-z)^{n+p}} ds\right]_{j,k=1}^n. \tag{3.10}$$

The integral in the determinant is of the form (3.4) which is up to a constant equal to $e^{iz(a_j-b_k)}(a_j-b_k)_+^{n+p-1}$. The exponential factors $e^{iz(a_j-b_k)}$ inside the determinant cancel with those in front of the determinant. The resulting expression is

$$J(A,B) \propto \frac{\det\left[(a_j-b_k)_+^{n+p-1}\right]}{\Delta(a)\Delta(b)}, \tag{3.11}$$

which is up to a constant indeed the right hand side of (2.6).

Whenever $A - UBU^*$ is positive definite for all $U \in \mathcal{U}(n)$, the formula (2.6) reduces to

$$\int_{\mathcal{U}(n)} \det\left(A - UBU^*\right)^p dU = c_{n,p} \frac{\det\left[(a_j-b_k)^{p+n-1}\right]_{j,k=1}^n}{\Delta(a)\Delta(b)}. \tag{3.12}$$

This is equivalent to an integral given by Gross and Richards in [32, formula (3.21)], namely

$$\frac{\det\left[(1-s_j t_k)^{-a}\right]_{j,k=1}^n}{\Delta(s)\Delta(t)} = \tilde{c}_{n,a} \int_{\mathcal{U}(n)} \det\left(I_n - SUTU^*\right)^{-(a+n-1)} dU \tag{3.13}$$

with $\tilde{c}_{n,a} = \prod_{j=0}^{n-1} (a)_j / j!$. The formula (3.13) is valid whenever S and T are Hermitian matrices with eigenvalues $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$, respectively, satisfying $|s_j t_k| < 1$ for all $j,k = 1, \ldots, n$. The formulas (3.12) and (3.13) are related if we take $-a = p + n - 1$, $S = A^{-1}$ and $T = B$. The constants are related by $c_{n,p} = 1/\tilde{c}_{n,a}$, and the formula (2.7) follows which completes the proof of Theorem 2.3. □

4 First proof of Theorem 2.1

As already said, our first proof of Theorem 2.1 only works if $m \geq 2n + \nu$. In that case there is an explicit formula for the distribution of a truncation T of size $(n + \nu) \times n$, namely

$$c \det(I_n - T^*T)^{m-2n-\nu} \theta(I_n - T^*T)\, dT \tag{4.1}$$

with $dT = \prod_{j=1}^{n+\nu} \prod_{k=1}^{n} d\operatorname{Re} T_{jk}\, d\operatorname{Im} T_{jk}$ the flat Lebesgue measure on the space of $(n + \nu) \times n$ rectangular complex matrices and c is a constant. The function θ is the Heaviside step function of a matrix argument defined in (2.5). The (unordered) eigenvalues $t_1, \ldots, t_n$ of T^*T are in the interval $[0, 1]$ and have the joint probability density

$$\frac{1}{Z_n} \prod_{j<k} (t_k - t_j)^2 \prod_{j=1}^{n} t_j^{\nu} (1 - t_j)^{m-2n-\nu}, \qquad 0 \leq t_1, \ldots, t_n \leq 1. \tag{4.2}$$

This is an example of a Jacobi ensemble [18]. When $m < 2n + \nu$, then T^*T has the eigenvalue 1 with multiplicity of at least $2n + \nu - m$ and the density (4.1) is not valid anymore.

We follow the proof of Lemma 2.2 in [47] except that at a certain stage in the proof the HCIZ integral (2.4) is replaced by the integral (2.6).

Proof of Theorem 2.1 in the case $m \geq 2n + \nu$ As already discussed in Remark 2.2 we may restrict to the case $l = n$ without loss of generality.

Consider a fixed square matrix X of size $n \times n$ which is assumed to be invertible. The change of variables $T \mapsto Y = TX$ has the Jacobian, see e.g. [51, Theorem 3.2],

$$\det(X^*X)^{-(n+\nu)} = \prod_{k=1}^{n} x_k^{-n-\nu}. \tag{4.3}$$

The distribution (4.1) on T (where T has size $(n + \nu) \times n$) then transforms into the distribution

$$\begin{aligned} &\propto \prod_{k=1}^{n} x_k^{-n-\nu} \det(I_n - (X^{-1})^*Y^*YX^{-1})^{m-2n-\nu} \theta(I_n - (X^{-1})^*Y^*YX^{-1})\, dY \\ &= \prod_{k=1}^{n} x_k^{-m+n} \det(X^*X - Y^*Y)^{m-2n-\nu} \theta(X^*X - Y^*Y)\, dY. \end{aligned} \tag{4.4}$$

