AN OUTLINE
OF
HOW MANIFOLDS RELATE TO ALGEBRAIC K-THEORY

Friedhelm Waldhausen

Consider manifolds without boundary. Allow taking product with R^k. What is left? Certainly the homotopy type and the tangent bundle. But, by an old theorem of Barry Mazur this is all that is left; and that is even true with parameters. In other words,

Thm. The forgetful map

space of (stable) manifolds ⟶ space of tangential spaces
(a suitable simplicial set) (another simplicial set)

is a homotopy equivalence.

(It goes without saying here, as they say, that we want our 'manifolds' and 'spaces' to have the homotopy types of finite CW complexes.)

We can reduce this to a simpler, but equivalent, statement by getting rid of the tangent bundles.

Equ. assertion. The map

space of framed (stable) manifolds ⟶ space of spaces

is a homotopy equivalence.

Mazur's theorem is strictly a result about non-compact manifolds. Namely define $C^{CAT}(X)$ as the homotopy fibre, at X, of the forgetful map

space of *compact* framed stable manifolds ⟶ space of spaces where CAT is one of DIFF, PL, TOP and where stabilization is given by product with $[0,1]^k$ (plus rounding of corners in the DIFF case).

Then $C^{CAT}(X) \not\simeq *$ (in general). In fact, its loop space is given by,

Exercise. $\Omega\, C^{CAT}(X) \simeq$ (stable) CAT pseudo-isotopy space of X (if X is a CAT manifold).

(Hints. Use immersion theory and general position; cf. lemma 5.2 of [W2] for a related argument.)

One knows that $C^{PL}(X) \xrightarrow{\simeq} C^{TOP}(X)$ (cf. [H2]) whereas the same is not true with $C^{DIFF}(X)$ instead (in fact, as Hatcher has phrased it, the latter is so named because it is different). Specifically, $C^{PL}(*) \simeq *$ (by the Alexander trick), but already $C^{DIFF}(*)$ is highly non-trivial. For example, there is an infinite cyclic summand in each of its homotopy groups in dimensions 4, 8, 12, ... It is related to the summand which Borel has found, one dimension higher up, in the algebraic K-theory of the ring of natural integers.

This relation to algebraic K-theory, how does it come about? To relate a manifold gadget, such as $C^{CAT}(X)$, to a non-manifold gadget one *must* at some point get rid of the manifolds. For $C^{PL}(X)$ this can be achieved by the following result,

Thm. Let X be a finite polyhedron. Then $C^{PL}(X)$ is homotopy equivalent (naturally, in X) to the simplicial category with

- objects: (locally trivial families of) finite polyhedra containing X as a deformation retract,
- morphisms: maps whose point inverses are contractible.

Remarks. 1. The theorem is very typical of the game: After a lot of effort, the only thing one has learned in the end is that two rather complicated definitions lead to the same thing, up to homotopy.

2. The theorem has variants, technical and otherwise; for example the finite polyhedra in its statement may be replaced by finite simplicial sets.

This theorem is more or less the same as the *parametrized h-cobordism theorem* of Hatcher [H1]. The proof given by Hatcher is not quite satisfactory. There is another proof [W5], it goes roughly as follows. For each compact polyhedron Y one considers the pairs (M,p) where M is a compact PL manifold of some fixed dimension n , and $p: M \to Y$ is a map having contractible point inverses. Let $S(Y,n)$ denote the 'space' of these pairs (a suitable simplicial set), and let $S(Y) = \varinjlim S(Y,n)$ (stabilize by allowing M to be replaced by $M\times[0,1]$). One shows,

(i) if $S(Y)$ is contractible for all Y then the theorem follows. This is more or less formal: one applies a fibration criterion in the spirit of Quillen's theorem B [Q1].

(ii) $S(Y)$ is contractible indeed. This is proved by a sort of induction on Y (the idea being to cut up Y into simpler pieces); the argument is based on a trick devised by Hatcher in the context of proving a theorem on 3-manifolds: the method of 'general position in patches' [H3].

As to algebraic K-theory now, let us begin by considering the *Euler characteristic*. It may be defined as an element of a certain universally defined group, namely the *class group* given in terms of generators and relations as follows,

- generators [Y] , where Y runs through the pointed spaces (of finite type)

- relations of two kinds
  (1) if $Y_1 \to Y_2$ is a cofibration , then $[Y_2] = [Y_1] + [Y_2/Y_1]$
  (2) if $Y \xrightarrow{\simeq} Y'$ is a weak homotopy equivalence, then $[Y] = [Y']$.

Obviously this definition of class group makes sense as soon as you have a category with notions of cofibration and weak equivalence. If moreover these notions have the usual familiar properties, and if you like to play with definitions, you will be able to write down a bisimplicial set in which these notions play a role, and which has the class group as its fundamental group, cf. [W1], [W3].

Def. The *algebraic K-theory* of that category (with cof. and w. eq.) is given by the loop space of (the geometric realization of) that bisimplicial set.

