THE β-FUNCTION OF THE SU(3) WILSON ACTION

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The quantitative behaviour of the β-function of the standard SU(3) Wilson action is studied with the help of a systematically optimized Monte Carlo Renormalization Group method.

1. Introduction. Classical Yang–Mills theory does not have a scale. A dimensionful parameter enters the theory only at the quantum level via the regularization–renormalization procedure. In the case of the lattice regularization this parameter is the lattice unit a (or the cut-off π/a).

The cut-off is an unphysical parameter and the physical predictions should be independent of it. Having another (unphysical) parameter in the theory – the bare coupling g – it can always be arranged that a specific quantity say, the mass gap, becomes independent of the cut-off. For this purpose the function $g = g(a)$ should be chosen appropriately. However, for a generic value of the cut-off the function $g(a)$ depends on the specific quantity which is kept fixed – there is no way to keep all the physical predictions unchanged as the cut-off is changed. It is only in the limit of large cut-offs (in the continuum limit) that a unique function $g(a)$ or a unique β-function $\beta(g) = -a\mathrm{d}g(a)/\mathrm{d}a$ can be defined. The β-function describes the way the bare coupling should be tuned in order to keep all the physical predictions independent of the cut-off in the continuum limit. The β-function is unique in this sense but not universal: it is different in different (lattice) formulations. In particular, the β-function depends on the lattice action chosen. Only the two leading terms in its perturbative expansion are universal:

$$\beta(g) = -b_0 g^3 - b_1 g^5 + \mathrm{O}(g^7), \qquad (1)$$

with

$$b_0 = 11N/(48\pi^2), \quad b_1 = \tfrac{34}{3}\left(N/16\pi^2\right)^2.$$

For large cut-offs (small bare coupling values) these terms dominate and define a universal scaling behaviour, "asymptotic scaling". Outside this region, but still in the continuum limit, the scaling behaviour is described by the full, and in general, completely unknown β-function.

It is basically important to reveal and understand the quantitative structure of the β-function – a fact which is underlined by recent, sometimes confusing, developments in both SU(2) and SU(3) gauge theories with the standard Wilson action.

The calculation of the glueball mass, string tension, etc., at different coupling values also gives the β-function immediately. However, this is an extremely – and unnecessarily – difficult way to proceed. These quantities reflect long-distance properties even in the continuum. Their analysis, at least at the present state of the art, always includes subjective elements. Additionally, nobody is going to measure these quantities at large correlation lengths (say, at $\xi \sim \mathrm{O}(100)$, or larger) in the near future. Monte Carlo Renormalization Group (MCRG) methods seem to be much more promising.

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2. The ratio method. In a recent paper we advocated two different MCRG techniques which worked remarkably well in $d = 2$, asymptotically free spin models [1] [‡1]. Both methods have the property that a systematic optimization method can be performed without changing the action. In this paper the SU(3) β-function will be studied with the help of the "improved ratio method".

The starting point of this method is simple and has already been discussed by Creutz earlier [3]: those ratios of Wilson loop expectation values from which the self mass (and corner) contributions cancel, satisfy the homogeneous renormalization group (RG) equation and are candidates for the study of the β-function. There are two problems, however. First, ratios composed of small loops are contaminated by lattice artifacts resulting in a systematic error which increases linearly with $\beta(=6/g^2)$ [1]. Second, the matching prediction is distorted by finite size effects if the correlation length (as defined in an infinite volume system) is comparable or larger than the lattice size.

The improved ratio method consists of the following steps. First the basic ratios are formed as

$$f(i_1,i_2;i_3,i_4) = \frac{W(i_1,i_2)}{W(i_3,i_4)},$$

$$i_1 + i_2 = i_3 + i_4,$$

$$g(i_1,i_2;i_3,i_4;i_5,i_6;i_7,i_8) = \frac{W(i_1,i_2)W(i_3,i_4)}{W(i_5,i_6)W(i_7,i_8)},$$

$$i_1, + i_2 + i_3 + i_4 = i_5 + i_6 + i_7 + i_8, \qquad (2)$$

and so on. Here $W(i_1,i_2)$ is the expectation value of a planar Wilson loop of size i_1,i_2. Of course, non-planar loops can be included as well. Apart from lattice artifacts, these functions satisfy the RG equation:

$$f(2i_1,2i_2;2i_3,2i_4;\beta,L) = f(i_1,i_2;i_3,i_4;\beta',L/2), \qquad (3)$$

and a similar equation for the other functions. Here $\beta - \beta' = \Delta\beta(\beta)$ is the change of the coupling constant required to increase the cut-off (or the correlation length) by a factor of 2. The function $\Delta\beta(\beta)$ is directly related to the integral of the inverse of the β-function and carries the same information:

$$\int_{\beta-\Delta\beta}^{\beta} \frac{dx}{x^{3/2}\beta_{\text{funct}}\left((6/x)^{1/2}\right)} = -\frac{2\ln 2}{\sqrt{6}}. \qquad (4)$$

In eq. (3) the lattice volume is scaled together with the correlation length – a standard way to reduce the finite size effects in RG studies.