In the next step we perform a singular value decomposition $Y = V\Sigma U$ with Jacobian [24]

$$dY \propto \left(\prod_{j=1}^{n} y_j^{\nu} \right) \Delta(y)^2 dU dV dy_1 \ldots dy_n \tag{4.5}$$

written in terms of the squared singular values $y_1, \ldots, y_n$ of Y. The measure dU is the Haar measure on $\mathcal{U}(n)$ and dV is the invariant measure on $\mathcal{U}(n + \nu)/[\mathcal{U}^n(1) \times \mathcal{U}(\nu)]$. We use this fact in (4.4) to perform the integration of V,

which only contributes to the constant. This yields a probability measure on $\mathcal{U}(n) \times [0,\infty)^n$ proportional to

$$\propto \left(\prod_{k=1}^{n} x_k^{-m+n}\right)\left(\prod_{j=1}^{n} y_j^{\nu}\right) \det(X^*X - U^*\Sigma^2 U)^{m-2n-\nu} \times \theta(X^*X - U^*\Sigma^2 U)\Delta(y)^2\, dU dy_1 \cdots dy_n. \quad (4.6)$$

The integral over U in (4.6) can be done with the help of the integral (2.6) with $A = X^*X$, $B = \Sigma^2$, and $p = m - 2n - \nu$. This leads to the density for the squared singular values $y_1, \ldots, y_n$ of Y, given those of X, which is proportional to (2.3) and Theorem 2.1 follows for $m \geq 2n + \nu$. □

5 Second proof of Theorem 2.1

We underline that the second approach to prove Theorem 2.1 does not rely on the restriction $m \geq 2n + \nu$. As before we denote the set of $n \times n$ unitary matrices and of $n \times n$ Hermitian matrices by $\mathcal{U}(n)$ and $\mathcal{H}(n)$, respectively. We also use $\mathcal{M}(m,n)$ for the space of $m \times n$ complex matrices and abbreviate $\mathcal{M}(m) = \mathcal{M}(m,m)$.

Also in the second approach we assume that $l = n$ because it does not restrict generality, see Remark 2.2. We assume X to be a fixed $n \times n$ matrix with non-zero squared singular values.

5.1 Preliminaries

Let f be a symmetric function in n variables. We extend f to Hermitian matrices A by defining $f(A) = f(a_1, \ldots, a_n)$ if $a_1, \ldots, a_n$ are the eigenvalues of A. With a slight abuse of notation we also define $f(B)$ for $(n+\nu) \times (n+\nu)$ matrices B having ν eigenvalues equal to 0, by putting $f(B) = f(b_1, \ldots, b_n)$ if $b_1, \ldots, b_n$ are the non-zero eigenvalues of B.

Then our aim is to prove that for all continuous symmetric functions f on $[0,\infty)^n$, we have

$$\mathbb{E}\left[f(Y^*Y)\right] = \int_{[0,\infty)^n} f(y_1, \ldots, y_n) p_{X,Y}(x,y) dy_1 \ldots dy_n \quad (5.1)$$

where, for a given X, $y \mapsto p_{X,Y}(x,y)$ denotes the density from (2.3). It will be enough to prove (5.1) for symmetric polynomial functions f, since the density $p_{X,Y}$ has a compact support and the symmetric polynomials are then obviously uniformly dense in the set of all continuous symmetric functions.

Note that, by our definition of f on matrices, we have $f(Y^*Y) = f(YY^*) = f(TXX^*T^*)$. Since T is the truncation of a Haar distributed unitary matrix U

of size $m \times m$, we have

$$\begin{aligned}\mathbb{E}\left[f(YY^*)\right] &= \int_{\mathcal{U}(m)} f\left(\begin{pmatrix} I_{n+\nu} & O_{n+\nu,m-n-\nu}\end{pmatrix} \widetilde{U} X X^* \widetilde{U}^* \begin{pmatrix} I_{n+\nu} \\ O_{m-n-\nu,n+\nu}\end{pmatrix}\right) dU \\ &\propto \int_{\mathcal{M}(m,n)} f\left(\begin{pmatrix} I_{n+\nu} & O_{n+\nu,m-n-\nu}\end{pmatrix} M X X^* M^* \begin{pmatrix} I_{n+\nu} \\ O_{m-n-\nu,n+\nu}\end{pmatrix}\right) \\ &\quad \times \prod_{1\le j<k\le n} \delta^{(2)}(\{M^*M\}_{jk}) \prod_{j=1}^{n} \delta(\{M^*M\}_{jj}-1) dM, \end{aligned} \tag{5.2}$$

where dU is the normalized Haar measure on the unitary group $\mathcal{U}(m)$. The matrix $O_{p,q}$ is the zero matrix of size $p \times q$. The complex $m \times n$ matrix

$$\widetilde{U} = U \begin{pmatrix} I_n \\ O_{m-n,n}\end{pmatrix} \tag{5.3}$$

is an element in the Stiefel manifold $\mathcal{U}(m)/\mathcal{U}(m-n)$. The orthonormality of the columns of $\widetilde{U}$ can be enforced by n^2 Dirac delta functions (recall that $\delta^{(2)}$ is the one for complex numbers). In this way we integrate in (5.2) over the larger space $\mathcal{M}(m,n)$. See also the discussion in [19].

