Time Series Analysis and Simulation

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1. INTRODUCTION

Since the stochastic nature of time series has become the main subject of research in time series analysis simulation techniques have been among the famous tools of time series analysts. Already the now classical papers by Slutzky [1937] and Yule [1926] use simulation experiments as basis for their reasoning.

Applying the theory of weak stationary processes to time series analysis creates a lot of methodical problems which can be solved in part only by simulation. Roughly one can classify the applications of simulation experiments as follows. First there are problems the theoretical background of which is worked out but one has not yet been able to deduce procedures which can be applied in empirical studies. Secondly problems where one hopes to find some hints by simulation experiments which relations would play a prominent rôle in explaining the theoretical background.

In the remainder of this paper we will discuss some examples of both kinds.

2. WHAT IS THE RIGHT DIVISOR WHEN ONE ESTIMATES THE COVARIANCE FUNCTION

Let x_t stand for the time series being the observations of the stochastic process[1] X_t . It is well known that

$$c(\tau) = \frac{1}{d} \sum_{t=1}^{n-|\tau|} x_{t+\tau}\, x_t \qquad (1)$$

is an estimator for the unknown covariance function[2] of X_t . There is

1) Unless otherwise stated by stochastic process we mean a weak stationary stochastic process having the ergodic property.

2) Strictly speaking we are estimating the covariance coefficients for $\tau = 0, \pm 1, \pm 2, \ldots$, we follow common practice to use the word covariance function for them.

quite a debate among time series analysts as how to choose d. It is easily seen that $d = n - |\tau|$ in eq.(1) will produce an unbiased estimator whilst $d = n$ comes out with a biased one. This makes some authors prefer the divisor $d = n - |\tau|$.

On the other hand applying some straight forward arithmetic to

$$Q(\mathscr{C}, u) = \frac{1}{2n} \sum_{j=0}^{n-1} \{ (\sum_{i=1}^{n} x_i u_k)^2 + (\sum_{i=1}^{n} (-1)^{\left[\frac{i+j}{n}\right]} x_i u_k)^2 \} \qquad (2)$$

at which $k \equiv i + j - 1 \pmod n$ and $[r]$ stands for the largest integer equal to or less than r one gets

$$Q(\mathscr{C}, u) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} c(i-j)\, u_i\, u_j \geq 0 \,. \qquad (3)$$

$c(i-j)$ is as defined in eq.(1) with divisor $d = n$. As eq.(3) shows in that case the estimator is a positive definite function.

If one replaces $c(i-j)$ in eq.(3) by the estimator with divisor $d = n - |\tau|$ it is easy to construct a counter example showing that in this case $Q(\mathscr{C}, u)$ is not positive definite. Parzen [1964] claims positive definitness to be the essential attribute of the covariance function for estimating spectra; therefore he recommends to use $d = n$ even though accepting a biased estimator.

If one tackles the question of the right divisor via simulation experiments one soon notices that this is a problem of minor practical importance. The following table presents a typical section of results one gets by simulation experiments.

Table 1

Power spectrum computed from $c(\tau)/c(0)$ using Tukey-window

frequency $\pi j/20$	divisor $d=n$	divisor $d=n-\|\tau\|$	circular	theoretical spectrum
j = 2	.593	.599	.593	.650
3	.487	.488	.488	.498
4	.359	.356	.359	.333
5	.233	.228	.232	.185
6	.130	.125	.129	.077
7	.063	.059	.061	.017

The time series was generated by a moving average of five with equal weights .2 over random numbers uniformly distributed over (0,1). "Circular" means that in computing $c(\tau)$, $\tau \neq 0$ always n values were used by putting $x_{t+\tau} = x_i$ for $t + \tau > n$ and $t + \tau \equiv i \pmod n$. The covariance function was estimated up to a lag of $m = 10$. The time series were $n = 200$ points long.

The table shows - as further results do - that the estimated spectra based on various methods of defining $c(\tau)$ are relative close together compared with their deviation from the theoretical values of the power spectrum.

The results are not due to the choice of the lag window as can be seen in table 2. There is shown a section of the results of the same simulation experiment which one gets using the lag window

$$\begin{array}{ll} 1 - \frac{6\tau^2}{m^2} \left(1 - \frac{|\tau|}{m}\right) & 0 \leq |\tau| \leq \frac{m}{2} \\ 2 \left(1 - \frac{|\tau|}{m}\right)^3 & \frac{m}{2} \leq |\tau| \leq m \end{array} \qquad (4)$$

as proposed by Parzen.

Table 2

Power spectrum computed from $c(\tau)/c(0)$ using Parzen-window

frequency $\pi j/n$	divisor $d=n$	divisor $d=n-\lvert\tau\rvert$	circular	theoretical spectrum
j = 2	.554	.560	.554	.650
3	.465	.467	.465	.498
4	.359	.358	.359	.333
5	.254	.251	.253	.185
6	.163	.159	.162	.077
7	.097	.094	.096	.017

In both tables frequency intervals centered around very low frequencies are omitted. The reason for this is a methodical problem which shall be demonstrated now.