Any linear combination of the functions $f, g, \ldots$ defined in eq. (2) satisfies eq. (3) also. In the improved ratio method, the mixing coefficients are determined by the requirement of cancelling the lattice artifact corrections to eq. (3) systematically order by order in perturbation theory. At the tree level, the linear combination of two basic ratios is formed and the mixing coefficient is determined by requiring eq. (3) to be correct on the tree level (i.e. by requiring $\Delta\beta = 0$; there is no scale generated, there is no shift in β at the tree level). At the one-loop level the linear combination of three basic ratios is formed, and the two mixing coefficients are determined by requiring $\Delta\beta = 132 \ln 2/(16\pi^2) \approx 0.579\ldots$ in eq. (3), which is the exact one-loop result (see eq. (4)), and so on. This procedure systematically eliminates the lattice artifacts in perturbation theory, which are relevant at large correlation lengths. Whether at moderate couplings non-perturbative lattice effects become important can be judged only through the consistency of the final results.

A large number of systematically improved, mixed ratios can be obtained this way. These mixed ratios are then used in the actual MC analysis to determine $\Delta\beta(\beta)$ via an equation analogous to eq. (3) written for the mixed ratios.

3. Numerical analysis. The ratio method requires a good quality measurement of different Wilson loop expectation values at β and β' on a lattice L^4 and $(L/2)^4$ respectively. Of course, β' is not known a priori. (The purpose of the calculation is just to determine it.) In practice, MC measurements are done at several approximately chosen β' values and a linear extrapolation between adjacent values of β' is used.

[‡1] One of the methods discussed in this paper (optimization of the block transformation for a given fixed action) has also been suggested by Swendsen [2].

A good statistics measurement of all the planar Wilson loop expectation values up to a size 8×8 at β values 5.8, 6.0,..., 6.6 on a $16^3 \times 32$ lattice was published recently by Barkai et al. [4]. By combining their numbers with our measurement at $\beta = 5.2$, 5.4, 5.7 and 5.85, 6.0 on lattices half as large, we studied the β-function ($\Delta\beta(\beta)$) in this coupling constant region.

In ref. [4] rectangular loops with both sides in the spatial direction were measured only. The numbers can be thought of as being obtained on a 16^4 lattice effectively. The Wilson loop values we obtained on the corresponding 8^4 lattices are given in table 1. At $\beta = 6.0$ all the Wilson loop values are larger on the 8^4 lattice than on the large lattice of ref. [4]. This is more than a 3 standard deviation effect. This finite size effect is expected to become smaller as β is decreased. By comparing our results at $\beta = 5.7$ with those obtained on a 16^4 lattice earlier [6] we conclude that here the size dependence is less than or of the order of 10^{-4} for the Wilson loops considered. For this reason we included the $\beta = 5.6$ results of ref. [4] in our analysis and did not recalculate these loops on the 8^4 lattice.

Using the 10 different Wilson loops (1×1, 1×2,..., 4×4) we formed 84 basic ratios of the type defined in eq. (2). (On the large lattice one forms the analogous ratios from 10 loops of even size: 2×2, 2×4,..., 8×8.) We used only those ratios where the total area of the loops in the numerator is different from (in our notation larger than) those in the denominator. (Those ratios where the area in the numerator and denominator is equal change so slowly with β that our statistics is not enough to use them in the matching conditions.)

The determination of the tree level mixing coefficients requires a rather trivial, tree level perturbative calculation. The one-loop calculation

Table 1
Wilson loop expectation values $W(I,J)$ measured on 8^4 lattices for different values of β by using the multihit method of ref. [5]. The loops were measured after every tenth pseudo heatbath sweep. The errors quoted in brackets are statistical errors corrected by the observed time correlations. The last row specifies the number of configurations analyzed at each value of β.