The complex matrix M can be partitioned into two blocks

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \tag{5.4}$$

with M_1 an $(n+\nu) \times n$ complex matrix and M_2 an $(m-n-\nu) \times n$ complex matrix. Then we have to calculate

$$\begin{aligned}\mathbb{E}\left[f(YY^*)\right] &= \int_{\mathcal{M}(n+\nu,m)} f\left(M_1 X X^* M_1^*\right) \\ &\quad \times \prod_{1\le j<k\le n+\nu} \delta^{(2)}(\{M^*M\}_{jk}) \prod_{j=1}^{n+\nu} \delta(\{M^*M\}_{jj}-1) dM. \end{aligned} \tag{5.5}$$

5.2 Proof of Theorem 2.1

To establish (5.1) we proceed in six steps.

Step 1: Matrix delta function In the first step we rewrite the Dirac delta functions in (5.2) as Fourier-Laplace transforms [19] where

$$\begin{aligned}&\prod_{1\le j<k\le n} \delta^{(2)}(\{M^*M\}_{jk}) \prod_{j=1}^{n} \delta(\{M^*M\}_{jj}-1) \\ &\quad = \lim_{t\to 0} \frac{1}{2^n \pi^{n^2}} \int_{\mathcal{H}(n)} \exp[\mathrm{Tr}(I_n - iK)(I_n - M^*M) - t\,\mathrm{Tr}\,K^2] dK \end{aligned} \tag{5.6}$$

with an integration over the space $\mathcal{H}(n)$ of Hermitian $n \times n$ matrices K. For an integration over the whole group $\mathcal{U}(m)$, i.e. $m = n$, this integration is equal to the one in [37, formula (13)]. The shift of the matrix K by iI_n ensures the

absolute integrability of the integral over M and the Gaussian incorporating the limit in the auxiliary variable t guarantees the absolute integrability in K. Note that the limit has to be understood as a limit in the weak topology meaning that we have to integrate over M, first, and, then, take the limit $t \to 0$. Hence the integral (5.5) reads

$$\mathbb{E}\left[f(YY^*)\right] \propto \lim_{t\to 0} \int_{\mathcal{M}(m,n)} f\left(M_1 X X^* M_1^*\right) \times \int_{\mathcal{H}(n)} \exp[\operatorname{Tr}(I_n - iK)(I_n - M_1^* M_1 - M_2^* M_2) - t \operatorname{Tr} K^2] dK dM \quad (5.7)$$

up to a constant only depending on m, n and ν.

Both integrals are absolutely integrable. Therefore we can interchange the integrals. The integral over the matrix M_2 is a Gaussian integral yielding

$$\mathbb{E}\left[f(YY^*)\right] \propto \lim_{t\to 0} \int_{\mathcal{H}(n)} \int_{\mathcal{M}(n+\nu,n)} f\left(M_1 X X^* M_1^*\right) \det(I_n - iK)^{-m+n+\nu} \times \exp[\operatorname{Tr}(I_n - iK)(I_n - M_1^* M_1) - t \operatorname{Tr} K^2] dM_1 dK. \quad (5.8)$$

Finally we take $t \to 0$. This can be done because f is polynomial. In the case $f = 1$ the integral over M_1 yields an additional factor $\det(I_{n+\nu} - iK)^{-n}$ ensuring the absolute integrability also at $t = 0$. Since the function f is polynomial we know that $f\left(M_1 X X^* M_1^*\right)$ is a polynomial in the matrix entries of M_1. Thus the Gaussian integral over M_1 yields a polynomial in $(I_{n+\nu} - iK)^{-1}$ with the lowest order to be $\det(I_{n+\nu} - iK)^{-n}$. Therefore the integrand of the K-integral after integrating over M_1, first, is indeed absolutely integrable also at $t = 0$ such that

$$\mathbb{E}\left[f(YY^*)\right] \propto \int_{\mathcal{H}(n)} \int_{\mathcal{M}(n+\nu,n)} f\left(M_1 X X^* M_1^*\right) \det(I_n - iK)^{-m+n+\nu} \times \exp[\operatorname{Tr}(I_n - iK)(I_n - M_1^* M_1)] dM_1 dK. \quad (5.9)$$

We underline that now the order of the integrals is crucial and cannot be interchanged.