3. CORRECTION FOR MEAN

If the process X_t has an expectation value $E[X_t] = \mu \neq 0$ this value would be unknown in most cases prior to empirical investigations. For an ergodic process X_t one has the unbiased estimator $\bar{x} = \frac{1}{n} \sum_{t=1}^{n} x_t$ for μ but it can be shown (e.g. Jenkins et.al. |1968|) that the use of data points $x_t - \bar{x}$ which are corrected for mean produces an additional bias of order $\frac{1}{n}$ if one takes an estimator for $C(\tau)$ as given by eq.(1).

Estimating the power spectrum this bias will result in an underestimation of the spectrum for low frequencies.

As can be seen from simulation studies made by Granger and Hughes [1968] and König and Wolters [1972] this bias can be very disturbant especially for short time series.

The following table lists some results of a simulation experiment by König et. al.[1972]. The estimating procedure used Parzen window with a lag up to 24. The series was 200 data points long.

Table 3

frequency $\pi j/n$	theoretical spectrum	no correct. for mean	correction for mean	Fishman procedure
j = 0	1.768	36.43	1.129	1.240
1	1.561	24.68	1.058	1.131
2	1.156	7.411	0.889	0.907
3	0.810	1.404	0.746	0.747
4	0.573	0.784	0.637	0.637
5	0.420	0.545	0.478	0.478

The last column presents corrected values following a proposal by Fishman[1969].

4. FURTHER METHODICAL DIFFICULTIES

It is evident that in practical application of spectral theory the prerequisite that one knows the covariance function for all real τ is never met. Moreover the time series at hand will be so short that one cannot rely on the asymptotic properties of the various estimators as proposed in the literature. So one has to face questions like:

What is the proper truncation point when estimating the covariance function?
Which window should be chosen when using an indirect estimating procedure?
For which frequency intervals should one take estimates?
How varies the bias of the spectral estimators with frequency?

One looks for procedures which enable those to get the right answer in their concrete data situation who would apply spectral theory. It revealed that simulation experiments (Jenkins et.al.[1968], Naeve [1969], König et.al.[1972]) can lead to a lot of rules as how to proceed. These rules have proved to be appropriate in numerous simulation studies and in many investigations of empirical time series.

Especially the book by Jenkins et.al.|1968| can be highly recommended to all who would apply spectral analysis.

5. THE NEED FOR A SPECTRAL SIMULATOR

The simulation studies mentioned so far were mostly designed to work like this. One produces a large number of simulation runs and computes spectral quantities by taking some sort of average. In comparing these quantities with their theoretical values one reaches some final conclusions or rules. A typical example was demonstrated in section 2. As has been mentioned one can get valuable procedures for practical applications going along this way.

Designing lectures on time series analysis one should make use of simulation techniques in quite another way. The wide spreading of computers and the installation of interactive computer languages give the opportunity to develop program systems which could take over a large part of the students practical training in time series analysis.

Such a system would confront the student with simulated time series. In a first step and with the help of built in procedures the student should try to find estimation values for covariance function, power spectra etc. which he considers to be "good". In a second step he then can compare his estimates with the theoretical values. Variation of the parameters for the estimation procedures will show him the influence of the various estimators on the final estimation values. If he goes through these steps for some time series generated by the same stochastic process he can get a feeling as how the finiteness of the time series will influence the results.

Training the students in this way could stop the bad habit just to use the program on time series analysis implemented at the computer center and then interpreting the results in a mix-up of methodical effects and the characteristics of the underlying process.

6. INVESTIGATION OF PHASE RELATIONS

Up to now only problems in univariate spectral analysis had been solved by simulation experiments. It is not necessary to point out that methodical problems will multiply in multivariate spectral analysis. Here too simulation studies can help to clarify the situation and do help to solve some of these problems.

Vicarious the problem of estimating the phase spectrum of two stochastic processes X_t and Y_t shall be demonstrated. The phase spectrum is especially valuable in economic applications for it reflects the temporal relations among economic variables.

Take for instance the saving relation of the Harrod-Domar growth model

$$S_t = s\,Y_{t-1} . \qquad (5)$$

Interpreting saving S_t and income Y_t as stochastic processes it follows from eq.(5) that their phase spectrum is

$$\phi(\omega) = \omega . \qquad (6)$$

Applying this theoretical concept on empirical time series one has to answer the question if, due to the relative shortness of the time series, one would be able to estimate the phase spectra so that relations like eq.(6) and more complex ones can be detected.