I	J	β				
		5.2	5.4	5.7	5.85	6.0
1	1	1.29520 (119)	1.41613 (179)	1.64745 (143)	1.72404 (81)	1.78351 (72)
1	2	0.57341 (185)	0.69686 (228)	0.97290 (224)	1.07557 (131)	1.15480 (105)
1	3	0.25610 (153)	0.34488 (177)	0.58446 (243)	0.68399 (163)	0.76241 (118)
1	4	0.11357 (98)	0.17103 (117)	0.35241 (206)	0.43735 (164)	0.50570 (107)
2	2	0.11735 (159)	0.18169 (215)	0.39431 (304)	0.49354 (191)	0.57499 (152)
2	3	0.02464 (80)	0.04903 (182)	0.17084 (217)	0.24444 (163)	0.30727 (211)
2	4	0.00529 (48)	0.01404 (88)	0.07630 (138)	0.12388 (123)	0.16910 (194)
3	3	0.00233 (80)	0.00732 (71)	0.05725 (136)	0.10166 (134)	0.14545 (213)
3	4	0.00058 (34)	0.0092 (41)	0.02117 (63)	0.04444 (90)	0.07222 (132)
4	4	– –	– –	0.00615 (42)	0.01620 (81)	0.03344 (112)
Configurations		32	32	64	48	64

Table 2
Illustration of the improvement procedure for three basic ratios. Tree level and one-loop improved ratios are formed from the three basic ratios R_1, R_2 and R_3. The mixing leads to a systematic improvement of the weak coupling behaviour of the observables considered. The last column shows the shift $\Delta\beta$ for the listed ratios obtained in the weak coupling limit.

Ratios		Weak coupling $\Delta\beta$
basic ratios	$R_1 = W(3,3)/W(2,4)$	-0.158β
	$R_2 = W(1,1)W(3,3)/W(1,2)W(2,3)$	-0.057β
	$R_3 = W(1,2)W(2,3)/W(1,3)W(1,3)$	0.046β
tree level improved ratios	$R_{13} = R_1 + 0.688298\, R_3$	0.582
	$R_{23} = R_2 + 0.523659\, R_3$	0.492
one-loop improved ratios	$R_{123} = R_1 + 0.027917\, R_2 + 0.702917\, R_3$	0.579

which is necessary to obtain the one-loop improved ratios has been completed recently [7,8]. The agreement between the two independent calculations (ref. [7] and ref. [8]) makes it probable that the results are free of algebraic errors. Although the "zero mode problem" is not treated correctly in these papers – it is a problem which awaits solution – the error induced by that goes to zero as $1/L^4$, and is expected to be very small [‡2].

A huge number of different, tree level and one-loop improved ratios can be formed. We introduced some reasonable cuts to reduce this set of observables: the mixing coefficients were required to be positive (to assure monotonic behaviour in β) and of the order of 1 (to avoid the dominance of certain basic ratios in the mixing procedure). This way we used O(1000) tree level improved and O(6000) one-loop improved ratios in the analysis. This mixing procedure is illustrated in table 2.

4. Results. For the final analysis we considered those subsets of ratios which gave a matching prediction with a reasonably small statistical error. The error cuts of table 3 were chosen in such a way as to let the matching prediction of a large number of ratios contribute to the final, average $\Delta\beta(\beta)$. To check that this final prediction is not biased by the overwhelming contribution of a few basic ratios, we also introduced a cut on the number of times a given basic ratio is allowed to appear. Table 3 shows that the result is insensitive to this cut. The stability of the predictions is remarkable. For instance at $\beta = 6.4$, 1541 one-loop improved ratios gave a matching prediction with a statistical error less than 0.075, and all of the 1541 matching values lie in the range $\Delta\beta(6.4) \in (0.472, 0.515)$! Of course, the numbers are

[‡2] A similar zero mode problem occurs in the non-linear σ model as discussed recently in ref. [9]. In this case one can check explicitly that the error induced by the naive treatment of the zero modes goes to zero rapidly when the lattice volume is increased.

Table 3
The average shift $\Delta\beta(\beta)$ obtained from one-loop improved ratios. Shown are the results for two different cuts on the error of the matching predictions of an individual ratio (error cut), $c_1 = 0.075(0.1)$ and $c_2 = 0.1(0.15)$ for $\beta = 6.6, 6.4, 6.2$ (6.0). The third and fifth columns specify the number of ratios analyzed in each case. Also given are the results for different cuts on the number of times a basic ratio is allowed to contribute (ratio cut).

β	Error cut				Ratio cut
	c_1		c_2		
	$\Delta\beta$	ratios	$\Delta\beta$	ratios	
6.6	0.558	2780	0.551	4461	∞
6.4	0.492	1541	0.490	3020	
6.2	0.462	773	0.443	1815	
6.0	0.324	212	0.323	1230	
6.6	0.559	493	0.554	539	50
6.4	0.492	414	0.490	487	
6.2	0.460	277	0.438	429	
6.0	0.323	170	0.335	394	
6.6	0.557	99	0.555	114	10
6.4	0.492	94	0.491	96	
6.2	0.460	61	0.435	96	
6.0	0.328	50	0.351	90	

correlated statistically – they are all composed of the same 10 + 10 Wilson loop values at a given β, but the small fluctuations suggest that the systematic errors are under control.