Step 2: Changes of variable The change of variables $M_1 \mapsto S = M_1 X$ has the Jacobian

$$\det(X^* X)^{-n-\nu} = \prod_{j=1}^{n} x_j^{-n-\nu} \quad (5.10)$$

and the change of variables $K \mapsto \tilde{K} = X^{-1} K (X^{-1})^*$ on the space of Hermitian matrices yields

$$\det(X^* X)^{n} = \prod_{j=1}^{n} x_j^{n}. \quad (5.11)$$

Applying this to (5.9) (and dropping the tilde from $\tilde{K}$), we obtain

$$\begin{aligned}\mathbb{E}\left[f(YY^*)\right] &\propto \prod_{j=1}^{n} x_j^{-\nu} \int_{\mathcal{H}(n)} dK \det(I_n - iXKX^*)^{-(m-n-\nu)} \\ &\quad \times \int_{\mathcal{M}(n+\nu,n)} dS f(SS^*) e^{\mathrm{Tr}(I_n - iXKX^*)(I_n - (X^{-1})^* S^* S X^{-1})} \\ &= \prod_{j=1}^{n} x_j^{-m+n} \int_{\mathcal{H}(n)} dK \det((X^*X)^{-1} - iK)^{-(m-n-\nu)} \\ &\quad \times \int_{\mathcal{M}(n+\nu,n)} dS f(SS^*) e^{\mathrm{Tr}((X^*X)^{-1} - iK)(X^*X - S^*S)} \\ &= \prod_{j=1}^{n} x_j^{-m+n} \int_{\mathcal{H}(n)} \Psi(K + i(X^*X)^{-1} - iI_n) dK, \end{aligned} \tag{5.12}$$

where Ψ is defined by

$$\Psi(K) := \det(I_n - iK)^{-(m-n-\nu)} \int_{\mathcal{M}(n+\nu,n)} f(SS^*) e^{\mathrm{Tr}(I_n - iK)(X^*X - S^*S)} dS. \tag{5.13}$$

Step 3: Shift in the matrix K In this step we prove the following Lemma.

Lemma 5.1. *For any matrix A with $\mathrm{Im}\, A := (A - A^*)/2i > -I_n$ (which means that $I_n + \mathrm{Im}\, A$ is a positive definite Hermitian matrix), we have*

$$\int_{\mathcal{H}(n)} \Psi(K + A) dK = \int_{\mathcal{H}(n)} \Psi(K) dK. \tag{5.14}$$

Proof. If A is Hermitian then we can simply apply the linear change of variables $K + A \mapsto K$ to obtain (5.14). Therefore we may restrict to the case $A = iB$ with B Hermitian and $B + I_n$ positive definite. We may also restrict to the case where B is a diagonal matrix. To see this we write Ψ_X to indicate that the definition (5.13) depends on X. Then for a unitary matrix U one has

$$\Psi_X(UKU^*) = \Psi_{U^*X}(K), \tag{5.15}$$

which follows from inserting UKU^* into the definition (5.13) and changing variables $S \mapsto SU^*$. Recall that $f(SS^*) = f(USS^*U^*)$ for all $U \in \mathcal{U}(n)$ because f only depends on the squared singular values of S. The invariance (5.15) implies by the unitary invariance of dK, see e.g. [21], that

$$\int_{\mathcal{H}(n)} \Psi_X(K + iB) dK = \int_{\mathcal{H}(n)} \Psi_{U^*X}(K + iU^*BU) dK. \tag{5.16}$$

and we may choose the unitary matrix U so that U^*BU is diagonal.

Let $p = m - n - \nu > 0$ and $A = iB = i\,\mathrm{diag}(b_1, \ldots, b_n)$ a diagonal matrix with $b_j > -1$, for $j = 1, \ldots, n$. Note that for $0 \le t \le 1$,

$$\begin{aligned}|\Psi(K + itB)| &\le |\det(I_n + tB - iK)|^{-p} \int_{\mathcal{H}(n)} |f(SS^*)|\, e^{\mathrm{Tr}(I_n + tB)(XX^* - S^*S)} dS \\ &\le C_0 \,|\det(I_n + tB - iK)|^{-p}. \end{aligned} \tag{5.17}$$

The prefactor is a finite constant $C_0 > 0$ depending on B and X but is independent of $t \in [0,1]$. Since all $b_j > -1$, it is clear that $I_n + tB$ is a positive definite matrix. Hence there exists a constant $C_1 > 0$, independent of t, such that $|\det(I_n + tB)| > C_1$ resulting in

$$|\det(I_n + tB - iK)| > C_1 \, |\det(I_n - iM_t)| \,, \tag{5.18}$$

where

$$M_t := (I_n + tB)^{-\frac{1}{2}} K (I_n + tB)^{-\frac{1}{2}} \tag{5.19}$$

is a Hermitian matrix. Let $\lambda_1(t), \ldots, \lambda_n(t)$ denote the eigenvalues of M_t. Then it is easy to check that, since the eigenvalues $\lambda_j(t)$ are real,