Various simulation studies [Naeve 1968, 1969] show that under certain circumstances this can be done. The studies were based on pairs of time series which were constructed to have a phase function of one of the following types

$$\begin{aligned} \phi(\omega) &= c && \text{fixed angle lag} \\ \phi(\omega) &= a\omega && \text{fixed time lag} \qquad (7) \\ \phi(\omega) &= a_1\omega + a_2 . && \end{aligned}$$

Let X_t be the first process of the pairs then the second one will be denoted in the sequence of eq.(7) X_t^c , X_{t-a} , $X_{t-a_1}^{a_2}$. In addition pairs of the form

$$X_t \;,\; \alpha X_{t-b}^{a} + \beta X_{t-d}^{c} \qquad (8)$$

were investigated. The procedures to generate such pairs are described in the paper by Naeve [1968].

Two typical results will be presented here. In the phase and Argand diagram the broken line stands for the graph of theoretical values.

Fig. 1 Phase Spectrum $X_t, Y_t = .5\ X_{t-1} + .5\ X_{t-2}$

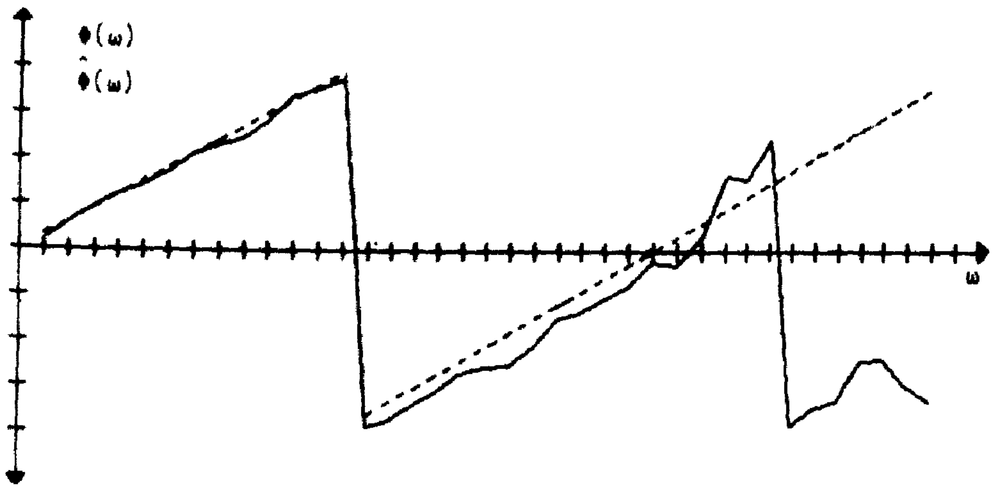
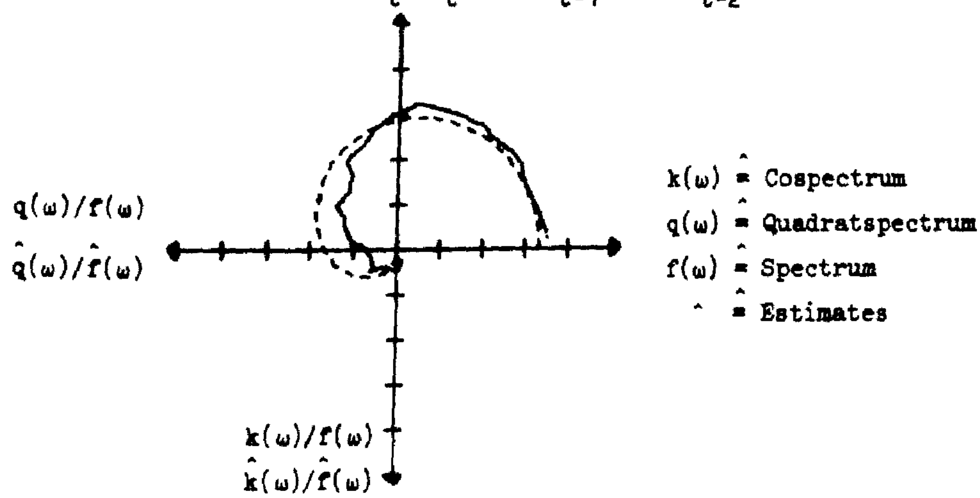


Fig. 2 Argand Diagram $X_t, Y_t = .5\ X_{t-1} + .5\ X_{t-2}$



It seems possible to detect simple relations between time series. But even for relations of the type $\Phi(\omega) = a_1\omega + a_2$ it is often not easy to determine a_2. Relations as in eq.(8) are very difficult to spy. One has to take great care when one is interpreting phase functions computed in empirical applications. Identification of phase relations requires a great deal of experience, needless to stress the usefulness of training programs as pointed out in section 5.