In table 4 we summarize our predictions obtained for the basic, tree level and one-loop improved ratios. Two different error estimates are quoted there. The first error estimate is the average statistical error of the individual matching values contributing to the average $\Delta\beta(\beta)$. This is presumably an overestimate, since averaging over many ratio predictions should also improve the statistics. The error given in the brackets is the average fluctuation of the matching predictions. This is presumably an underestimate of the real error. Having only the final numbers of the measurement of ref. [4], we could not do more reliable error estimates. The average fluctuation of the matching predictions obtained from different basic ratios is 3–5 times larger than those of the one-loop improved ratios – at least at the larger β values. Around $\beta = 6.0$ the perturbative improvement does not seem to be effective any more.

The available information on $\Delta\beta(\beta)$ is summarized in fig. 1. The points related to T_c were deduced from the results of refs. [10] and [11], while those related to the string tension are taken from refs. [6] and [4]. We did not include the string tension result of ref. [4] at $\beta = 6.6$ since even a $16^3 \times 32$ lattice is too small at this large value of β to extract the asymptotic slope of the static quark–antiquark potential. Presumably the string tension extracted at $\beta = 6.4$ is also influenced by finite size effects [‡3]. There is a recent high statistics measurement for T_c at $N_T = 2$, 4, 6 and 8 which indicates a very similar pattern to that which we found in this paper [12]. Some information on $\Delta\beta(\beta)$ also exists from measurements of fermionic observables ($\langle\bar{\psi}\psi\rangle$, m_p) in the quenched approximation [13], which shows the same qualitative behaviour. However, the statistics for these observables is quite poor.

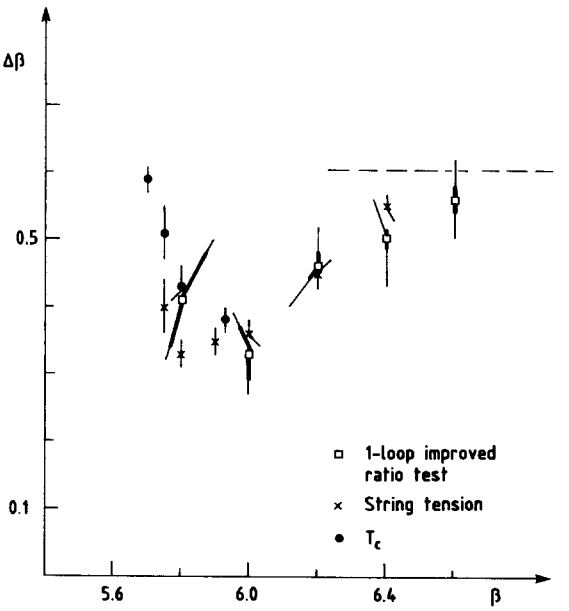

Fig. 1. The average shift $\Delta\beta$ as a function of β obtained from the analysis of one-loop improved ratios (squares). (At $\beta = 5.8$ the basic ratios are used.) The error bars refer to the statistical error (thin bars) and the average fluctuations (thick bars) quoted in table 4. Also shown are the predictions for $\Delta\beta$ obtained from the string tension (crosses) and the critical temperature (full points).

Table 4
Predictions for the average shift $\Delta\beta(\beta)$ obtained from basic, tree level and one-loop improved ratios. The first error quoted is the average statistical error of the ratios analyzed, while the error in brackets gives the average fluctuations of the matching predictions. At $\beta = 5.8$ only the basic ratios could be analyzed with reasonable statistical accuracy.

β	One-loop mixing	Tree level mixing	Basic ratios
6.6	$0.56 \pm 0.06(0.02)$	$0.55 \pm 0.07(0.02)$	$0.57 \pm 0.05(0.05)$
6.4	$0.49 \pm 0.06(0.01)$	$0.47 \pm 0.07(0.02)$	$0.50 \pm 0.05(0.05)$
6.2	$0.46 \pm 0.06(0.02)$	$0.42 \pm 0.08(0.03)$	$0.45 \pm 0.04(0.05)$
6.0	$0.33 \pm 0.07(0.04)$	$0.34 \pm 0.07(0.05)$	$0.38 \pm 0.06(0.06)$
5.8	–	–	$0.41 \pm 0.09(0.07)$

Our prediction

$$\Delta\beta(6.6) = 0.56 \pm 0.06 \qquad (5)$$

shows only a slight deviation from asymptotic scaling for $\beta \geq 6.0$ (see eq. (4)). What we consider

[‡3] Measuring the long-distance part of the potential is a difficult task and at the present state of the art the extraction of a string tension is not free of subjective elements, as is discussed in detail in ref. [4].

most relevant, however, is that a quantitative understanding of the rather non-trivial way the standard action approaches continuum seems to be emerging. This clearly is not in agreement with the present status of glueball mass calculations [14] which indicate asymptotic scaling already for $\beta \gtrsim 5.1$.

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