$$\begin{aligned} |\det(I_n - iM_t)| &= \prod_{k=1}^{n} |1 - i\lambda_k(t)| = \prod_{k=1}^{n} \sqrt{1 + \lambda_k^2(t)} \\ &\geq \max_{k=1,\ldots,n} |\lambda_k(t)| = \max_{x \in \mathbb{C}^n \setminus \{0\}} \frac{|\langle M_t x, x \rangle|}{\langle x, x \rangle} \end{aligned} \tag{5.20}$$

by the properties of Rayleigh quotients. Taking $x = x_j = (I_n + tB)^{\frac{1}{2}} e_j$ in (5.20) and noting that $\langle M_t x_j, x_j \rangle = K_{jj}$, see (5.19), we obtain

$$|\det(I_n - iM_t)| \geq \max_{j=1,\ldots,n} \frac{|K_{jj}|}{\langle x_j, x_j \rangle}. \tag{5.21}$$

The vectors x_j depend on t, but their norms are uniformly bounded from below $\langle x_j, x_j \rangle \geq 1 + t b_{\min} > C > 0$ for $t \in [0,1]$ where $b_{\min} = \min_{j=1,\ldots,n} b_j$ and $C = 1$ if $b_{\min} \geq 0$ or $C = 1 - b_{\min}$ if $0 \geq b_{\min} > -1$. Then, combining (5.17), (5.18) and (5.21), we obtain

$$|\Phi(K + tiB)| \leq \frac{C_2}{\max_{j=1,\ldots,n} |K_{jj}|^p}, \tag{5.22}$$

for some constant $C_2 > 0$, independent of t.

In the integral $\int_{\mathcal{H}(n)} \Psi(K) dK$ we first do the integration over the diagonal elements K_{jj} for $j = 1, \ldots, n$. The integrand is analytic in each of the K_{jj}. Because of the estimate (5.22) with $p \geq 1$, the integral can be deformed from the real line to the horizontal line in the complex K_{jj}-plane with imaginary part equal to b_j. We do this for all diagonal entries resulting in (5.14). □

Lemma 5.1 can be applied to (5.12) because $A = i(X^*X)^{-1} - iI_n$ satisfies $\operatorname{Im} A = (X^*X)^{-1} - I_n > -I_n$. Hence we have

$$\begin{aligned} \mathbb{E}\left[f(YY^*)\right] \propto \prod_{j=1}^{n} x_j^{-m+n} \int_{\mathcal{H}(n)} dK \, \det(I_n - iK)^{-(m-n-\nu)} \\ \times \int_{\mathcal{M}(n+\nu,n)} dS f(SS^*) e^{\operatorname{Tr}(I_n - iK)(X^*X - S^*S)}. \end{aligned} \tag{5.23}$$

Step 4: Singular value decomposition of S We take the singular value decomposition $S = V_1 \Sigma V_2$ where $V_1 \in \mathcal{U}(n+\nu)/[\mathcal{U}^n(1) \times \mathcal{U}(\nu)]$, $V_2 \in \mathcal{U}(n)$, and Σ is a diagonal matrix with the singular values of S. The Jacobian of this

transformation is proportional to $\Delta(y)^2 \prod_{j=1}^n y_j^\nu$, see [24], where $y_1, \ldots, y_n$ are the squared singular values. Recall that f is defined as a symmetric function on the eigenvalues. Then (5.13) reads

$$\Psi(K) \propto \frac{e^{\mathrm{Tr}(I_n - iK)(X^*X)}}{\det(I_n - iK)^{m-n-\nu}} \int_{[0,\infty)^n} f(y_1, \ldots, y_n) \prod_{j=1}^n y_j^\nu \Delta(y)^2 dy_1 \ldots dy_n \\ \times \int_{\mathcal{U}(n)} e^{-\mathrm{Tr}(I_n - iK)V_2^* \Sigma^2 V_2} dV_2 \quad (5.24)$$

The integral over $V_2 \in \mathcal{U}(n)$ is a HCIZ integral (2.4). Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of K, then the HCIZ integral yields a term proportional to $\det[e^{-(1-i\lambda_j)y_k}]$ $/[\Delta(\lambda)\Delta(y)]$. We end up with

$$\Psi(K) \propto \frac{e^{\mathrm{Tr}(I_n - iK)(X^*X)}}{\det(I_n - iK)^{m-n-\nu}\Delta(\lambda)} \\ \times \int_{[0,\infty)^n} f(y_1, \ldots, y_n) \prod_{j=1}^n y_j^\nu \Delta(y) \det\left[e^{-(1-i\lambda_j)y_k}\right]_{j,k=1}^n dy_1 \ldots dy_n. \quad (5.25)$$