7. CROSS SPECTRAL ANALYSIS IN THE PRESENCE OF FEEDBACK

It can be shown [Granger 1964] that cross spectral analysis is less appropriate if feedback is present in the system generating the time series to be studied. But how can one identify feedback without knowledge of the generating system? In two papers Garbers [1970, 1971] offers a rule of thumb. I quote from his latest paper: "Vorausgesetzt zwischen zwei Variablen besteht überhaupt ein Zusammenhang, so dominiert in dieser Beziehung dann eine Richtung, wenn die aus den Originalwerten geschätzten "Kohärenzen" und "Phasen" mit den aus den 1. Differenzen berechneten weitgehend übereinstimmen. Diese Dominanz ist schwächer, wenn die "Phasen" kaum, die "Kohärenzen" jedoch weitgehend übereinstimmen. Besteht diese Ähnlichkeit schließlich weder für die "Phasen" noch für die "Kohärenzen", so existiert zwischen den Variablen ein ausgeprägter "Feedback"." 1)

Due to theoretical considerations there is little reason to accept this rule of thumb, for one can show [Jenkins et.al.1968] that phase spectra and coherence spectra remain unaltered by linear prefiltering operations, - and computing first differences is just such kind of filtering. However, there are two aspects of practical importance which might be in favour of Garbers' suggestion. His empirical series are in some sort of feedback relation, provided the economic reasoning is valid. Also it is wellknown [Jenkins et.al 1968, Naeve 1969] that one has to make special efforts to discover the structure of the underlying processes if only short series are at hand, - and economic time series must be considered to be very short according to the theory of weak stationary random processes. Combining these two aspects, one might suspect that they cause the difference between the spectra computed from the original series and their first differences respectively.

Since Garbers substantiates the existance of feedback relations between time series strictly on a qualitative basis, it seems to be a natural step to look out for a definition of feedback that allows its measurement prior to spectral procedures.

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1) Translated quotation: "On the assumption that two variables are connected with each other, one direction of this mutual influence is dominating if the coherence and phase spectra, estimated from the original time series, correspond extensively to those spectra computed from the first differences. This dominance is of less weight if the phase spectra hardly correspond, although the coherences still do fairly well. Finally, there is confirmed feedback between the variables if neither the phases nor the coherences correspond to one another."

We will adopt the definition of feedback and causality in the book of Granger [1964]. Granger's definition seems to fit best into spectral theory: both resting on second moment properties. So if we have two processes labelled j and k we take C(k,j) as a measure of the strength of the feedforward relation from process k to process j where C(k,j) is defined as

$$C(k,j) = 1 - V_j(j,k) \,/\, V_j(j) \,. \qquad (9)$$

$V_j(j)$ is the prediction error variance of the best linear predictor of the process j using only past values of this process whereas $V_j(j,k)$ stands for the corresponding variance if one uses the past values of both processes. If one interchanges j and k one gets the measure C(j,k) for the feedforward relation from process j to k. The product C(k,j)C(j,k) will be taken as measure for feedback between process j and process k.

The following table summarizes some results. It can be seen that there is on the ground of the adopted definition of feedback, no evidence for Garbers' rule of thumb. The first pair of time series shows results which are in full agreement with his rule of thumb, but the second pair brings quite the opposite results one should expect.

Table 4

Time Series	1→2	2→1	1<->2	difference in phase	difference in coherence
1 Auftragseingang Industrie 1955-1968 2 Umsatz Industrie	yes	yes	yes	yes	yes
1 Kurzfristige Kredite an Nichtbanken 1957-1968 2 Bankenliquidität 2. Grades	no	no	no	yes	yes

j → k means feedforward from j to k ; j <-> k means feedback relation.

Now one could rise the question whether the proposed procedure would work at all in practical applications as predicted by theoretical considerations. A natural way to find it out would be to start simulation experiments.

Simulation experiments made by Birkenfeld [1973] prove two things. First the adequacy of our tools in feedback situations and secondly that Garbers' rule is wrong. Birkenfeld experimented with simulated time series generated by processes with known feedback relations. In the presence of feedback his results never revealed any considerable deviations between the coherence spectra or phase spectra estimated from the original time series and their first differences. Beyond that, he was always able to redetect the feedback relations applying a procedure based on Granger's definition.

8. CONCLUSION

It has been shown by various examples that simulation is a valuable aid in preparing for empirical studies on time series analysis. But prior to any simulation experiment one thing should be made distinct. If one checks on methods all facts deduced from theory must be correct. This seems obvious but unfortunately one can find counter examples.

For instance Schips and Stier [1973] spent quite a lot of effort to invent a theory about the possibility to estimate the phase spectrum in dependence of the type of the power spectra. They back their results by simulation studies. Unfortunately they have overseen just one small point: their theoretical phase function is wrong. If one tackles the problem using the correct phase function all difficulties are gone with the wind. It was just much ado about nothing.

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