In particular, given a space X there is associated to it its K-theory A(X) via a model category for an equivariant homotopy theory, attached to X , of spaces of finite type. In making this precise, one needs to make choices, viz.

space: topological space or simplicial set,

finite: finite on the nose or finite up to homotopy,

equivariant: spaces over X or spaces with an action of ΩX .

Thm. All these choices don't matter (if they are made right). In addition, $A(X)$ can also be expressed by the 'plus' construction of Quillen, using matrices over the 'ring up to homotopy' $Q(\Omega X_+)$ (cf. below).

The relation between K-theory and PL pseudo-isotopy theory is now given as follows. One introduces a connected de-loop $Wh^{PL}(X)$ of $C^{PL}(X)$ (this is to get rid of the dimension shift with respect to algebraic K-theory).

Thm. There is a map $A(X) \to Wh^{PL}(X)$ whose fibre is a homology theory (that is, as a functor of X it satisfies the excision axiom and, of course, the homotopy axiom).

The proof of this result, along with the aforementioned foundational material on K-theory, makes up the content of the rather long paper [W3]. The proof has nothing to do with manifolds: it uses the non-manifold translation of $C^{PL}(X)$, and hence $Wh^{PL}(X)$, described above.

There is nothing wrong with manifolds, however, and one can in fact write down manifold models for all the spaces involved, and natural maps between them, to represent the whole fibration of the theorem [W2]. (There is just one thing which is not clear from the manifold point of view, namely that one of the terms in the fibration happens to be a homology theory. In other words, the detour through non-manifolds is only required here to recognize a homology theory when one sees one!) The manifold models are essentially independent of the category of definition (their construction, that is, not necessarily of course the homotopy types that they represent). So one can use the natural forgetful map to compare categories. Smoothing theory now tells us that, in the situation at hand, the difference between DIFF and PL is itself only a homology theory. It thus results from the theorem that its analogue for smooth manifolds is also valid; that is [W2], there is a fibration, natural in X ,

$$h(X) \longrightarrow A(X) \longrightarrow Wh^{DIFF}(X)$$

where $X \mapsto h(X)$ is a homology theory. Now comes,

<u>1st surprise</u>. That fibration splits.

Reason. Functors can be *stabilized*, $F^S(X) = \varinjlim \Omega^n \mathrm{fibre}(F(S^n \wedge X_+) \to F(*))$. Now it is known, as a consequence of Morlet's disjunction lemma [H2], that

$$(Wh^{DIFF})^S(X) \simeq * .$$

So one gets a diagram of fibrations

$$\begin{array}{ccccc} h(X) & \longrightarrow & A(X) & \longrightarrow & Wh^{DIFF}(X) \\ \simeq\downarrow & & \downarrow & & \downarrow \\ h^S(X) & \xrightarrow{\simeq} & A^S(X) & \longrightarrow & (Wh^{DIFF})^S(X) \end{array}$$

where the arrow $h(X) \to h^S(X)$ is a homotopy equivalence since stabilization doesn't change a homology theory and where, of course, $h^S(X) \to A^S(X)$ is a homotopy equivalence since $(Wh^{DIFF})^S(X)$ is contractible. Thus there is a splitting $A(X) \simeq A^S(X) \times Wh^{DIFF}(X)$. Next comes,

<u>2nd surprise</u>. $A^S(X)$ in turn splits as $Q(X_+) \times ?$.

Reason. By using an explicit description of $A^S(*)$ in terms of smooth manifolds one can check [W2] that the usual map $B\Sigma_\infty \to QS^0$ factors through $A^S(*)$; the argument in the general case is similar.

Putting the two surprises together one obtains a double splitting

$$A(X) \simeq Q(X_+) \times Wh^{DIFF}(X) \times \mu(X) .$$

The mysterious third factor has informally become known as the 'mystery homology theory'. Accordingly the next result might then be called the theorem of the vanishing mystery homology,

<u>Thm</u>. $\mu(X) \simeq *$.

An account of it may be found in [W4]. Rationally, that vanishing can be deduced [W1] from a vanishing in group homology, namely the rational vanishing of the kernel of the trace map

$$H_*(\,GL(Z),\ M(Z)\,) \longrightarrow H_*(\,GL(Z),\ Z\,) .$$

This was established by L^2-cohomology methods by Borel and by Farrell and Hsiang; more recently, Goodwillie has succeeded in giving an algebraic proof. Conversely it is now possible to turn the situation around and to use topology to compute the homology of the adjoint representation, including torsion. This goes as follows.

Given a ring R and a bimodule A over it, one can define a sort of particularly primitive K-theory, the so-called *stable K-theory* $K^S(R,A)$ (cf. [W6]). It is an elementary fact [W6], [K1] that this 'computes' the homology of the adjoint representation in the sense that there is a spectral sequence

$$H_p(\ GL(R),\ \pi_q K^S(R,A)\ ) \implies H_{p+q}(\ GL(R),\ M(A)\ )$$

where the homology on the left is ordinary (i.e. untwisted). Moreover, as Bökstedt has pointed out, the spectral sequence will collapse in certain interesting cases, for example if $R = Z$ (this is due to the presence of a product structure). One is thus led to try and compute stable K-theory.