Step 5: Eigenvalue decomposition of K Next, we decompose $K = V_0 \Lambda V_0^*$ in a unitary matrix $V_0 \in \mathcal{U}(n)/\mathcal{U}^n(1)$ and its eigenvalues $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. The Jacobian is proportional to $\Delta(\lambda)^2$, see e.g. [21], and by (5.23) and (5.25) we have

$$\mathbb{E}\left[f(YY^*)\right] \propto \prod_{j=1}^n x_j^{-m+n} \int_{\mathbb{R}^n} \prod_{j=1}^n (1 - i\lambda_j)^{-m+n+\nu} \Delta(\lambda) d\lambda_1 \cdots d\lambda_n \\ \times \int_{[0,\infty)^n} f(y_1, \ldots, y_n) \prod_{j=1}^n y_j^\nu \Delta(y) \det\left[e^{-(1-i\lambda_j)y_k}\right]_{j,k=1}^n dy_1 \cdots dy_n \\ \times \int_{\mathcal{U}(n)} e^{\mathrm{Tr}(I_n - iV_0 \Lambda V_0^*)(X^*X)} dV_0. \quad (5.26)$$

The V_0 integral is again a HCIZ integral (2.4), and it gives a contribution proportional to $\det[e^{(1-i\lambda_j)x_k}]/[\Delta(\lambda)\Delta(x)]$. Plugging this term into (5.26) and noting that we may change the order of integration at this stage, we find

$$\mathbb{E}\left[f(YY^*)\right] = \int_{[0,\infty)^n} f(y_1, \ldots, y_n) p_{X,Y}(x, y) dy_1 \cdots dy_n \quad (5.27)$$

where

$$p_{X,Y}(x, y) \propto \left(\prod_{j=1}^n x_j^{-m+n}\right) \left(\prod_{j=1}^n y_j^\nu\right) \frac{\Delta(y)}{\Delta(x)} \times \\ \int_{\mathbb{R}^n} \prod_{j=1}^n (1 - i\lambda_j)^{-(m-n-\nu)} \det\left[e^{(1-i\lambda_j)x_k}\right]_{j,k=1}^n \det\left[e^{-(1-i\lambda_j)y_k}\right]_{j,k=1}^n d\lambda_1 \cdots d\lambda_n. \quad (5.28)$$

Step 6: Andréief formula We calculate the integral over $\lambda_1, \ldots, \lambda_n$ in (5.28) via the Andréief formula (2.8). This gives the determinant

$$n! \det \left[\int_{-\infty}^{\infty} \frac{e^{(x_k - y_j)(1-i\lambda)}}{(1-i\lambda)^{m-n-\nu}} d\lambda \right]_{j,k=1}^{n} \tag{5.29}$$

with integrals of the form (3.4). The result is

$$\int_{-\infty}^{\infty} \frac{e^{(x_k - y_j)(1-i\lambda)}}{(1-i\lambda)^{m-n-\nu}} d\lambda = \frac{2\pi}{(m-n-\nu-1)!} (x_k - y_j)_+^{m-n-\nu-1}. \tag{5.30}$$

We use this in (5.27) and (5.28) and obtain (5.1). This concludes concludes the proof of the Theorem 2.1. □

6 Proofs of Proposition 2.7 and Theorem 2.8

We first prove Proposition 2.7 and Theorem 2.8 afterwards.

Proof of Proposition 2.7 Recall that P_k and Q_k are defined by (2.31) and (2.32), respectively, for $k = 0, \ldots, n-1$. Then Q_k is in the linear span of $w_1^{(r)}, \ldots, w_n^{(r)}$ since it is of the form (2.27) with $q(s) = (s-k)_k$ a polynomial of degree $k \leq n-1$. Also note that the integrand in (2.31) has poles at $t = 0, \ldots, k$, and no other poles inside Σ_k. Thus if we evaluate (2.31) by means of the residue theorem and find contributions from $t = 0, \ldots, k$, only. Hence the function P_k is a polynomial of degree k. It remains to verify the biorthogonality (2.29).

Let $l, k = 0, 1, \ldots, n-1$. From (2.31) we have

$$\int_0^1 P_l(x) Q_k(x) dx$$
$$= \frac{1}{2\pi i} \oint_{\Sigma_l} \frac{1}{(t-l)_{l+1}} \prod_{j=1}^{r} \frac{\Gamma(t+1+m_j-n)}{\Gamma(t+1+\nu_j)} \int_0^1 x^t Q_k(x) dx dt. \tag{6.1}$$

The moments of Q_k are given by the general identity for the moments of the Meijer G-function,

$$\int_0^1 x^{s-1} Q_k(x) dx = (s-k)_k \prod_{j=1}^{r} \frac{\Gamma(s+\nu_j)}{\Gamma(s+m_j-n)}. \tag{6.2}$$

This identity can be plugged into (6.1) which cancels a lot of Γ-factors,

$$\int_0^1 P_l(x) Q_k(x) dx = \frac{1}{2\pi i} \oint_{\Sigma_l} \frac{(t+1-k)_k}{(t-l)_{l+1}} dt = \frac{1}{2\pi i} \oint_{\Sigma_l} \frac{\Gamma[t-l]}{\Gamma[t+1-k]} dt. \tag{6.3}$$

For $l < k$, the integrand in (6.3) is a polynomial and the integral is zero by Cauchy's theorem. For $l \geq k$, there are poles at $t = k, \ldots, l$ which are all inside the contour Σ_l. The integrand is a rational function that behaves like $O(t^{k-l-1})$ as $|t| \to \infty$. Therefore by simple residue calculation at infinity, the integral also vanishes if $l > k$ and it is equal to 1 if $l = k$. Thus (2.29) is satisfied.