Now suppose that k is a ground ring over which R is an algebra. Then one can define the *Hochschild homology* $H_k(R,A)$ and one can construct a natural transformation

$$K^S(R,A) \longrightarrow H_k(R,A) \ .$$

It turns out that this natural transformation will be happy to become an equivalence as soon as one is prepared to give it a chance.

'Giving it a chance' means that one should not attempt to do something which, from a broader perspective perhaps, is openly unreasonable; for example, to try to take $k = Z$ here. Indeed, if the algebraic K-theory of spaces is looked at from the algebraic K-theory of rings point of view, one is forced to look at 'rings' which are a little unusual to the taste of many (e.g. $A(*) = K(R)$ where R is the 'ring up to homotopy' QS^0). Such 'rings' are not algebras over Z, in general, so that $H_Z(R,A)$ is not even defined.

One is thus led to try and take $k = QS^0$. Not surprisingly, this leads one into constructions which are technically rather involved (for example, it would be easy to mimick the usual definition of Hochschild homology to obtain a simplicial object in the homotopy category; but that is not enough, of course). Granting that all of this makes sense, unravelling of the definitions shows that, for this k, the statement " $\mu(*) \simeq *$ " is equivalent to the statement that the map

$$K^S(k,k) \longrightarrow H_k(k,k) \simeq k$$

is an equivalence. To proceed from this special case to the general case one has to use, among other things, a decomposition theorem for the K-theory of 'rings' which are free products, of sorts.

The 'topological Hochschild homology' $H_k(R,A)$, $k = QS^o$, has been computed by Marcel Bökstedt in the case $A = R = Z$ [B1]. The argument is difficult, but the result is easy enough to state. The homotopy groups of $H_k(Z,Z)$, $k = QS^o$, are the cyclic groups

$$Z , 0 , 0 , Z/2 , 0 , Z/3 , 0 , Z/4 , 0 , Z/5 , \ldots .$$

We see that we have a problem about ordinary rings here (for example the ring Z); namely *the problem of how to compute the stable K-theory* and hence the homology of the adjoint representation. But to solve the problem we must first consider it in the rather extended framework of 'rings up to homotopy'.

One wonders if perhaps a similar thing is true with regard to the problem of computing algebraic K-theory.

To conclude, here are a few remarks on the notion of *ring up to homotopy*. It is doubtful if there is one technical choice which is equally well suited for everything that one wants to do with it. There happens to exist a simple notion which suffices for many purposes [G1]. Namely let a quote-abelian group denote a functor from spaces to spaces which preserves connectivity and satisfies approximate excision. This may be regarded as another 'coordinate-free' way of specifying a (connected) homology theory, by stabilization, and hence a spectrum; that excision is asked only approximately here has the reason that (1) this is good enough, and (2) it allows one to keep functors such as the identity functor. By definition then a quote-ring is just a quote-abelian group together with a monad structure in the sense of category theory; and to define K-theory, say, is just a matter of writing it down [G1].

From a naive point of view, a 'ring up to homotopy' is nothing but a topological space R together with structure maps add: $R{\times}R \to R$, mult , and so on, so that the ring axioms are satisfied up to homotopy. In this situation one can, for example, define the 'space of homotopy invertible matrices' as the pullback in the diagram

$$\begin{array}{ccc} \hat{GL}(R) & \longrightarrow & M(R) \\ \downarrow & & \downarrow \\ GL(\pi_0 R) & \longrightarrow & M(\pi_0 R) \end{array} .$$

It is a multiplicative H-space by means of matrix multiplication. But it is plausible, on the other hand, that one needs more information to ensure the existence (or even functorial existence) of a classifying space for this H-space. Here is a neat example of a bad failure.

The determinant map $\hat{GL}(QS^o) \longrightarrow \hat{GL}_1(QS^o) = G$ is an H-map, and a retraction. Suppose it could be de-looped. Then the de-looped map would extend to the 'plus' construction (since BG has abelian fundamental group). That is, it would induce a map $A(*) \to BG$, and this map would be left inverse, more or less, to the (doubtlessly existing) map in the other direction, $BG \to A(*)$. It would therefore follow that the latter map is injective on homotopy groups. But this is not true as one sees by using the splitting $A(*) \simeq QS^o \times Wh^{DIFF}(*)$ together with the following facts,

(i) the composite map $BO \to BG \to A(*) \to Wh^{DIFF}(*)$ is trivial [W2],

(ii) the composite map $BG \to A(*) \to QS^o$ is non-trivial only at the prime 2 (and here just barely so, being 'multiplication by η') [B2].

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FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT BIELEFELD
4800 BIELEFELD, FRG.