Inserting (2.31) and (2.32) into (2.28), and noting that we can take the same contour Σ_n for every k in (2.31), we obtain a double integral for K_n,

$$K_n(x,y) = \frac{1}{(2\pi i)^2}\int_C ds \oint_{\Sigma_n} dt \prod_{j=1}^{r} \frac{\Gamma(s+\nu_j)\Gamma(t+1+m_j-n)}{\Gamma(s+m_j-n)\Gamma(t+1+\nu_j)} \sum_{k=0}^{n-1} \frac{(s-k)_k}{(t-k)_{k+1}} x^t y^{-s}$$
$$= \frac{1}{(2\pi i)^2}\int_C ds \oint_{\Sigma_n} dt \prod_{j=1}^{r} \frac{\Gamma(s+1+\nu_j)\Gamma(t+1+m_j-n)}{\Gamma(s+1+m_j-n)\Gamma(t+1+\nu_j)} \sum_{k=0}^{n-1} \frac{(s+1-k)_k}{(t-k)_{k+1}} x^t y^{-s-1}, \tag{6.4}$$

where we made the change of variables $s \mapsto s+1$. The summation can be simplified, because of the telescoping sum

$$\sum_{k=0}^{n-1} \frac{(s+1-k)_k}{(t-k)_{k+1}} = \frac{1}{s-t}\left(\frac{\Gamma(s+1)}{\Gamma(t+1)}\frac{\Gamma(t+1-n)}{\Gamma(s+1-n)} - 1\right), \tag{6.5}$$

which can be readily checked by complete induction. The contours Σ_n and C do not intersect. Therefore the term $t = s$ is not a pole inside the contour Σ_n. The double integral (6.4) splits into two terms due to (6.5). For the second term the contour integral over t does not encircle any pole because of $m_j - n > \nu_j$ and $m_j - n, \nu_j \in \mathbb{N}_0$. Thus this term vanishes. The remaining term is the one shown in (2.33).

The expression (2.33) in terms of Meijer G-functions is obtained by noticing that

$$\frac{1}{s-t} = -\int_0^1 u^{t-s-1} du. \tag{6.6}$$

Interchanging the s and t integral with the u integral and using the definition of the Meijer G-functions (2.19) the identity follows, see also the proof of Theorem 5.3 in [48]. This concludes the proof. □

Proof of Theorem 2.8 We employ the following asymptotic behavior of a ratio of Gamma functions

$$\frac{\Gamma(t+1+m_j-n)}{\Gamma(s+1+m_j-n)} = \begin{cases} \dfrac{\sin \pi s}{\sin \pi t}\, n^{t-s}\left(1 + O\left(\dfrac{1}{n}\right)\right), & \text{for } j = 0, \\ (m_j - n)^{t-s}\left(1 + O\left(\dfrac{1}{m_j - n}\right)\right), & \text{for } j \in \{1,\dots,r\} \setminus J \end{cases} \tag{6.7}$$

as $n \to \infty$. This follows as in the proof of Theorem 5.3 in [48], since $m_0 = 0$ and $m_j - n \to \infty$ as $n \to \infty$ for $j \in \{1,\dots,r\} \setminus J$.

In the double integral formula in (2.33) we deform the contour Σ_n to Σ, and obtain

$$\frac{1}{c_n} K_n\left(\frac{x}{c_n}, \frac{y}{c_n}\right)$$
$$= \frac{c_n^{s-t}}{(2\pi i)^2}\int_C ds \oint_{\Sigma} dt \prod_{j=0}^{r} \frac{\Gamma(s+1+\nu_j)}{\Gamma(t+1+\nu_j)} \prod_{j=0}^{r} \frac{\Gamma(t+1+m_j-n)}{\Gamma(s+1+m_j-n)} \frac{x^t y^{-s-1}}{s-t}. \tag{6.8}$$

Because of definition (2.35) we have

$$c_n^{s-t}\prod_{j=0}^{r}\frac{\Gamma(t+1+m_j-n)}{\Gamma(s+1+m_j-n)}=\left(n^{s-t}\frac{\Gamma(t+1+m_0-n)}{\Gamma(s+1+m_0-n)}\right)$$
$$\times\prod_{j\notin J}\left((m_j-n)^{s-t}\frac{\Gamma(t+1+m_j-n)}{\Gamma(s+1+m_j-n)}\right)\prod_{j\in J}\frac{\Gamma(t+1+m_j-n)}{\Gamma(s+1+m_j-n)}. \quad (6.9)$$

Each of the factors in the product has a finite limit as $n\to\infty$, cf. (6.7), and the full product tends to

$$\frac{\sin\pi s}{\sin\pi t}\prod_{k=1}^{q}\frac{\Gamma(t+1+\mu_k)}{\Gamma(s+1+\mu_k)} \quad (6.10)$$

since $m_j-n=\mu_k$ if $j=j_k\in J$. It is then allowed to take the limit inside the double integral in (6.8) and deform the contour C to the vertical line $\operatorname{Re}s=-1/2$ by analyticity and the decay of the integrand at infinity in the s-variable. This leads to (2.36).

The expression in terms of Meijer G-functions is obtained by using the reflection formula of the Gamma function, $\sin\pi z=\pi/[\Gamma(1-z)\Gamma(z)]$ together with (6.6). The identity now follows along the same lines as in the proof of Proposition 2.7. □

7 Conclusions and Outlook

We analyzed the singular value statistics of a matrix X (fixed as well as randomly chosen) multiplied by a truncated unitary matrix T which is distributed by the induced Haar measure. Though we only considered a multiplication from the left side TX one can easily generalize the results to a multiplication from both sides $T_{\mathrm{L}}XT_{\mathrm{R}}$ where T_{L} and T_{R} are truncations of two independent unitary matrices. The reason for such a simple generalization is the determinantal point process fulfilled by the joint probability density of the singular values. We proved that the joint probability density of the squared singular values of TX satisfies a polynomial ensemble if the joint probability density of the squared singular values of X does. In particular with the help of our results one can calculate the squared singular value statistics of any product $T_{\mathrm{L},1}\cdots T_{\mathrm{L},\mathrm{r_L}}XT_{\mathrm{R},1}\cdots T_{\mathrm{R},\mathrm{r_R}}$ with $r_{\mathrm{L}},r_{\mathrm{R}}\in\mathbb{N}_0$ and $T_{\mathrm{L,j}}$ and $T_{\mathrm{R,j}}$ truncations of independent unitary matrices and X either fixed or another random matrix. Indeed one can also mix products of truncated unitary matrices with Ginibre matrices. The combination of Theorem 1.1 and Corollary 2.4 yields a polynomial ensemble for the squared singular values of such a mixed product. We expect that also for such a mixed product the statistics are governed by Meijer G-functions since this particular kind of functions is closed under Mellin convolution as it was shown here for a pure product of truncated unitary matrices and studied in [5, 6] for a product of Ginibre matrices.

Our study shows that the determinantal point process also applies to a product with truncated unitary matrices. In particular one needs a group integral replacing the Harish Chandra/ Itzykson-Zuber integral [35, 42] for which it is not immediate that the result can be written in terms of determinants, cf. Theorem 2.3. This particular result is even more astounding when noticing that

one does not effectively integrate over the whole unitary group but only over a subset. The reason for this is a Heaviside step function in the integrand. Harish Chandra made contact between group integrals and representation theory [35]. It would be interesting if also something like this exists for the integral considered by us and can be explained by group theory.

Additionally we looked at the spectral statistics of a product of truncated unitary matrices in detail. For this product we calculated the kernel of the corresponding polynomial ensemble of the squared singular values at finite matrix size and in the hard edge limit at the origin. In a forthcoming publication we also derive the bulk, the soft and the hard edge statistics as it was very recently done for the Ginibre ensemble in [49]. The latter two statistics may appear at the lower and the upper bound of the support because the squared singular values of a product of truncated unitary matrices always live on the interval $[0, 1]$. If the support touches either the origin or the upper bound 1 one would expect hard edge statistics at these points.

Another generalization of our results refers to the restriction that the first matrix in the product of truncated unitary matrices has to satisfy $m_1 > 2n+\nu_1$, see Corollary 2.4. This matrix has not generally to be the first matrix T_1. With the help of the discussion above it can be any matrix in the product. Nevertheless we have to assume that at least one truncated unitary matrix multiplied has to satisfy the condition to prove Corollary 2.4 in the way we have done. An interesting question would be: What happens if this restriction is not met? From the spectral statistics of one truncated unitary matrix, see e.g. [58, 25], we know that some singular values are exactly located at 1. Numerical simulations performed by us hint that this seems to be true also for a product of truncated unitary matrices. In the case that this is indeed true, the question arises about the algebraic structure. Does the determinantal point process carry over to a product $T_r \cdots T_1$ where the restriction $m_j > 2n_j + \nu_j$ is not met for all $j = 1, \ldots, r$?

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